MATHEMATICS

COURSE – II

Vectors, Three Dimensional Geometry,
Theory of Equations

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PREFACE

This book deals with the topics in Vectors, Three Dimensional Geometry, Theory of ations included in the syllabus for the second year of the B.Sc. Course offered by the dhra Pradesh Open University. These topics cover the "core" area of the subject to be died in the Second Year of the Three Year Degree Course in Science. The syllabus for sake of convenience is divided into Blocks, each of which comprises a number of units. ch Block generally covers a specific area of the subject. The units are prepared by spe-lists in accordance with a format so designed as to enable the student to read and under-ind them without much difficulty. Each unit begins with a statement of its objectives and s at its end assignments intended to test the students comprehension of its subject atter.

In Block–1, vector is taken as a quantity which has magnitude and direction to make student to apply the concepts to geometry and physics. Analytical Geometry provides a neral method for solving geometric problems. This analytic geometry was explained in ocks 2, 3, 4 and 5. Roots of an equation, relation between the roots and coeffcients, ure of roots, and solutions of cubic and biquadratic equations are introduced through the eory of equations in Block 6.

The university hopes that the course material will help the student to get acquainted the concepts and principles of Mathematics.

Contents

BLOCK-1 : VECTOR ANALYSIS

Introduction

Vectors are useful tools in mathematical and physical problems. Many physical quantities like velocities and forces may be represented by vectors. Use of vector methods simplifies many calculations considerably. Further, vectors help us in visualising many physical and geometrical quantities and relations between them. In this Block we shall consider the basic concepts and methods of Vector Analysis and their applications to various physical and geometrical problems. The algebra of vectors mostly follows the same lines as the algebra of real numbers with occasional deviations, especially, in multiplication. It is assumed that the reader is familiar with elementary concepts of geometry like, lines, planes, triangles, quadrilateral, tetrahedron, parallellism etc.

UNIT-1 : VECTORS AND SCALARS

1.0 Contents

1.1 Aims and Objectives

After going through unit you should be able to :

i) distinguish between scalar quantities and vectors quantities,

ii) acquaint with the elementary operations of addition, subtraction and multiplication by a scalar of vector quantities,

iii) formulate the geometric and physical problems in vectorial form.

1.2 Introduction

In physics a quantity which has magnitude but no direction is called a scalar. Volume, mass, density are some examples of scalar quantities. A scalar is represented by a real number (infact it is a real number) which denotes its magnitude; eg. volume of a sphere, mass or density of a body, temperature of a body etc.

A non-negative real number associated with a direction is called a vector [the direction from A to B is nothing but the ordered pair (A, B)]. Displacement, velocity, force are some examples of vectors. A vector is represented by a directed line segment whose length represents the magnitude of the vector and whose direction is same as that of the directed line segment. In Fig. 1, $\overrightarrow{AB}$ is the directed line segment with length AB and direction from A to B. A is called the initial point and B is called the terminal point. Line segments $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are different since they have opposite

Fig. 1

directions eventhough they have the same length. A vector is usually denoted by a single letter such as a, b, c etc. Thus we may write $a = \overrightarrow{AB}$. Then $|a|$ denotes the length of the vector a. Length of a vector is also known as magnitude or modulus of the vector.

1.3 Types of Vectors

Equality of Vectors : Two vectors are said to be equal if and only if they have the same

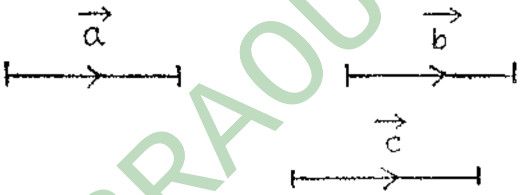

Fig. 2

magnitude and same or parallel directions (see Fig. 2). If **a** and **b** are equal we write **a** = **b**. In Fig. 2, **a** = **b** = **c**.

SAQ 1: Is weight of a body a vector or a scalar ?

Zero Vector : A vector whose magnitude is zero is called zero vector or a null vector and is denoted by 0 (It may be noted that all zero vectors are equal). The initial and terminal points of a zero vector coincide. Thus $\overrightarrow{AA}$, $\overrightarrow{BB}$ etc., are null vectors.

Unit vector : A vector whose magnitude is unity (i.e., 1) is called a unit vector. The unit vector in the direction of a non-zero vector a is denoted by $\hat{a}$ and is read as 'a cap'. Thus $\hat{a} = \frac{a}{|a|}$.

Negative of a vector : The vector whose magnitude is the same as a, but whose direction is opposite to that of a is called the negative of a and is denoted by **–a**. Thus in Fig. 3, $\overrightarrow{AB}$ = **a** and $\overrightarrow{BA}$ = **–a** .

Fig. 3

Coinitial vectors : Vectors having the same initial point are called coinitial vectors. In Fig.4 **a, b, c** are coinitial vectors.

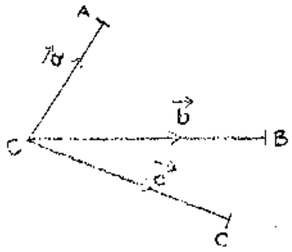

Fig. 4

Notation : Let O be any arbitrary point taken as origin. If A is any point different from O, $\overrightarrow{OA}$ is usually denoted by **a** where **a** = length $\overrightarrow{OA}$.

Prallelogram Law of Forces

According to this law, if two forces, acting at any point, are represented by the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram (see Fig. 5). This law motivates the definition of addition of vectors.

1.4 Addition of vectors

The sum of two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is defined as the vector $\overrightarrow{AC}$ (Fig. 5). Thus if a and b are any two given vectors (not necessarily coinitial) and b is moved parallel to itself so as to make its initial point to coincide with the terminal point of **a**, then the vector sum **a + b** is the vector whose

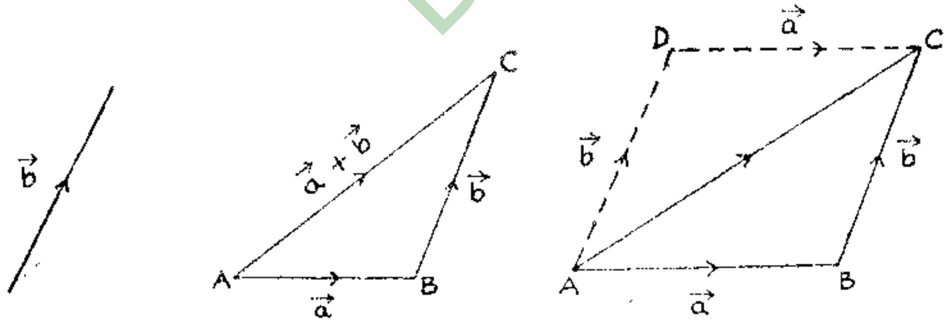

Fig. 5

initial point is the same as that of **a** and the terminal point as that of **b**. It is clear from Fig. 5, that if $\overrightarrow{AB}$, $\overrightarrow{BC}$ represent vectors **a** and **b** respectively then the third side $\overrightarrow{AC}$ of the triangle ABC gives the vector sum **a + b**. For this reason, this definition of addition of vectors is called the triangular law of addition. Completing the parallelogram ABCD and noting that $\overrightarrow{BC} = \overrightarrow{AD} = \mathbf{b}$, we have

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD}$$

That is, the sum of two coinitial vectors is the vector represented by the diagonal of the parallelogram formed with the component vectors as the adjacent sides. This method of addition of vectors is known as the parallelogram law of addition and is identical with the triangular law.

Note : If $\overrightarrow{AB} = \overrightarrow{A'B'}$ and $\overrightarrow{BC} = \overrightarrow{B'C'}$ (Fig. 6), then it can be seen from elementary geometrical

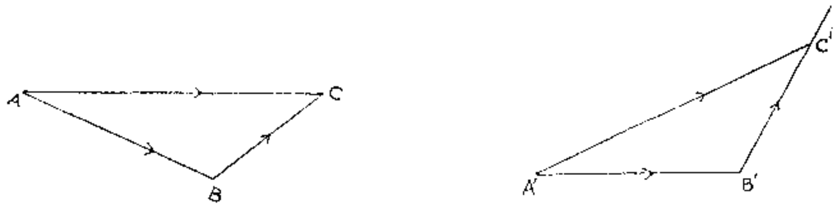

Fig. 6

considerations as indicated in the Fig. 6 that $\overrightarrow{AC} = \overrightarrow{A'C'}$. It may be noted that triangle ABC is congruent with the triangle A′B′C′ and AC is parallel to A′C′.

(i) **Vector addition is commutative** : For any vectors **a** and **b**, **a** + **b** = **b** + **a**.

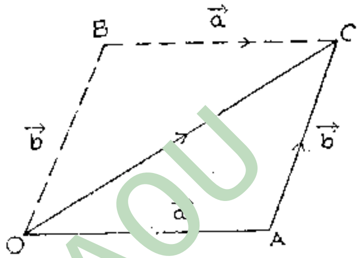

Fig.7

Let $\overrightarrow{OA}$ = a and $\overrightarrow{OB}$ = b. Then by triangular law of addition we have

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ = a + b ... (1)

Completing the parallelogram OABC, we have

$\overrightarrow{OB} = \overrightarrow{AC}$ = b, $\overrightarrow{BC} = \overrightarrow{OA}$ = a.

Now by definition

$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$ = b + a ... (2)

From (1) and (2) **a** + **b** = **b** + **a**

(ii) **Vector addition is associative**: For any three vectors **a**, **b**, **c**, (**a**+**b**)+**c** = **a**+(**b**+**c**)

Let $\overrightarrow{OA}$ = a, $\overrightarrow{AB}$ = b and $\overrightarrow{BC}$ = c. Complete

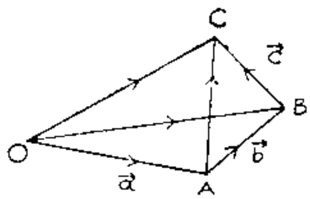

Fig. 8

the quadrilateral OABC (and draw the diagonals). Then, we have

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$\text{Also } \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\mathbf{b} + \mathbf{c})$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \qquad \ldots (1)$$

$$\text{Again } \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$$

$$\text{But } \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = (\mathbf{a} + \mathbf{b})$$

$$\therefore \overrightarrow{OC} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \qquad \ldots (2)$$

From (1) and (2) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

In view of the equality of vectors $(\mathbf{a}+\mathbf{b})+\mathbf{c}$, $\mathbf{a}+(\mathbf{b}+\mathbf{c})$ we can write each of these as $\mathbf{a}+\mathbf{b}+\mathbf{c}$ without any ambiguity. Further, it may be noted that the result holds good whether **a**, **b**, **c** are coplanar or not.

1.5 Subtraction of vectors

If **a**, **b** be any two vectors then their difference is defined by $\mathbf{a} - \mathbf{b} = \mathbf{a}+(-\mathbf{b})$. Thus to subtract a vector **b** from **a** reverse the direction of **b** and add it to **a**. In particular, $\mathbf{a} - \mathbf{a} = \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (In fact that is the definition of **0**). It may be noted that $\mathbf{a} + \mathbf{b} = \mathbf{c} \Rightarrow \mathbf{a} = \mathbf{c} - \mathbf{b}$ ($\Rightarrow$ stands for 'implies').

1.6 Multiplication of a vector by a scalar

Let **a** be any non zero vector and m a non zero scalar. Then the product of the vector **a** by the scalar m, denoted by $m\mathbf{a}$, is defined as the vector whose magnitude is $|m|$ times the magnitude of **a** and whose direction is the same as that of **a** if m is positive or opposite to that of **a** if m is negative.

We also define $0.\mathbf{a} = \mathbf{0}$ and $m.\mathbf{0} = \mathbf{0}$. The reader is advised to verify that

$$n\,(m\,\mathbf{a}) = m\,(n\mathbf{a}) = mn\,\mathbf{a}$$

$$\text{and } (m+n)\,\mathbf{a} = m\,\mathbf{a} + n\,\mathbf{a}.$$

where m and n are scalars.

1.7 Collinear vectors

Non zero vectors having the same (parallel) direction are called collinear vectors. If **a** is any non zero vector, every vector **r** which is collinear with **a** can be written as $\mathbf{r} = x\,\mathbf{a}$, where x is a scalar. This follows from the definition of product of a vector by a scalar.

Theorem-1: If **a** and **b** are any two non-collinear vectors and l, m are any two scalars, then

$$l\,\mathbf{a} + m\,\mathbf{b} = \mathbf{0} \;\Rightarrow\; l = 0, m = 0$$

Proof : Suppose $l \neq 0$, then

$$l\,\mathbf{a} + m\,\mathbf{b} = \mathbf{0} \;\Rightarrow\; \mathbf{a} = -\frac{m}{l}\,\mathbf{b},$$

which shows that **a** and **b** are collinear, which is a contradiction.

$\therefore\; l = 0$, similarly $m = 0$.

Position vector : Let O be any point of reference. The position vector of any point P in space with respect to O (as origin of reference) is defined to be $\overrightarrow{OP} = \mathbf{r}$.

SAQ 2 : Is $\mathbf{i} + \mathbf{j} = \mathbf{0}$? Here **i** and **j** are unit vectors along x and y axes respectively.

1.8 Coplanar vectors

Vectors which lie in the same plane are called coplanar vectors. Here we assume that the vectors a: co-initial.

Theorem-2 : If **a**, **b** be two non collinear position vectors and if **r** be the position vector of any point in the plane OAB, then **r** can be written as

$$\mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} \qquad \dots (1)$$

where x and y are unique scalars. Conversely, any vector of the form (1) stands for the position vector of point in the plane OAB.

Proof : Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OP} = \mathbf{r}$.

From P draw PL and PM parallel to OB and OA to meet OA and OB in L and M respectively.

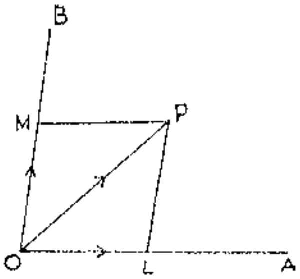

Fig. 9

Then $\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OL} + \overrightarrow{OM}$

Since $\overrightarrow{OL}$ is collinear with $\overrightarrow{OA} = \mathbf{a}$, we have $\overrightarrow{OL} = x\mathbf{a}$ where x is a suitable scalar.

Similarly $\overrightarrow{OM} = y\,\mathbf{b}$.

$\therefore\ \mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b}$.

The converse is obvious and its proof is left to the reader.

Uniqueness : Let $\mathbf{r} = x'\mathbf{a} + y'\mathbf{b} = x\,\mathbf{a} + y\,\mathbf{b}$

Then $(x - x')\,\mathbf{a} + (y - y')\,\mathbf{b} = 0$, and hence

$x - x' = 0,\ y - y' = 0$, since **a**, **b** are non collinear (cf. Theorem 1).

$\therefore\ x = x',\ y = y'$.

Hence **r** can be written as $x\,\mathbf{a} + y\,\mathbf{b}$ uniquely.

SAQ 3 : Are the vectors $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{k} + \mathbf{i}$ coplanar ?

Theorem-3 : If **a**, **b**, **c** are non coplanar vectors and l, m, n are scalars, then

$$l\,\mathbf{a} + m\,\mathbf{b} + n\,\mathbf{c} = 0 \Rightarrow l = 0,\ m = 0,\ n = 0$$

Proof : If possible let $l \neq 0$, then $\mathbf{a} = -\frac{m}{l}\,\mathbf{b} - \frac{n}{l}\,\mathbf{c}$ which shows that **a** is coplanar with **b** and **c** (cf Th. 2) which is a contradiction.

Therefore $l = 0$. Similarly, $m = 0$ and $n = 0$.

Theorem-4 : If **a**, **b**, **c** are three given non-coplanar vectors then any vector **r** can be written as

$$\mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c}$$

where x, y, z are unique scalars.

Proof : Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OP} = \mathbf{r}$

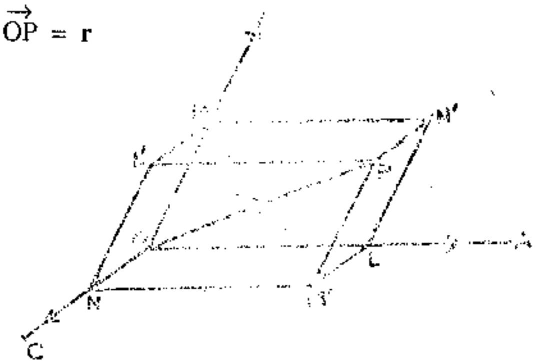

Fig. 10

Lines OA, OB, OC being non-coplanar, define three different planes BOC, COA, AOB when taken in pairs. Let P be any point. Through P draw planes parallel to these three planes meeting OA, OB and OC in L, M and N respectively, so that we obtain a parallelopiped having OP as a diagonal (see Fig.10). We have

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OL} + \overrightarrow{LN'} + \overrightarrow{N'P}$$

$$= \overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON}$$

Since $\overrightarrow{OL}, \overrightarrow{OM}, \overrightarrow{ON}$ are collinear with **a**, **b**, **c** respectively, there exist scalars x, y, z such that

$$\overrightarrow{OL} = x\,\mathbf{a},\ \overrightarrow{OM} = y\,\mathbf{b},\ \overrightarrow{ON} = z\,\mathbf{c}$$

$$\therefore\ \mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c}$$

Uniqueness : Let $\mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$.

Then $(x - x')\,\mathbf{a} + (y - y')\,\mathbf{b} + (z - z')\,\mathbf{c} = \mathbf{0}$

and so $x - x' = 0,\ y - y' = 0,\ z - z' = 0$ (by Th. 3)

Thus $x = x',\ y = y',\ z = z'$.

1.9 Vector r in terms of unit vectors i, j and k

If **i, j, k** are unit vectors along three mutually perpendicular coordinate axes OX, OY, OZ respectively. Then using theorem 4, any vector $\overrightarrow{OP} = \mathbf{r}$ can be expressed as

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

where x, y, z are called the coordinates of the point P. Usually, **i, j, k** are taken to form a right handed system in the sense that viewed from end point of **k**, the rotation from **i** to **j** through 90°

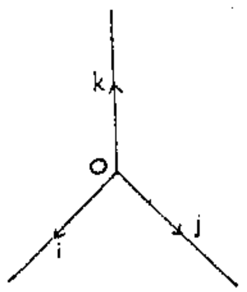

Fig. 11

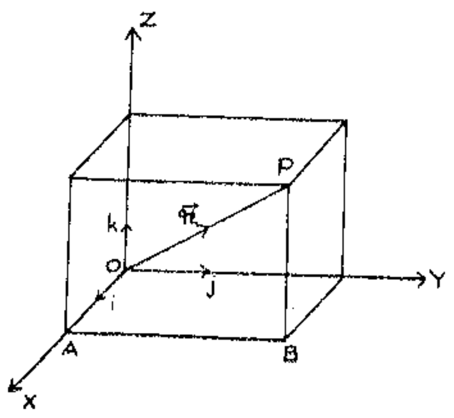

Fig. 12

appears anti-clock wise (see fig.11). Here $\mathbf{r} = \overrightarrow{OP}$ is the position vector of the point P which has the coordinates (x, y, z) with reference to the set of rectangular axes passing through O.

We know that $OP^2 = OB^2 + PB^2 = OA^2 + AB^2 + PB^2$ (see fig. 12)

$$= x^2 + y^2 + z^2$$

And so $|\mathbf{r}| = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

1.10 Vector $\overrightarrow{AB}$ in terms of the position vectors of A and B

Let O be the origin and let **a**, **b** be the position vectors of A, B. Then we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

(by Triangular law of addition)

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

Thus $\overrightarrow{AB}$ = position vector of B − position vector of A.

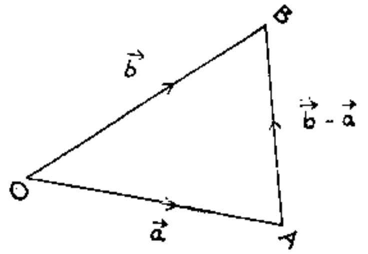

Fig. 13

Examples

Ex.1 : Find the position vector of a point which divides the line joining two given points in a given ratio.

Sol : Let O be the origin of reference. Let **a** and **b** be the position vectors of the points A and B. i.e., $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Let C divide AB in the ratio $l : m$ so that

$$\frac{AC}{CB} = \frac{l}{m}$$

Therefore $l\,CB = m\,AC$.

Hence $\quad l\,\overrightarrow{CB} = m\,\overrightarrow{AC}$

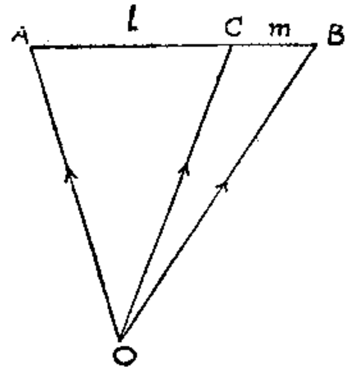

Fig. 14

Expressing the vectors $\overrightarrow{BC}$ and $\overrightarrow{AC}$ in terms of the position vectors of the end points, we have

$$l\left(\overrightarrow{OB} - \overrightarrow{OC}\right) = m\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$$

$$\text{i.e., } (l+m)\,\overrightarrow{OC} = l\,\overrightarrow{OB} + m\,\overrightarrow{OA} = l\,\mathbf{b} + m\,\mathbf{a}$$

$$\therefore \overrightarrow{OC} = \frac{l\,\mathbf{b} + m\,\mathbf{a}}{l + m}.$$

The reader is advised to observe that the result holds even if C divides AB externally in the ratio $l : m$.

In particular, if C is the midpoint of AB, then $\overrightarrow{OC} = \dfrac{\mathbf{a} + \mathbf{b}}{2}$

Ex.2 : Show that the points whose position vectors are $-2\,\mathbf{a} + 3\,\mathbf{b} + 5\,\mathbf{c}$, $\mathbf{a} + 2\,\mathbf{b} + 3\,\mathbf{c}$ and $7\,\mathbf{a} - \mathbf{c}$ are collinear.

Sol : Let us denote the given points by D, E, F. Taking any point as the origin of reference, we have

$$\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = (\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}) - (-2\mathbf{a} + 3\mathbf{b} + 5\mathbf{c})$$

$$= 3\mathbf{a} - \mathbf{b} - 2\mathbf{c}$$

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD} = (7\mathbf{a} - \mathbf{c}) - (-2\mathbf{a} + 3\mathbf{b} + 5\mathbf{c})$$

$$= 9\mathbf{a} - 3\mathbf{b} - 6\mathbf{c} = 3(3\mathbf{a} - \mathbf{b} - 2\mathbf{c}) = 3\,\overrightarrow{DE}$$

$$\text{Thus } \overrightarrow{DF} = 3\,\overrightarrow{DE}$$

Therefore $\overrightarrow{DF}$ and $\overrightarrow{DE}$ are collinear.

Hence the points D, E, F are collinear.

Ex. 3 : Find the vector equation of a straight line passing through a given point and parallel to a given vector.

Sol : Let O be the origin of reference. Let A be the given point and **b** be the given vector. Let **a** be the position vector of A and **r** the position vector of any point P on the straight line drawn through A parallel to **b**.

Since AP is parallel to **b**, we can write $\overrightarrow{AP} = t\,\mathbf{b}$, where t is a scalar.

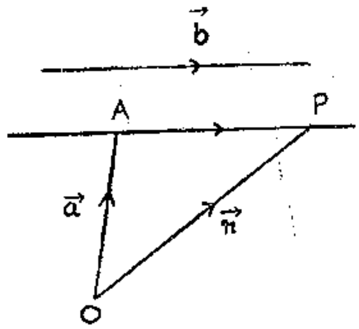

Fig. 15

Then $\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + t\,\mathbf{b}$

Therefore $\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$ is the equation of a given straight line in vector form.

Ex. 4 : Prove that the straight line joining the mid points of two sides of a triangle is parallel to the third side and half of its length.

Sol : Let ABC be the given triangle in which D and E are the middle points of AB and AC respectively. Let A be taken as the origin of refrence and let **b** and **c** be the position vectors of B and C respectively.

Then the position vectors of D and E are $\frac{\mathbf{b}}{2}$ and $\frac{\mathbf{c}}{2}$ respectively.

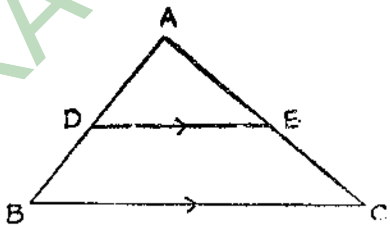

Fig. 16

Now $\overrightarrow{BC}$ = (position vector of C) – (position vector of B) = $\mathbf{c} - \mathbf{b}$

$\overrightarrow{DE}$ = (position vector of E) – (position vector of D) = $\frac{1}{2}(\mathbf{c} - \mathbf{b})$

Therefore DE is parallel to BC and DE $= \frac{1}{2}$ BC.

Ex. 5 : Prove that the diagonals of a parallelogram bisect each other.

Sol : Let ABCD be a parallelogram.

Take A as origin of reference.

Let position vectors of B and D be **b** and **d**.

Then the position vector of

$C = \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \mathbf{b} + \mathbf{d}$

Position vector of E,

the middle point of BD $= \frac{1}{2}(\mathbf{b} + \mathbf{d})$

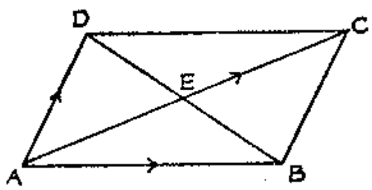

Fig. 17.

Again position vector of middle point of AC $= \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\mathbf{b} + \mathbf{d})$

$\therefore$ Mid Point of AC is same as midpoint of BD.

Ex. 6 : Five forces act at one vertex A of a regular hexagon in the directions of the other vertices and proportional to the distances of these vertices from A. Find their resultant.

Sol : The forces are represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ and $\overrightarrow{AF}$.

If **R** is their resultant (vector sum), then

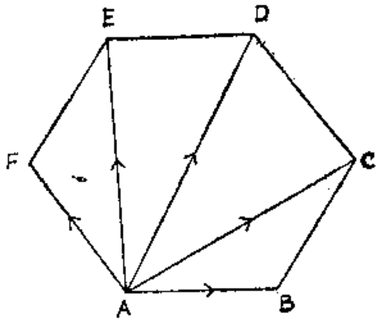

Fig. 18

$$\mathbf{R} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$$

$$= \overrightarrow{AB} + \left(\overrightarrow{AD} + \overrightarrow{DC}\right) + \overrightarrow{AD} + \left(\overrightarrow{AD} + \overrightarrow{DE}\right) + \overrightarrow{AF}$$

$$= \left(\overrightarrow{AB} + \overrightarrow{DE}\right) + \left(\overrightarrow{DC} + \overrightarrow{AF}\right) + 3\,\overrightarrow{AD}$$

$= 3\,\overrightarrow{AD}$ (It can be easily seen that $\overrightarrow{AB}$ and $\overrightarrow{DE}$ being equal and opposite vectors, their sum is zero. Similarly $\overrightarrow{DC} + \overrightarrow{AF} = 0$)

$\therefore$ The resultant $\mathbf{R} = 3\,\overrightarrow{AD}$

Ex. 7 : ABCD is a parallelogram and O is any point. Show that the forces represented by $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are equivalent to the forces represented by $\overrightarrow{OB}$ and $\overrightarrow{OD}$.

Sol. : Let P be the point of intersection of the diagonals. Let us take O as origin of reference.

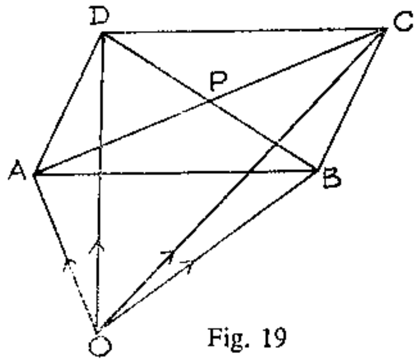

Fig. 19

Clearly P is the midpoint of AC as well as of BD (cf. Ex. 5)

$\therefore$ The position vector of P $= \mathbf{p} = \dfrac{\mathbf{a} + \mathbf{c}}{2} = \dfrac{\mathbf{b} + \mathbf{d}}{2}$

$\therefore \mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$.

1.11 Summary

A physical quantity which has both a manitude and a direction in space is called a vector quantity It is represented by a directed line segment. **a** = **b** implies and implied by the direction of **a** is same as the direction of **b** and the magnitude of **a** = $|\mathbf{a}|$ is same as the magnitude of **b** = $|\mathbf{b}|$. $\frac{\mathbf{a}}{|\mathbf{a}|} = \hat{\mathbf{a}}$ denotes a unit vector in the directon of the vector **a**. Vectors obey the triangular law of addition. Addition of vectors is commutative. Multiplication of a vector **a** by a scalar α is a vector α **a** whose magnitude is α times the magnitude of **a** and the direction is same as that of **a** if α is positive and opposite to that of **a** if α is negative. Vector **a**, α **a** are said to be collinear vectors. If a vector **a** can be expressed in the form $x\,\mathbf{b} + y\,\mathbf{c}$ where x, y are scalars, then the vectors **a**, **b**, **c** are said to be coplanar. If **i**, **j**, **k** denote the unit vectors along the rectangular coordinate axes x, y, z directions then any vector **r** in space can be expressed in the form $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ where x, y, z are scalars. Vector $\overrightarrow{AB}$ is equal to $\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$. If a point C divides the line joining two points A, B in the ratio $m : n$ then position vector of C is $= \frac{m\,\overrightarrow{OB} + n\,\overrightarrow{OA}}{m + n}$.

1.12 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Define vector, addition of vectors, multiplication of vectors by a scalar.

b) Show that the vectors **a**, **b**, 3**a** – 2**b** are collinear.

(ii) a) Define coplanar vectors. Show that any vector **r** can be written as $\mathbf{r} = x\,\mathbf{a} + y\,\mathbf{b} + z\,\mathbf{c}$ where **a**, **b**, **c** are three given vectors.

b) If G is the centroid of the triangle ABC show that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \mathbf{0}$.

II. *Briefly answer the following questions.*

i) Show that the vectors **a** – 2**b** + 3**c**, –2**a** + 3**b** – **c**, 4**a** – 7**b** + 7**c** are collinear.

ii) The position vectors of the points A, B, C, D are given by $2\mathbf{i} + 4\mathbf{k}$, $5\mathbf{i} + 3\sqrt{3}\,\mathbf{j} + 4\mathbf{k}$, $-2\sqrt{3}\,\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$. Show that AB is parallel to CD, and $\frac{3}{2}$ times CD.

iii) Find the vector equation of the line joining the points $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{k} - 2\mathbf{j}$.

iv) Forces p, 2p and 3p act along the sides AB, BC, CA of an equilateral triangle ABC. Find their resultant.

v) Find the unit vector parallel to the sum of the vectors $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

1.13 Answers to Self Assessment Questions

SAQ 1: Yes, weight of a body **w** is a vector quantity. It has mass as its magnitude and has direction vector vertically downwards.

SAQ 2: $\mathbf{i} + \mathbf{j} \neq 0$, since **i** and **j** are non-collinear vectors.

SAQ 3: They are not coplanar vectors, since (they are linearly independent) one cannot be expressed as a linear combination of the other two.

UNIT–2 : SCALAR AND VECTOR PRODUCTS OF VECTORS

2.0 Contents

2.1 Aims and objectives

After going through this unit you will be able to :

(i) define the scalar and vector products of two vectors and interpret them geometrically,

(ii) prove that the scalar and vector products of vectors are distributive over addition of vectors,

(iii) obtain expressions for scalar and vector products of three or more vectors.

2.2 Introduction

In unit 1, we have dealt with addition and subtraction of vectors and also the multiplication of vectors with scalars. In this unit we shall deal with the idea of the "products" of two or three vectors. Since the origin of vector algebra lies in physical and geometrical problems, the definitions for the products of vectors will be so devised that they will be consistent with such products which occur in problems of physics, mechanics and geometry. Here after, **i, j, k** stand for a right handed system of unit vectors along three mutually perpendicular axes.

2.3 Scalar (dot) product of two vectors

The scalar product of two vectors can be defined by considering the following physical situation.

If a constant force **F** acting on a particle displaces it from the position A to the position B, then

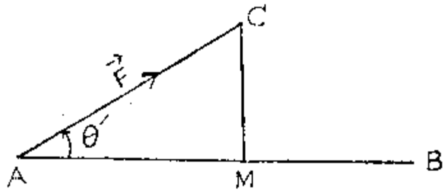

Fig. 1

$$\text{work done by } \mathbf{F} = (AB)\,(\text{Resolved part of } \mathbf{F} \text{ along } AB)$$

$$= (AB)\,(AM) = (AB)\,(|\mathbf{F}|\cos\theta) \qquad \dots (1)$$

Thus work done by a force **F** is the scalar product of vectors representing the force and displacement.

Definition : The scalar product of two vectors **a** and **b** is defined as the scalar, ab cos θ, where a, b are the magnitudes of the vectors **a** and **b** and θ is the angle between them such that $0 \le \theta \le \pi$. The scalar product is denoted by **a** . **b** by placing a dot between the vectors. For this reason the scalar product is also known as the dot product. It is also known as the inner product or direct product of the vectors **a** and **b**.

Thus, inview of (1), the scalar product of two vectors, physically, represents work done by a force in moving a particle from A to B.

2.3.1 Properties of dot product

(i) $\mathbf{a} \cdot \mathbf{b} = a\, b \cos\theta = b\, a \cos\theta = \mathbf{b} \cdot \mathbf{a}$

This shows that dot product is commutative.

(ii) If $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow a\, b \cos\theta = 0$. It follows that

$\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\cos\theta = 0$. If a and b be non - zero vectors, then $\cos\theta = 0 \Rightarrow \theta = \pi/2$ That is $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a}$ and $\mathbf{b}$ are perpendicular ($\mathbf{a} \neq 0, \mathbf{b} \neq 0$)

(iii) If $a = b = 1$, then

$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 1 \cos\theta = \cos\theta$

Hence the dot product of two unit vectors gives the cosine of the angle (hence the angle) between them.

(iv) If a and b be parallel, then $\theta = 0$ and so

$\mathbf{a} \cdot \mathbf{b} = ab \cos 0 = ab$

In particular if $\mathbf{a} = \mathbf{b}$, we get

$\mathbf{a} \cdot \mathbf{a} = a\, a = a^2$

$\therefore$ we write $\mathbf{a} \cdot \mathbf{a} = a^2$

(v) If m, n be two scalars and $\mathbf{a}$, $\mathbf{b}$ be two vectors then it can be seen, easily, that

$(m\, \mathbf{a}) \cdot (n\, \mathbf{b}) = m\, n\, (\mathbf{a} \cdot \mathbf{b}) = (n\, \mathbf{a}) \cdot (m\, \mathbf{b})$

The reader is advised to check the correctness of this in the cases $m \geq 0$ and $m < 0$ etc. As a particular case,

$$(l\, \mathbf{a}) \cdot \mathbf{b} = l\, ab \cos\theta = a\, (l\, b) \cos\theta = \mathbf{a} \cdot (l\, \mathbf{b}) = l\, (\mathbf{a} \cdot \mathbf{b})$$

(vi) we have $\mathbf{i} \cdot \mathbf{i} = 1 \cdot 1 \cos 0 = 1 \cdot 1 \cos 0 = 1$.

Similarly $\mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

Also $\mathbf{i} \cdot \mathbf{j} = 1 \cdot 1 \cos 90° = 0$

Similarly, $\mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

2.3.2 Geometrical Interpretation of Scalar Product

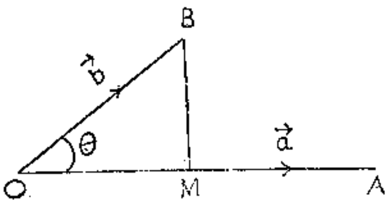

Fig. 2

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\angle AOB = \theta$, so that

$a = |\mathbf{a}| = OA$ and $b = |\mathbf{b}| = OB$.

Then, we have $\mathbf{a} \cdot \mathbf{b} = ab \cos\theta$, by definition

$= a\, (OB \cos\theta) = a\, (\pm\, OM)$, the + or – sign being taken according as $\cos\theta$ is positive or negative respectively.

$\therefore$ $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|$ multiplied by projection of $\mathbf{b}$ in the direction of $\mathbf{a}$. Similarly,

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|$ multiplied by the projection of $\mathbf{a}$ in the direction of $\mathbf{b}$.

Hence the scalar product can be interpreted geometrically as follows.

The scalar product of two vectors is the product of the magnitude of one vector a and the projection of the other vector in the direction of a.

Theorem-1 : (Distributive Law) : The scalar product is distributive with respect to vector addition. That is $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

Proof : Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{BC} = \mathbf{c}$.

Then we have

$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \mathbf{b} + \mathbf{c}$ (see fig. 3).

Draw BM and CN perpendicular to OA. Then OM, ON and MN are the projections of **b**, **b** + **c** and **c** respectively on the vector **a**.

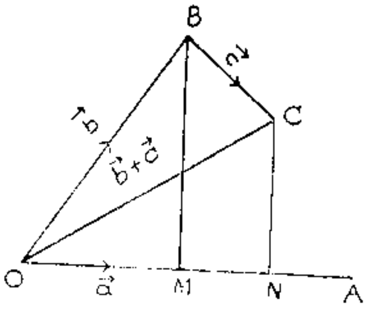

Fig. 3

$$
\begin{aligned}
\text{We have } \mathbf{a} . (\mathbf{b} + \mathbf{c}) &= \overrightarrow{OA} . \overrightarrow{OC} \\
&= \text{OA (projection of OC on OA)} \\
&= \text{OA (ON)} = \text{OA (OM + MN)} \\
&= \text{(OA) (OM)} + \text{(OA) (MN)} \\
&= \text{(OA) (proj. of } \overrightarrow{OB} \text{ on } \overrightarrow{OA}) + \text{OA (proj. of } \overrightarrow{BC} \text{ on } \overrightarrow{OA}) \\
&= \mathbf{a} . \mathbf{b} + \mathbf{a} . \mathbf{c}
\end{aligned}
$$

Thus $\mathbf{a} . (\mathbf{b} + \mathbf{c}) = \mathbf{a} . \mathbf{b} + \mathbf{a} . \mathbf{c}$

The reader is advised to draw other figures where angle between **a** and **b** is obtuse.

SAQ 1 : If **a** , **b** represent the sides of a right angled triangle, the angle included being 90°, what is the value of **a** . **b** ?

Remarks :

(i) Since scalar product is commutative, we have

$$\mathbf{a} . (\mathbf{b} + \mathbf{c}) = (\mathbf{b} + \mathbf{c}) . \mathbf{a}$$

(ii) Since scalar product is distributive, we have

$$
\begin{aligned}
\mathbf{a} . (\mathbf{b} - \mathbf{c}) &= \mathbf{a} . [\mathbf{b} + (-\mathbf{c})] = \mathbf{a} . \mathbf{b} + \mathbf{a} . (-\mathbf{c}) \\
&= \mathbf{a} . \mathbf{b} + [-(\mathbf{a} . \mathbf{c})] = \mathbf{a} . \mathbf{b} - \mathbf{a} . \mathbf{c}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
(\mathbf{a} + \mathbf{b})^2 &= (\mathbf{a} + \mathbf{b}) . (\mathbf{a} + \mathbf{b}) = (\mathbf{a} + \mathbf{b}) . \mathbf{a} + (\mathbf{a} + \mathbf{b}) . \mathbf{b} \\
&= \mathbf{a} . \mathbf{a} + \mathbf{b} . \mathbf{a} + \mathbf{a} . \mathbf{b} + \mathbf{b} . \mathbf{b} \\
&= \mathbf{a} . \mathbf{a} + 2\mathbf{a} . \mathbf{b} + \mathbf{b} . \mathbf{b} \quad [\because \mathbf{a} . \mathbf{b} = \mathbf{b} . \mathbf{a}] \\
&= \mathbf{a}^2 + 2\mathbf{a} . \mathbf{b} + \mathbf{b}^2 \\
\text{Similarly, } (\mathbf{a} - \mathbf{b})^2 &= \mathbf{a}^2 - 2\mathbf{a} . \mathbf{b} + \mathbf{b}^2
\end{aligned}
$$

(iv)

$$
\begin{aligned}
(\mathbf{a} + \mathbf{b}) . (\mathbf{a} - \mathbf{b}) &= (\mathbf{a} + \mathbf{b}) . \mathbf{a} - (\mathbf{a} + \mathbf{b}) . \mathbf{b} \\
&= \mathbf{a} . \mathbf{a} + \mathbf{b} . \mathbf{a} - \mathbf{a} . \mathbf{b} + \mathbf{b} . \mathbf{b} \\
&= \mathbf{a}^2 - \mathbf{b}^2 \quad [\because \mathbf{a} . \mathbf{b} = \mathbf{b} . \mathbf{a}]
\end{aligned}
$$

Scalar Product in terms of the components

$$
\begin{aligned}
\text{Let } \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and } \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \\
\text{Then } \mathbf{a} . \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) . (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 \text{ (Refer Result vi of 2.1 and Th. 1)}
\end{aligned}
$$

Thus when two vectors are known in their component forms the scalar product is equal to the sum of the products of their corresponding components.

In particular, $\mathbf{a} \cdot \mathbf{a} = a^2 = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$

$$= a_1^2 + a_2^2 + a_3^2$$

$$\therefore a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

2.4 Vector product of two vectors

The following physical situation motivates the definition of vector product of two vectors.

Let O be any point. If a force **F** acts at the point A, Draw OL perpendicular to the line of action of F.

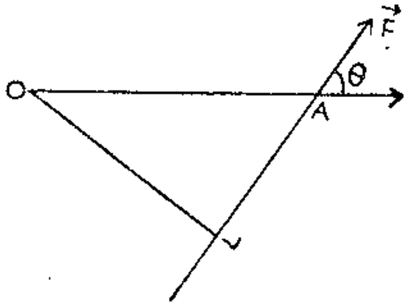

Fig. 4

If θ be the angle between $\overrightarrow{OA}$ and F, then

$|\overrightarrow{OA} \times \mathbf{F}| = OA\,|\mathbf{F}|\,\sin\theta = |\mathbf{F}|\; OL$ = magnitude of moment of F about O.

Hence the measure of moment of F about O = $|\mathbf{F} \times \overrightarrow{OA}|$.

It may be mentioned that moment of **F** about O is a vector.

Definition : The vector product of two vectors **a** and **b** is defined as the vector, **v**, which has the following properties.

(i) $|\mathbf{v}| = |\mathbf{a}|\;|\mathbf{b}|\;\sin\theta$, where θ is the angle between the two vectors **a** and **b**, $0 \le \theta \le \pi$.

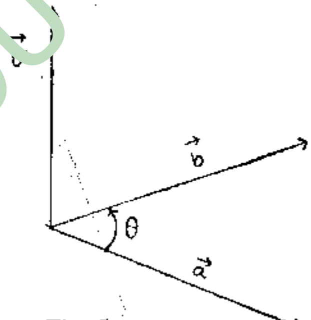

Fig. 5

(ii) **v** is perpendicular to both **a** and **b**. Thus v is perpendicular to the plane of **a** and **b** (i.e., $\mathbf{v} \cdot \mathbf{a} = 0$ and $\mathbf{v} \cdot \mathbf{b} = 0$).

(iii) **v** is directed in such a way that, a rotation about v (which will release a right handed screw forward in the direction of v) through an angle θ will bring **a** to the direction of **b**. The vector product v is written as **a x b** and hence is also called "The Cross Product" of the two vectors.

Some Useful Results : Some simple and useful results which follow immediately from the definition of the vector product, are given below.

(i) **a x b** and **b x a** have the same magnitude ab sin θ ; both are perpendicular to **a** and **b**, yet are opposite in directions (see fig. 6).

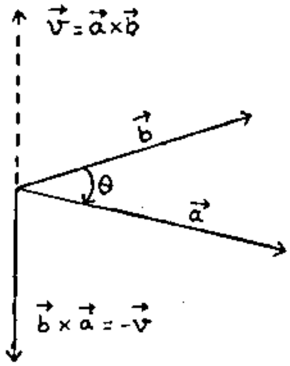

Fig. 6

Thus, we have $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

∴ The vector product is not commutative.

(ii) If $\mathbf{a} \times \mathbf{b} = 0$, that is if $ab \sin \theta = 0$, it follows that either $a = 0$, or $b = 0$ or $\theta = 0$ or π. In case a and b are non zero vectors, they are parallel to each other.

(iii) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

(iv) $\mathbf{a} \times m\,\mathbf{a} = m\,(\mathbf{a} \times \mathbf{a}) = m\,\mathbf{0} = \mathbf{0}$

(v) $\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = \mathbf{j}$

$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \mathbf{k} \times \mathbf{j} = -\mathbf{i},\ \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

and $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

(vi) Distributive Law $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

Proof : Let $\mathbf{p} = \mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$

For the dot product of **p** with an arbitrary vector **q**

i.e., $\mathbf{q} \cdot \mathbf{p} = \mathbf{q} \cdot \{\mathbf{a} \times (\mathbf{b} + \mathbf{c})\} - \mathbf{q} \cdot (\mathbf{a} \times \mathbf{b}) - \mathbf{q} \cdot (\mathbf{a} \times \mathbf{c})$

In section 2.5 we will show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

$$\begin{aligned} \text{Hence } \mathbf{q} \cdot \mathbf{p} &= (\mathbf{q} \times \mathbf{a}) \cdot (\mathbf{b} + \mathbf{c}) - (\mathbf{q} \times \mathbf{a}) \cdot \mathbf{b} - (\mathbf{q} \times \mathbf{a}) \cdot \mathbf{c} \\ &= (\mathbf{q} \times \mathbf{a}) \cdot \mathbf{b} + (\mathbf{q} \times \mathbf{a}) \cdot \mathbf{c} - (\mathbf{q} \times \mathbf{a}) \cdot \mathbf{b} - (\mathbf{q} \times \mathbf{a}) \cdot \mathbf{c} \end{aligned}$$

(since the scalar multiplication is distributive)

$$= 0.$$

Thus either $\mathbf{q} = \mathbf{0}$ or **q** is ⊥r to **p** or $\mathbf{p} = \mathbf{0}$. Since **q** is arbitray, we can choose it to be both non zero and not ⊥r to **p**. Hence $\mathbf{p} = \mathbf{0}$ and

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(vii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{i} + (a_3 b_1 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k}$ which can be written in the form

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{ symbolically.}$$

This follows from (v) and (vi)

Vector Area

Given a plane area bounded by a simple closed directed curve C, we associate a vector **A** with the given area such that

(i) the magnitude of **A** is equal to the area,

(ii) **A** is perpendicular to the plane of the area,

and (iii) the sense of **A** is such that the direction of description of the boundary of the curve and the sense of A correspond to the rotation of a right handed screw. In other words viewed from the end of **C** the rotation is anti-clock wise.

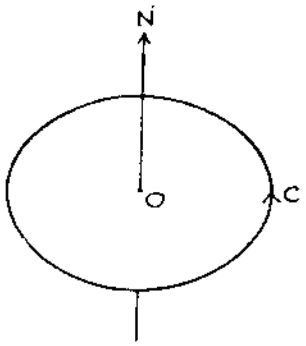

Fig. 7

Fig. 8

2.4.1 Geometrical Interpretation of Vector Product

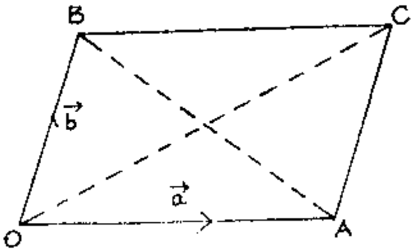

Fig. 8

Let OACB be a parallelogram as shown in the figure 8. We know that the area of Δ OAB $= \frac{1}{2}$ (OA) (OB) sin θ.

$$= \frac{1}{2} |\mathbf{a}| \, |\mathbf{b}| \sin\theta = \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$$

$\therefore$ The area of the parallelogram

$= 2 \times$ area of Δ OAB $= |\mathbf{a} \times \mathbf{b}|$

Thus the cross product of **a** and **b** is in magnitude the area of the parallelogram whose adjacent sides are represented by **a** and **b**.

Thus vector area of the parallelogram OACB $= \vec{OA} \times \vec{OB}$

SAQ 2 : What is the vector perpendicular to the vectors **a** and **b**?

SAQ 3 : In how many ways can you multiply two given vectors?

2.5 Products of Three or More Vectors

Given three vectors **a**, **b** and **c**, we can form the product of the following type (i) (**a . b**) **c**, (ii) (**a** × **b**) . **c** and (iii) (**a** × **b**) × **c**. The first being the product of the scalar **a . b** and the vector **c**, represents a vector parallel to **c**, the second being the scalar product of the vectors **a** × **b** and **c** is a scalar and is usually called the scalar triple product; the third is the vector product of the vectors **a** × **b** and **c** and so is a vector.

2.5.1 Scalar Triple Product

If **a**, **b**, **c** be any three vectors, then the scalar product of **a** × **b** with **c**, that is, (**a** × **b**) . **c** is called the scalar triple product of the vectors **a**, **b**, **c** and is denoted by [**a b c**]. There should be no ambiguity in omitting the brackets in (**a** × **b**) . **c** since **a** × (**b . c**) has no meaning.

Geometrical meaning of a Triple product : The product [**a b c**]represents in magnitude the volume of a parallelopiped having **a**, **b**, **c** as coterminus edges.

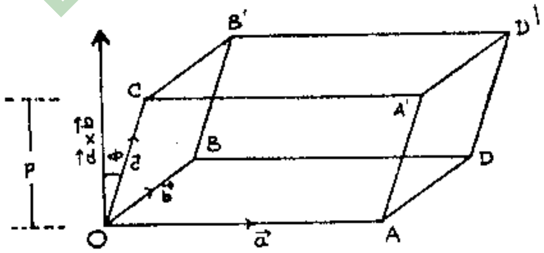

Fig. 9

Consider a parallelopiped with $\vec{OA}$ = **a**, $\vec{OB}$ = **b** and $\vec{OC}$ = **c** as coterminus edges where **a**, **b**, **c** form a right handed system. Let V be its volume and α be the area of the faces OADB or CA′ D′ B′ which are parallel to the vectors **a** and **b** and let p be the distance between these faces. Let ϕ be the angle between **c** and **a** × **b**.

$$\text{Then } |\mathbf{a} \times \mathbf{b}| = \alpha \text{ and } |\mathbf{c}| \cos\phi = p$$

$$\text{Volume V} = \alpha p = |\mathbf{a} \times \mathbf{b}| \, |\mathbf{c}| \cos\phi$$

$$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

Hence the result.

Note that [**a b c**] = − V, if **a**, **b**, **c** form a left - handed triad. In particular, we have

[**i j k**] = [**i** × **j**] . **k** = **k . k** = 1 we have the following results :

(i) A necessary and sufficient condition for three coterminus vectors to be coplanar is that their scalar triple product should vanish. If three vectors **a**, **b** and **c** are coplanar, the volume of the parallelopiped is zero. That is there is no volume. Hence [**a b c**] = 0.

(ii) If two vectors of a scalar triple product are equal, then the scalar triple product becomes zero. That is, there is no parallelopiped. If **a** = **b** or **b** = **c** or **c** = **a**, the parallelopied has zero volume and so [**a b c**] = 0.

(iii) The scalar triple product is independent of the position of dot and cross and is unaltered as long as the cyclic order of the factors remain the same. The sign of the scalar triple product changes when the cyclic order is changed.

Let **a, b, c** form a right – handed system of vectors such that $\mathbf{a} = \overrightarrow{OA}$; $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Then if V be the volume of the parallelopiped with OA, OB and OC as coterminus edges, we have $V = (\mathbf{a} \times \mathbf{b}) . \mathbf{c}$.

Again **b, c, a** and **c, a, b**, form right - handed system of triads and so

$V = (\mathbf{b} \times \mathbf{c}) . \mathbf{a} = (\mathbf{c} \times \mathbf{a}) . \mathbf{b}$

Thus $V = (\mathbf{a} \times \mathbf{b}) . \mathbf{c} = (\mathbf{b} \times \mathbf{c}) . \mathbf{a} = (\mathbf{c} \times \mathbf{a}) . \mathbf{b}$ (1)

Since dot product is commutative, we have

$V = \mathbf{c} . (\mathbf{a} \times \mathbf{b}) = \mathbf{a} . (\mathbf{b} \times \mathbf{c}) = \mathbf{b} . (\mathbf{c} \times \mathbf{a})$ (2)

From (1), it follows that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$

From (1) and (2) we have $(\mathbf{a} \times \mathbf{b}) . \mathbf{c} = \mathbf{a} . (\mathbf{b} \times \mathbf{c})$,

Showing that the scalar triple product is independent of the position of dot and cross.

Further, we see that **a, c, b** ; **b, a, c** ; **c, b, a** are left - handed system of triads, so that

$-V = (\mathbf{a} \times \mathbf{c}) . \mathbf{b} = (\mathbf{b} \times \mathbf{a}) . \mathbf{c} = (\mathbf{c} \times \mathbf{b}) . \mathbf{a}$

$= \mathbf{b} . (\mathbf{a} \times \mathbf{c}) = \mathbf{c} . (\mathbf{b} \times \mathbf{a}) = \mathbf{a} . (\mathbf{c} \times \mathbf{b})$ (3)

From (2) and (3) we get $\mathbf{a} . (\mathbf{b} \times \mathbf{c}) = V = -\mathbf{a} . (\mathbf{c} \times \mathbf{b})$

$[\because \mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})]$

$\therefore [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$, which shows that the change in the cyclic order changes the sign of the scalar triple product

(iv) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof: $(\mathbf{a} \times \mathbf{b}) . \mathbf{c} = [(a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}] \cdot (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$

$$= (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2.5.2 The Vector Triple Product

If **a, b, c** be any three vectors, then the vector product of $\mathbf{a} \times \mathbf{b}$ with **c** is called the vector triple product of the vectors **a, b** and **c** (in this order) and is written as $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. In this expression the parenthesis or some separating symbol, is necessary, as $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. That is vector triple product is not associative.

The vector $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$ and **c**. But since $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane of **a** and **b**, it follows that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ lies in the plane of **a** and **b**. Thus $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ can be expressed in the form $l\mathbf{a} + m\mathbf{b}$ (cf. Theorem - 2 of unit-1).

SAQ 4 : How many types of products can you formulate with the given three vectors ? What are they ?

Expansion Formula : If **a, b, c** are any three vectors then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\, \mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\, \mathbf{a}.$$

Proof : Let **i** be the unit vector along **a**. Let **j** be a unit vector perpendicular to **a** in the plane OAB (fig. 9 a). Then

$$\mathbf{a} = a_1 \mathbf{i}$$

$$\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j}$$

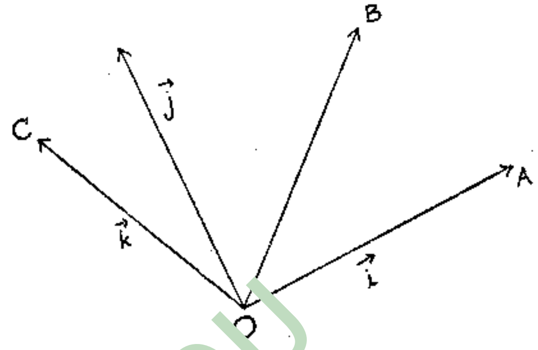

Fig. 9a

Let **k** be chosen $\perp$ r to **i** and **j**, so that **i, j, k** form a right-handed system, then

$$\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}$$

Now $\mathbf{a} \times \mathbf{b} = a_1 b_2 (\mathbf{i} \times \mathbf{j}) = a_1 b_2 \mathbf{k}$

$$\begin{aligned} \text{so that } (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= (a_1 b_2 \mathbf{k}) \times (c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}) \\ &= (a_1 b_2 c_1)(\mathbf{k} \times \mathbf{i}) + a_1 b_2 c_2 (\mathbf{k} \times \mathbf{j}) \\ &= a_1 b_2 c_1 \mathbf{j} - a_1 b_2 c_2 \mathbf{i} \\ &= a_1 c_1 (b_1 \mathbf{i} + b_2 \mathbf{j}) - (b_1 c_1 + b_2 c_2) a_1 \mathbf{i} \end{aligned}$$

(by adding and subtracting $a_1 b_1 c_1 \mathbf{i}$)

$$= (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}$$

Examples

Ex. 1 : Find the value of λ such that **a** and **b** are orthogonal, where $\mathbf{a} = 2\mathbf{i} + \lambda \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

Sol. : If **a** and **b** are at right angles, their dot product is zero.

$$\therefore\ \mathbf{a} \cdot \mathbf{b} = (2\mathbf{i} + \lambda \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 0$$

$$\Rightarrow 2 - 2\lambda + 3 = 0$$

$$\Rightarrow 2\lambda = 5$$

$$\therefore\ \lambda = 5/2$$

Ex. 2 : Find the projection of $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ on the vector $\mathbf{b} = \mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$

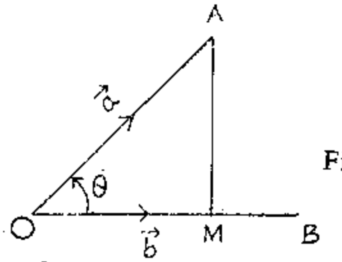

Fig. 10

Sol. : OM is the projection of **a** on **b**

$$\therefore \; OM = OA \cos\theta = OA\left(\frac{\mathbf{a}\,.\,\mathbf{b}}{a\,b}\right) \qquad \because \; \mathbf{a}\,.\,\mathbf{b} = ab\cos\theta$$

$$= \frac{\mathbf{a}\,.\,\mathbf{b}}{b} \qquad \because \; OA = a = |\mathbf{a}|$$

$$\mathbf{a}\,.\,\mathbf{b} = (2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\,.\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2 + 6 + 2 = 10$$

$$b = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

$$\therefore \; OM = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}.$$

Ex. 3 : Prove that the altitudes of a triangle are concurrent.

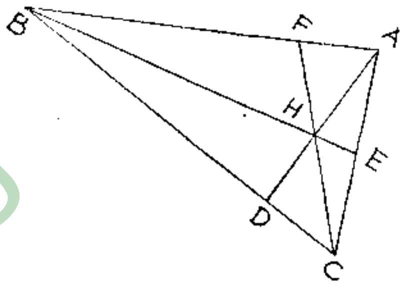

Fig. 11

Sol. : Let H be the point of interesction of the altitudes from A and B to the opposite sides. Let **a, b, c** and **h** be the position vectors of A, B, C and H respectively relative to some origin. Since AH is ⊥r to BC we have

$$(\mathbf{h} - \mathbf{a})\,.\,(\mathbf{c} - \mathbf{b}) = 0$$

i. e., $\mathbf{h}\,.\,\mathbf{c} - \mathbf{h}\,.\,\mathbf{b} - \mathbf{a}\,.\,\mathbf{c} + \mathbf{a}\,.\,\mathbf{b} = 0$ ... (1)

Again, since BH is ⊥r to CA, we have

$$(\mathbf{h} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c}) = 0$$

i. e., $\mathbf{h}\,.\,\mathbf{a} - \mathbf{h}\,.\,\mathbf{c} - \mathbf{b}\,.\,\mathbf{a} + \mathbf{b}\,.\,\mathbf{c} = 0$ ... (2)

Adding (1) and (2), we get

$$\mathbf{h}\,.\,\mathbf{a} - \mathbf{h}\,.\,\mathbf{b} - \mathbf{c}\,.\,\mathbf{a} + \mathbf{c}\,.\,\mathbf{b} = 0$$

i. e., $(\mathbf{h} - \mathbf{c}) \cdot (\mathbf{a} - \mathbf{b}) = 0$ ... (3)

Equation (3) shows that CH is perpendicular to BA.

This shows that the altitudes of a triangle are concurrent.

Ex. 4 : Prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$

Sol. : we have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\,.\,\mathbf{c})\,\mathbf{b} - (\mathbf{a}\,.\,\mathbf{b})\,\mathbf{c}$$
$$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b}\,.\,\mathbf{a})\,\mathbf{c} - (\mathbf{b}\,.\,\mathbf{c})\,\mathbf{a}$$
$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}\,.\,\mathbf{b})\,\mathbf{a} - (\mathbf{c}\,.\,\mathbf{a})\,\mathbf{b}$$

Adding and noting that the dot product is commutative we get

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

Ex. 5 : Find α such that the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \alpha\,\mathbf{j} + 5\mathbf{k}$ are coplanar.

Sol. : Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$,

$\mathbf{c} = 3\mathbf{i} + \alpha\mathbf{i} + 5\mathbf{k}$

If a, b, c be complanar, then [a b c] = 0

$$\text{i.e., } \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \alpha & 5 \end{vmatrix} = 0$$

i.e., $2(10 + 3\alpha) + 1(5 + 9) + 1(\alpha - 6) = 0$

i.e., $7\alpha = -28$

$\therefore \alpha = -4$

Ex. 6 : Show that $[\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Sol : We have $[\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}] = (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{b})$

put $\mathbf{b} \times \mathbf{c} = \mathbf{p}$

$$\begin{aligned} \therefore \text{L.H.S} &= \{\mathbf{p} \times (\mathbf{c} \times \mathbf{a})\} \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \{(\mathbf{p} \cdot \mathbf{a})\, \mathbf{c} - (\mathbf{p} \cdot \mathbf{c})\, \mathbf{a}\} \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \{(\mathbf{b} \times \mathbf{c} \cdot \mathbf{a})\, \mathbf{c} - (\mathbf{b} \times \mathbf{c} \cdot \mathbf{c})\, \mathbf{a}\} \cdot (\mathbf{a} \times \mathbf{b}) \\ &= \{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\, \mathbf{c} - 0\} \cdot (\mathbf{a} \times \mathbf{b}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{c}\ \mathbf{a}\ \mathbf{b}] \\ &= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 \end{aligned}$$

Ex. 7 : Due to the application of a force $\mathbf{F} = 6\mathbf{j} + 8\mathbf{k}$, a particle changes its position from A (1, –1, 2), to B (–1, 1, 2). Find the work done.

Sol : Work done = $\mathbf{F} \cdot \overrightarrow{AB}$

$\overrightarrow{AB}$ = position vector of B – position vector of A

$= (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -2\mathbf{i} + 2\mathbf{j}$

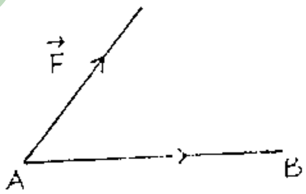

Fig. 12

$\therefore$ Work done $= \mathbf{F} \cdot \overrightarrow{AB} = (6\mathbf{j} + 8\mathbf{k}) \cdot (-2\mathbf{i} + 2\mathbf{j}) = 12$ units

Ex. 8 : A force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ is applied at the point (1, –1, 2). Find the moment of force about the point (2, –1, 3).

Sol : Moment of the force $|\mathbf{F} \times \overrightarrow{OA}|$

$\overrightarrow{OA}$ = position vector of A – position vector of O

$= (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= -\mathbf{i} - \mathbf{k}$

Moment of Force $= |(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \times (-\mathbf{i} - \mathbf{k})|$

$= |-2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}|$

$= \sqrt{4 + 49 + 4} = \sqrt{57}$

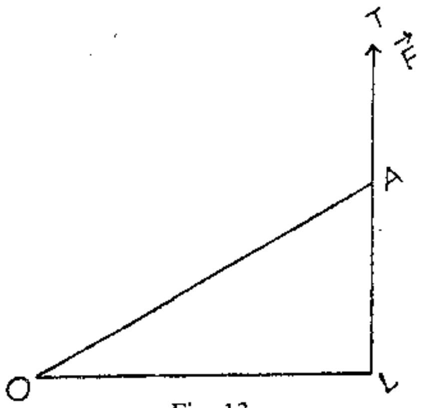

Fig. 13

Ex. 9 : If $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{BC} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{CA} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ form a right angled traingle. Find the remaining angles of the triangle.

Sol : We have $\overrightarrow{BC} + \overrightarrow{CA} = (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + (2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$

$= 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} = \overrightarrow{AB}$

Hence $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ form the sides of a triangle.

Also $\overrightarrow{AB} \cdot \overrightarrow{CA} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) =$ $6 - 2 - 4 = 0$, showing that $\overrightarrow{AB}$ is $\perp$ r to $\overrightarrow{CA}$.
$\therefore \angle A = 90°$.

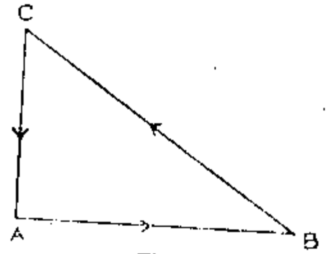

Fig. 14

$\therefore \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ form a right angled traingle.

$$\cos B = \frac{\left|-\overrightarrow{AB} \cdot \overrightarrow{BC}\right|}{(AB)(BC)} = \frac{3+6+5}{\sqrt{14}\sqrt{35}} = \sqrt{\frac{2}{5}}$$

$$\therefore B = \cos^{-1}\sqrt{\frac{2}{5}}$$

$$\cos C = \frac{\left|\left(-\overrightarrow{BC} \cdot \overrightarrow{CA}\right)\right|}{(BC)(CA)} = \frac{|2-3-20|}{\sqrt{35}\sqrt{21}} = \sqrt{\frac{21}{35}} = \sqrt{\frac{3}{5}}$$

$$\therefore C = \cos^{-1}\sqrt{\frac{3}{5}}$$

2.6 Summary

Scalar product of two vectors **a**, **b** is **a** . **b** is = |**a**| |**b**| cos θ, where θ is the angle between the vectors **a** and **b**. Geometrically it is the product of the magnitude of one vector **a** and the projection of other vector in the direction of the vector **a**. Scalar product of vectors is distributive over the addition of vectors. It means that **a** . (**b** + **c**) = **a** . **b** + **a** . **c**. Vector product of two vectors **a** and **b** is equal to |**a**| |**b**| sin θ $\hat{n}$ where θ is the angle between the vectors **a** and **b** and $\hat{n}$ is the unit vector in the direction of the outward drawn normal to the plane containing **a** and **b**. Geometrically, it represents the area of a parallelogram with sides **a**, **b** in magnitude and the direction is in the direction perpendicular to the plane containing **a** and **b**.**a**.(**b** × **c**) is called the scalar triple product. It is a scalar quantity. It represents the volume of a parallelopiped with sides **a**, **b**, **c**. It is also denoted by [**a b c**]·[**a b c**] = 0 ⇔ **a**, **b**, **c** are coplanar. Vector triple product is a vector quantity. It is not associative in general; (**a** × **b**) × **c** = (**c** . **a**) **b** – (**c** . **b**) **a**.**a** × (**b** × **c**) = (**a** . **c**) **b** – (**a** . **b**) **c**.

2.7 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Define dot product and give its geometrical interpretation.

b) If |**a** + **b**| = |**a** – **b**|, show that **a** and **b** are mutually perpendicular.

(ii) a) Define cross product of two vectors and interpret it geometrically.

b) Find the vector whose length is 3 and perpendicular to both 3**i** + **j** – 4**k**, 6**i** + 5**j** + 2**k**.

(iii) a) Give the geometrical interpretation of triple scalar product.

b) Show that 2**i** – **j** – **k**, 3**i** – 4**j** + 5**k**, 2**i** + 4**j** – 6**k** are coplanar.

II. *Briefly answer the following questions.*

i) The coordinates of two points are (3, 1, 2) and (2, –2, 4); Find the cosine of the angle between the vectors joining the points to the origin.

ii) Show that the vectors 9**i** + **j** – 6**k**, 4**i** – 6**j** + 5 **k** are mutually perpendicular.

iii) Obtain the unit vector perpendicular to both 4**i** – **j** + 3**k** and –2**i** + **j** – 2**k**.

iv) If **a** + **b** + **c** = 0 show that **a** × **b** = **b** × **c** = **c** × **a**.

v) A Particle acted upon by constant forces $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is displaced from the point (1, 2, 3) to the point (5, 4, 1). Find the work done by the forces.

vi) A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ acts at a point A, whose position vector is $2\mathbf{i} - \mathbf{j}$, Find the moment of $\mathbf{F}$ around the origin.

2.8 Answers to Self Assessment Questions

SAQ 1: If **a** and **b** includes the right angle. then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos 90° = 0$.

SAQ 2: Vector $\perp$ r to both **a** and **b** is $\mathbf{a} \times \mathbf{b}$.

SAQ 3 : If the given two vectors are **a** and **b**, you can multiply them in two ways; one is the dot product i.e., $\mathbf{a} \cdot \mathbf{b}$ and the other is cross product i.e., $\mathbf{a} \times \mathbf{b}$.

SAQ 4 : The Possible products of three vectors **a**, **b** and **c** are

(i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, (ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and (iii) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(i) is called triple scalar product.

(ii), (iii) are called vector triple products. In general, the vector product will not obey the associative law.

UNIT–3 : VECTOR DIFFERENTIATION

3.0 Contents

3.1 Aims and Objectives

After going through this unit, you will be able to :

i) differentiate the vector valued functions

ii) differentiate partially, the given vector valued functions of more than two variables.

3.2 Introduction

In units 1 and 2, we have learnt about the addition and multiplication of vectors. In several physical situations we often come across vectors which alter with position or time. For example, the velocity **v** of a particle may change with time t, or the strength **E** of electric field may depend on position at which we measure it. We know that a real variable y being a function of the real variable x is usually represented by $y : R \rightarrow R$ or simply as $y\,(x)$. Similarly if a vector **F** depends on a real variable t, this relationship is denoted by **F** = **F** (t). In this unit we will be concerned with the concept of differentiation of **F** with respect to t.

3.3 Differentiation of Vector Function of a real variable

Vector Function of a Real Variable

Definition : If for every real number t of a set S of real numbers, there corresponds one value **F** (t), then we say that **F** is a vector function of the real variable t. S is called the domain of **F**.

We know that any vector **r** in three dimensional space can be written as **r** = x**i** + y**j** + z**k**, where **i**, **j**, **k** are three mutually perpendicular unit vectors. Now any vector function **F** (t) can be expressed as

$$\mathbf{F}\,(t) \;=\; F_x(t)\,\mathbf{i} + F_y(t)\,\mathbf{j} + F_z(t)\,\mathbf{k} \qquad \text{... (1)}$$

where F_x, F_y, F_z are real functions of t and are called the components of **F**.

Differentiation of Vector Function of a Real Variable

Let **F** be a vector function of the real variable 't' with the domain S. If t_0 and $t_0 + \delta t_0$ are points in S, we write

$$\delta\,\mathbf{F}\left(t_0\right) \;=\; \mathbf{F}\left(t_0 + \delta t_0\right) - \mathbf{F}\left(t_0\right) \qquad \text{... (2)}$$

If $\lim\limits_{\delta t_0 \to 0} \dfrac{\delta \mathbf{F}}{\delta t_0}$ exists we denote it by $\dfrac{d\mathbf{F}}{dt}$ at $t = t_0$ and call it the derivative of **F** at t_0. In general

$$\frac{d\mathbf{F}}{dt} = \lim_{\delta t_0 \to 0} \frac{\mathbf{F}(t + \delta t) - \mathbf{F}(t)}{\delta t} \qquad \text{... (3)}$$

stands for the derivative (differential coefficient) at a general point $t \in S$ and we write

$$\delta\,\mathbf{F}(t) \;=\; \mathbf{F}\,(t + \delta t) - \mathbf{F}\,(t) \qquad \text{... (4)}$$

From fig. 1 we see that $\overrightarrow{PQ} = \mathbf{F}(t+\delta t) - \mathbf{F}(t)$.

Since PQ is a chord of the curve described by the terminal point P of $\mathbf{F}$, it is clear that as $\delta t \to 0$, $\overrightarrow{PQ}$ tends to a vector along the tangent at P and hence the direction of $\frac{d\mathbf{F}}{dt}$ is along a tangent to the locus of P.

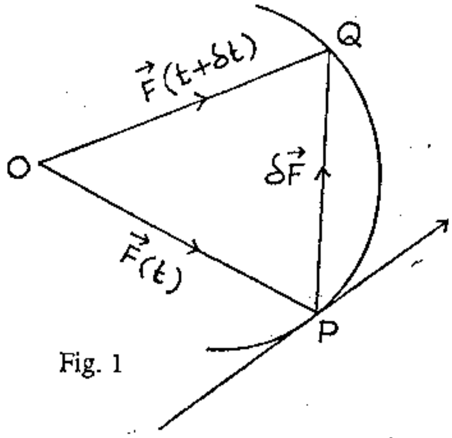

Fig. 1

From fig-1 we see that $\overrightarrow{PQ} = \mathbf{F}(t+\delta t) - \mathbf{F}(t)$.

Since PQ is chord of the curve described by the terminal point P of $\mathbf{F}$, it is clear that as $\delta t \to 0$, $\overrightarrow{PQ}$ tends to a vector along the tangent at P and hence the direction of $\frac{d\mathbf{F}}{dt}$ is along a tangent to the locus of P.

Since $\frac{d\mathbf{F}}{dt}$ is itself a vector function of the real variable t, it is clear that definition (3) may be applied repeatedly to define higher order differential coefficients as long as the corresponding limits exist. Thus for example,

$$\frac{d^2\mathbf{F}}{dt^2} = \lim_{\delta t \to 0} \frac{\frac{d\mathbf{F}}{dt}(t+\delta t) - \frac{d\mathbf{F}(t)}{dt}}{\delta t}$$

t stands for a real variable which is usually referred to as a scalar variable.

Example : **Velocity and Acceleration** : If the real variable t stands for time $\mathbf{r}(t) = \mathbf{F}(t)$, $\overrightarrow{PQ} = \delta \mathbf{r}$ gives the displacement of the point P in time δt or $\frac{\delta \mathbf{r}}{\delta t}$ gives the average velocity during the interval δt.

Taking limit as $\delta t \to 0$, we get the velocity $\mathbf{v}$ as

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}$$

$$\therefore \mathbf{V} = \frac{d\mathbf{r}}{dt}$$

where $\mathbf{V}$, the velocity, is a vector function of t. Similarly, accelaration being the rate of change of velocity, we have

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}$$

which is a vector function of t.

3.3.1 Differentiation of Sums and Products

If $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are differentiable vector function of t and ϕ is a differentiable real function of t, then we shall prove the following results, which follow the same procedure as in classical calculus.

1. $\frac{d}{dt}(\mathbf{F} \pm \mathbf{G}) = \frac{d\mathbf{F}}{dt} \pm \frac{d\mathbf{G}}{dt}$

   Let $\mathbf{f} = \mathbf{F} \pm \mathbf{G}$

   i.e., $\mathbf{f}(t) = \mathbf{F}(t) \pm \mathbf{G}(t)$

$$\therefore \frac{\mathbf{f}(t+\delta t) - \mathbf{f}(t)}{\delta t} = \frac{\mathbf{F}(t+\delta t) - \mathbf{F}(t)}{\delta t} \pm \frac{\mathbf{G}(t+\delta t) - \mathbf{G}(t)}{\delta t}$$

Taking limit as $\delta t \to 0$, we have $\frac{d\mathbf{f}}{dt} = \frac{d\mathbf{F}}{dt} \pm \frac{d\mathbf{G}}{dt}$

i.e., $\frac{d}{dt}(\mathbf{F} \pm \mathbf{G}) = \frac{d\mathbf{F}}{dt} \pm \frac{d\mathbf{G}}{dt}$

2. $\frac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \frac{d\mathbf{G}}{dt} + \mathbf{G} \cdot \frac{d\mathbf{F}}{dt}$

Let $f = \mathbf{F} \cdot \mathbf{G}$; $f(t) = \mathbf{F}(t) \cdot \mathbf{G}(t)$

Here f is not a vector function, but a scalar function of t.

$$\therefore \frac{f(t+\delta t) - f(t)}{\delta t} = \frac{\mathbf{F}(t+\delta t) \cdot \mathbf{G}(t+\delta t) - \mathbf{F}(t) \cdot \mathbf{G}(t)}{\delta t}$$

Subtracting and adding $\mathbf{F}(t+\delta t) \cdot \mathbf{G}(t)$ to the numerator of right hand side we get

$$\frac{f(t+\delta t) - f(t)}{\delta t} = \frac{\mathbf{F}(t+\delta t) \cdot [\mathbf{G}(t+\delta t) - \mathbf{G}(t)] + \mathbf{G}(t) \cdot [\mathbf{F}(t+\delta t) - \mathbf{F}(t)]}{\delta t}$$

Taking limit as $\delta t \to 0$ and also noting that

$$\lim_{\delta t \to 0} \mathbf{F}(t+\delta t) = \mathbf{F}(t)$$

we get $\frac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \frac{d}{dt}[f(t)] = \mathbf{F} \cdot \frac{d\mathbf{G}}{dt} + \mathbf{G} \cdot \frac{d\mathbf{F}}{dt}$

3. $\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times \mathbf{G}$

It should be noted that the order of the vectors while differentiating the vector product is, of course, important.

Let $\mathbf{f} = \mathbf{F} \times \mathbf{G}$. Here $\mathbf{f}$ is a vector function of t.

$$\therefore \frac{\mathbf{f}(t+\delta t) - \mathbf{f}(t)}{\delta t} = \frac{\mathbf{F}(t+\delta t) \times \mathbf{G}(t+\delta t) - \mathbf{F}(t) \times \mathbf{G}(t)}{\delta t}$$

Substracting and adding, as before, $\mathbf{F}(t+\delta t) \times \mathbf{G}(t)$ to the numerator of R.H.S we get

$$\mathbf{f}(t+\delta t) - \mathbf{f}(t) = \mathbf{F}(t+\delta t) \times \left[\frac{\mathbf{G}(t+\delta t) - \mathbf{G}(t)}{\delta t}\right] + \left[\frac{\mathbf{F}(t+\delta t) - \mathbf{F}(t)}{\delta t}\right] \times \mathbf{G}(t)$$

Taking limit as $\delta t \to 0$, we have

$$\frac{d}{dt}[\mathbf{f}(t)] = \frac{d}{dt}[\mathbf{F} \times \mathbf{G}] = \mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times \mathbf{G}$$

4. $\frac{d}{dt}(\phi \mathbf{F}) = \phi \frac{d\mathbf{F}}{dt} + \frac{d\phi}{dt} \mathbf{F}$

where ϕ is a scalar function of t. We can prove this as in 2.

5. $\frac{d}{dt}[\mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})] = \frac{d}{dt}[\mathbf{F}\ \mathbf{G}\ \mathbf{H}]$

$$= \left[\frac{d\mathbf{F}}{dt}\ \mathbf{G}\ \mathbf{H}\right] + \left[\mathbf{F}\ \frac{d\mathbf{G}}{dt}\ \mathbf{H}\right] + \left[\mathbf{F}\ \mathbf{G}\ \frac{d\mathbf{H}}{dt}\right]$$

$$= \frac{d\mathbf{F}}{dt} \cdot (\mathbf{G} \times \mathbf{H}) + \mathbf{F} \cdot \left(\frac{d\mathbf{G}}{dt} \times \mathbf{H}\right) + \mathbf{F} \cdot \left(\mathbf{G} \times \frac{d\mathbf{H}}{dt}\right)$$

[F G H] is a scalar triple product of three vectors and we know that its value remains unaltered provided we maintain the cyclic order of the vectors **F, G, H**.

Now $[\mathbf{F}\ \mathbf{G}\ \mathbf{H}] = \mathbf{F}\cdot(\mathbf{G}\times\mathbf{H}) = \mathbf{G}\cdot(\mathbf{H}\times\mathbf{F}) = \mathbf{H}\cdot(\mathbf{F}\times\mathbf{G})$

$$\therefore \frac{d}{dt}[\mathbf{F}\ \mathbf{G}\ \mathbf{H}] = \frac{d}{dt}[\mathbf{F}\cdot(\mathbf{G}\times\mathbf{H})]$$

$$= \mathbf{F}\cdot\frac{d}{dt}(\mathbf{G}\times\mathbf{H}) + \frac{d\mathbf{F}}{dt}\cdot(\mathbf{G}\times\mathbf{H}) \qquad \text{(using result 2)}$$

$$= \mathbf{F}\cdot\left[\frac{d\mathbf{G}}{dt}\times\mathbf{H} + \mathbf{G}\times\frac{d\mathbf{H}}{dt}\right] + \frac{d\mathbf{F}}{dt}\cdot(\mathbf{G}\times\mathbf{H}) \text{ (using result 3)}$$

$$= \frac{d\mathbf{F}}{dt}\cdot(\mathbf{G}\times\mathbf{H}) + \mathbf{F}\cdot\left(\frac{d\mathbf{G}}{dt}\times\mathbf{H}\right) + \mathbf{F}\cdot\left(\mathbf{G}\times\frac{d\mathbf{H}}{dt}\right)$$

$$= \left[\frac{d\mathbf{F}}{dt}\mathbf{G}\ \mathbf{H}\right] + \left[\mathbf{F}\ \frac{d\mathbf{G}}{dt}\ \mathbf{H}\right] + \left[\mathbf{F}\mathbf{G}\ \frac{d\mathbf{H}}{dt}\right]$$

6. $$\frac{d}{dt}\{\mathbf{F}\times(\mathbf{G}\times\mathbf{H})\} = \frac{d\mathbf{F}}{dt}\times(\mathbf{G}\times\mathbf{H}) + \mathbf{F}\times\frac{d}{dt}(\mathbf{G}\times\mathbf{H})$$

$$= \frac{d\mathbf{F}}{dt}\times(\mathbf{G}\times\mathbf{H}) + \mathbf{F}\times\left\{\frac{d\mathbf{G}}{dt}\times\mathbf{H} + \mathbf{G}\times\frac{d\mathbf{H}}{dt}\right\}$$

(using result 3)

$$= \frac{d\mathbf{F}}{dt}\times(\mathbf{G}\times\mathbf{H}) + \mathbf{F}\times\left(\frac{d\mathbf{G}}{dt}\times\mathbf{H}\right) + \mathbf{F}\times\left(\mathbf{G}\times\frac{d\mathbf{H}}{dt}\right)$$

(Note that the order of the factors must be maintained)

7. *Function of a function* : Let t be a real differentiable function in a domain S and **F** be real differentiable vector function defined over the range of t and we write $t(s) = \phi(s)$, then **F** is a derivable function of s and

$$\frac{d\mathbf{F}}{ds} = \frac{d\mathbf{F}}{dt}\frac{dt}{ds} = \frac{d\mathbf{F}}{dt}\frac{d\phi}{ds}$$

Let δt be a small increment in t which produces corresponding increment $\delta\mathbf{F}$ and δs in **F** and s respectively. Also as $\delta t \to 0$, both $\delta\mathbf{F}$ and $\delta s \to 0$.

Also $$\frac{\delta\mathbf{F}}{\delta s} = \frac{\delta\mathbf{F}}{\delta s}\frac{\delta t}{\delta s}$$

Proceeding to the limit as $\delta t \to 0$ consequently we get as in calculus

$$\frac{d\mathbf{F}}{ds} = \frac{d\mathbf{F}}{dt}\frac{dt}{ds} = \frac{d\mathbf{F}}{dt}\frac{d\phi}{ds}, \text{ since } t = \phi(s).$$

Constant Vectors

Definition : **F** is said to be a constant vector if F_x, F_y, F_z are constant functions where

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

Theroem-1 : A necessary and sufficient condition for **F** (t) to be contant is $\frac{d\mathbf{F}}{dt} = 0$

Proof : The condition is necessary. Suppose **F** (t) is constant. Then

$$\frac{d\mathbf{F}}{dt} = \lim_{\delta t \to 0}\frac{\mathbf{F}(t+\delta t) - \mathbf{F}(t)}{\delta t} = 0$$

since $\mathbf{F}(t+\delta t) = \mathbf{F}(t)$

Conversely, if $\frac{d\mathbf{F}}{dt} = \mathbf{0}$, we have to show **F** is constant.

If $\mathbf{F}(t) = F_1(t)\,\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k}$,

then $\frac{d\mathbf{F}}{dt} = 0 \Rightarrow \frac{dF_1}{dt}\mathbf{i} + \frac{dF_2}{dt}\mathbf{j} + \frac{dF_3}{dt}\mathbf{k} = 0$

Equating to zero the coefficients of **i**, **j** and **k** we get

$$\frac{dF_1}{dt} = 0,\quad \frac{dF_2}{dt} = 0,\quad \frac{dF_3}{dt} = 0$$

Hence F_1, F_2, F_3 are constants.

Therefore **F** (t) is a constant vector function.

Theorem-2 : A necessary and sufficient condition for **F** (t) to have constant magnitude is $\mathbf{F}\cdot\frac{d\mathbf{F}}{dt} = 0$

Proof :

$$\frac{d}{dt}\left(|\mathbf{F}|^2\right) = \frac{d}{dt}(\mathbf{F}\cdot\mathbf{F}) = \frac{d\mathbf{F}}{dt}\cdot\mathbf{F} + \mathbf{F}\cdot\frac{d\mathbf{F}}{dt}$$

$$= 2\,\mathbf{F}\cdot\frac{d\mathbf{F}}{dt}$$

Hence $\mathbf{F}\cdot\frac{d\mathbf{F}}{dt} = 0 \Leftrightarrow \frac{d}{dt}\left(|\mathbf{F}|^2\right) = 0 \Leftrightarrow |\mathbf{F}|^2 = \text{constant}$

$$\Leftrightarrow |\mathbf{F}| = \text{constant.}$$

SAQ 1 : Give an example of a vector with constant magnitude and verify the theorem 2.

Theorem-3 : A necessary and sufficient condition for **F** (t) to have constant direction is $\mathbf{F}\times\frac{d\mathbf{F}}{dt} = \mathbf{0}$.

Proof : Let $\mathbf{F} = \phi\,\mathbf{f}$ where ϕ is a scalar function and **f** is a vector function of unit modulus. Now **F** has a constant direction ⇔ **f** has a constant direction.

⇔ **f** has constant magnitude and direction

$\Leftrightarrow$ **f** is constant $\Leftrightarrow d\mathbf{f} = 0 \Leftrightarrow \frac{d\mathbf{F}}{dt} = \frac{d\phi}{dt}\mathbf{f}$

$\Leftrightarrow \quad \mathbf{F}\times\frac{d\mathbf{F}}{dt} = \phi\,\mathbf{f}\times\frac{d\phi}{dt}\mathbf{f} = \frac{d\phi}{dt}\,\mathbf{f}\times\mathbf{f} = \mathbf{0}$

SAQ 2 : Give an example of a vector with constant direction. Verify the theorem 3.

Components of a derivative

If **F** is written in terms of its cartesian components, ie., $\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$, then

$$\frac{d\mathbf{F}}{dt} = \frac{dF_x}{dt}\mathbf{i} + \frac{dF_y}{dt}\mathbf{j} + \frac{dF_z}{dt}\mathbf{k} \qquad \text{... (6)}$$

For, in its cartesian form **F** is written as the sum of three vectors i.e., $F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ each of which is the product of a variable scalar and a constant vector. Applying results 1 and 4 of last section (6) follows immediately.

3.4 Partial Differentiation

We have defined the derivative of a vector **F** which is a function of single real variable t. A vector may, however, depend on more than one real variable in just the same way as in ordinary calculus. When this is the case, we can define the partial derivatives of the vector with respect to each of the variables. If to each point in a subset S of the t - s plane we associate a vector **F**. Then **F** is called a function of the two variables t and s. If **F** is a function of the two variables t and s we write $\mathbf{F} = \mathbf{F}(t, s)$ and define

$$\frac{\partial \mathbf{F}}{\partial t} = \lim_{\delta t \to 0} \frac{\mathbf{F}(t + \delta t, s) - \mathbf{F}(t, s)}{\delta t} \qquad \dots (7)$$

$$\frac{\partial \mathbf{F}}{\partial s} = \lim_{\delta t \to 0} \frac{\mathbf{F}(t, s + \delta s) - \mathbf{F}(t, s)}{\delta s}$$

as the partial derivatives of **F** with respect to t and s.

Higher derivatives may, similarly, be defined as in calculus. Thus

$\frac{\partial^2 \mathbf{F}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}}{\partial t}\right)$, $\frac{\partial^2 \mathbf{F}}{\partial t\, \partial s} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}}{\partial s}\right)$ etc. and for well behaved functions $\frac{\partial^2 \mathbf{F}}{\partial t\, \partial s} = \frac{\partial^2 \mathbf{F}}{\partial s\, \partial t}$.

(The details are not necessary here). The results in section 3.3 for differentiation of sums and products may be easily shown to remain true for partial differentiation. Thus for example,

$$\frac{\partial}{\partial t}(\mathbf{F} \times \mathbf{G}) = \frac{\partial \mathbf{F}}{\partial t} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial t} \text{ etc.}$$

Similarly, the effect of a partial differential operator acting on a vector is given by the same operator acting on its components. For example

$$\frac{\partial^2 \mathbf{F}}{\partial t\, \partial s} = \frac{\partial^2 F_x}{\partial t\, \partial s}\,\mathbf{i} + \frac{\partial^2 F_y}{\partial t\, \partial s}\,\mathbf{j} + \frac{\partial^2 F_z}{\partial t\, \partial s}\,\mathbf{k} \qquad \dots (8)$$

3.5 Total Differential

$$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial t}\, dt + \frac{\partial \mathbf{F}}{\partial s}\, ds \qquad \dots (9)$$

is known as the total differential of **F**. This is of course, the vector analogus of the scalar notion of a differential

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy$$

for $z = z(x, y)$.

SAQ. 3 : If $\mathbf{F} = 2t\mathbf{i} + 3t^2\mathbf{j} + 4t^2\,\mathbf{k}$ then find the components of the vector $\frac{d\mathbf{F}}{dt}$.

Examples

Ex. 1: If $\mathbf{F} = e^{3t}\,\mathbf{i} + t^2\,\mathbf{j} - \log(1 + t)\,\mathbf{k}$, find

(a) $\frac{d\mathbf{F}}{dt}$, (b) $\frac{d^2\mathbf{F}}{dt^2}$, (c) $\left|\frac{d^2\mathbf{F}}{dt^2}\right|$ and $\frac{d}{dt}(\mathbf{F} \cdot \mathbf{F})$ at $t = 0$.

Sol : Given $\mathbf{F} = e^{3t}\,\mathbf{i} + t^2\,\mathbf{j} - \log(1 + t)\,\mathbf{k}$ ... (1)

(a) Differentiating (1) with respect to t, we get

$$\frac{d\mathbf{F}}{dt} = 3e^{3t}\,\mathbf{i} + 2t\,\mathbf{j} - (1 + t)^{-1}\,\mathbf{k} \qquad \dots (2)$$

$\therefore$ At $t = 0$, $\frac{d\mathbf{F}}{dt} = 3\mathbf{i} - \mathbf{k}$

(b) Differentiating (2), again with respect to t, we have

$$\frac{d^2\mathbf{F}}{dt^2} = 9e^{3t}\,\mathbf{i} + 2\,\mathbf{j} + (1 + t)^{-2}\,\mathbf{k}$$

$\therefore$ At $t = 0$, $\frac{d^2\mathbf{F}}{dt^2} = 9\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ... (3)

(c) Taking modulus of (3), we get

at t = 0, $\left|\frac{d^2\mathbf{F}}{dt^2}\right| = (9^2 + 2^2 + 1^2)^{1/2} = \sqrt{86}$

(d) $\mathbf{F} \cdot \mathbf{F} = \left[e^{3t}\,\mathbf{i} + t^2\,\mathbf{j} - \log(1+t)\,\mathbf{k}\right] \cdot \left[e^{3t}\,\mathbf{i} + t^2\,\mathbf{j} - \log(1+t)\,\mathbf{k}\right]$

$$= e^{6t} + t^4 + [\log(1+t)]^2$$

$$\frac{d}{dt}(\mathbf{F} \cdot \mathbf{F}) = 6e^{6t} + 4t^3 + \frac{2}{(1+t)} \log(1+t)$$

$\therefore$ At $t = 0, \dfrac{d}{dt}(\mathbf{F} \cdot \mathbf{F}) = 6 + 0 + 0 = 6$

Ex. 2: A particle moves along the curve $x = 3t^2, y = t^2 - 2t, z = t^3$. Find the velocity and acceleration at $t = 1$.

Sol : Let $\mathbf{r}$ be the position vector of any point P (x, y, z) on the curve, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 3t^2\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + t^3\,\mathbf{k} \qquad \ldots (1)$$

$$\therefore \text{The velocity } \mathbf{v} = \frac{d\mathbf{r}}{dt} = 6t\mathbf{i} + (2t - 2)\,\mathbf{j} + 3t^2\,\mathbf{k} \qquad \ldots (2)$$

At $t = 1$, the velocity $\mathbf{v} = 6\mathbf{i} + 3\mathbf{k}$

$$|\mathbf{v}| = \sqrt{6^2 + 3^2} = 3\sqrt{5}$$

Acceleraion $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 6\mathbf{i} + 2\mathbf{j} + 6t\mathbf{k}$

At $t = 1$, $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$$|\mathbf{a}| = \sqrt{6^2 + 2^2 + 6^2} = \sqrt{76} = 2\sqrt{19}$$

Ex. 3: A particle moves along the curve, $x = 3t^2, y = t^2 - 4t, z = 3t - 5$

where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Sol: Let $\mathbf{r}$ be the position vector of any point P (x, y, z) on the curve. Then

$$\mathbf{r} = 2t^2\,\mathbf{i} + (t^2 - 4t)\,\mathbf{j} + (3t - 5)\,\mathbf{k}$$

$$\therefore \frac{d\mathbf{r}}{dt} = \text{vel} = 4t\mathbf{i} + (2t - 4)\,\mathbf{j} + 2\mathbf{k}$$

$$= 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \text{ at } t = 1.$$

Now we know $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$

$\therefore \dfrac{\mathbf{a} \cdot \mathbf{b}}{a} = b\cos\theta$ = projection of $\mathbf{b}$ in the direction of $\mathbf{a}$

$\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{A}|} = \hat{a} \cdot b = b\cos\theta$ = components of b in the direction of a where $\hat{a}$ is a unit vector.

Hence the component of velocity in the direction of $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is

$$\frac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 9 + 4}} \cdot (4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$$

Again acceleration $= \dfrac{d^2\mathbf{r}}{dt^2} = 4\mathbf{i} + 2\mathbf{j}$

Hence the component of acceleration in the given direction is

$$\frac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{14}} \cdot (4\mathbf{i} + 2\mathbf{j}) = \frac{4 - 6}{\sqrt{14}} = + \frac{\sqrt{14}}{7}$$

Ex. 4: If $\mathbf{F} = 5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$ and $\mathbf{G} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$ find (a) $\frac{d}{dt}(\mathbf{F}\cdot\mathbf{G})$ (b) $\frac{d}{dt}(\mathbf{F}\times\mathbf{F})$.

Sol : (a) $\mathbf{F}\cdot\mathbf{G} = 5t^2 \sin t - t\cos t,$

$$\frac{d}{dt}(\mathbf{F}\cdot\mathbf{G}) = \frac{d}{dt}(5t^2 \sin t - t\cos t)$$

$$= 10\,t\sin t + 5t^2\cos t - (\cos t - t\sin t)$$

$$= (5t^2 - 1)\cos t + 11\,t\sin t.$$

(b) $$\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= t^3\cos t\,\mathbf{i} - t^3\sin t\,\mathbf{j} - (5t^2\cos t + t\sin t)\,\mathbf{k}$$

$$\therefore \frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = (t^3\sin t - 3t^3\cos t)\,\mathbf{i} - (t^3\cos t + 3t^2\sin t)\,\mathbf{j}$$

$$+ (5t^2\sin t - 11\,t\cos t - \sin t)\,\mathbf{k}$$

Ex. 5: Find the unit tangent vector at any point on the curve $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$. Also determine the unit tangent at the point when $t = 2$.

Sol: Let $\mathbf{r}$ be the position vector of any point on the curve so that

$$\mathbf{r} = (t^2 + 1)\,\mathbf{i} + (4t - 3)\,\mathbf{j} + (2t^2 - 6t)\,\mathbf{k}$$

Then a tangent vector to the curve at any point is

$$\frac{d\mathbf{r}}{dt} = 2t\,\mathbf{i} + 4\mathbf{j} + (4t - 6)\,\mathbf{k}$$

The magnitude of the vector is $\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t - 6)^2}$

Then the required unit tangent vector is $= \dfrac{2t\,\mathbf{i} + 4\mathbf{j} + (4t - 6)\,\mathbf{k}}{\sqrt{4t^2 + 16 + (4t - 6)^2}}$

At $t = 2$, the unit tangent vector is

$$\frac{4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{4^2 + 4^2 + 2^2}} = \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}.$$

Ex. 6 : If $\mathbf{F}$ has constant magnitude show that $\mathbf{F}$ and $\frac{d\mathbf{F}}{dt}$ are perpendicular provided $\left|\frac{d\mathbf{F}}{dt}\right| \neq 0$.

Sol : Since $\mathbf{F}$ has constant magnitude,

$\mathbf{F}\cdot\mathbf{F} =$ constant.

Then $\frac{d}{dt}(\mathbf{F}\cdot\mathbf{F}) = \mathbf{F}\cdot\frac{d\mathbf{F}}{dt} + \frac{d\mathbf{F}}{dt}\cdot\mathbf{F} = 2\mathbf{F}\cdot\frac{d\mathbf{F}}{dt} = 0.$

Thus $\mathbf{F}\cdot\frac{d\mathbf{F}}{dt} = 0$ and $\mathbf{F}$ is perpendicular to $\frac{d\mathbf{F}}{dt}$ prvided $\left|\frac{d\mathbf{F}}{dt}\right| \neq 0.$

(Note : we can use theorem –1 directly to show this).

Ex. 7: Evaluate $\frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)$

Sol : Using result 5 of sec. 3.3., we get

$$\frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) = \frac{d}{dt}\left[\mathbf{r}\ \frac{d\mathbf{r}}{dt}\ \frac{d^2\mathbf{r}}{dt^2}\right]$$

$$= \left[\frac{d\mathbf{r}}{dt}\ \frac{d\mathbf{r}}{dt}\ \frac{d^2\mathbf{r}}{dt^2}\right] + \left[\mathbf{r}\ \frac{d^2\mathbf{r}}{dt^2}\ \frac{d^2\mathbf{r}}{dt^2}\right] + \left[\mathbf{r}\ \frac{d\mathbf{r}}{dt}\ \frac{d^3\mathbf{r}}{dt^3}\right]$$

The first two scalar triple products vanish because two vectors are equal.

$$\therefore \frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) = \left[\mathbf{r}\ \frac{d\mathbf{r}}{dt}\ \frac{d^3\mathbf{r}}{dt^3}\right] = \mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}$$

3.6 Summary

If S is a set of real numbers and if $\forall\, t \in S$, there corresponds a unique vector as the value of the vector valued function $\mathbf{F}(t)$, then $\mathbf{F}(t)$ is called the vector function of the real varialble t. S is called the domain of $\mathbf{F}$. $\frac{d(\mathbf{F}(t))}{dt} = \lim_{\delta t\to 0}\frac{\mathbf{F}(t+\delta t)-\mathbf{F}(t)}{\delta t}$. If $\mathbf{r}(t)$ denote the position vector of a moving point then velocity $\mathbf{V} = \frac{d\mathbf{r}}{dt}$; acceleration $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{V}}{dt}$. If $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are differentiable vector functions of t and ϕ is a differentialble real function of t, then

(1) $\frac{d}{dt}(\mathbf{F}\pm\mathbf{G}) = \frac{d\mathbf{F}}{dt}\pm\frac{d\mathbf{G}}{dt}$ (2) $\frac{d}{dt}(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\cdot\frac{d\mathbf{G}}{dt}+\frac{d\mathbf{F}}{dt}\cdot\mathbf{G}$

(3) $\frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\times\frac{d\mathbf{G}}{dt}+\frac{d\mathbf{F}}{dt}\times\mathbf{G}$, (4) $\frac{d}{dt}(\phi\,\mathbf{F}(t)) = \phi\frac{d\mathbf{F}}{dt}+\frac{d\phi}{dt}\mathbf{F}$

(5) $\frac{d}{dt}(\mathbf{F}\cdot(\mathbf{G}\times\mathbf{H})) = \frac{d[\mathbf{F}\ \mathbf{G}\ \mathbf{H}]}{dt} = \frac{d\mathbf{F}}{dt}\cdot(\mathbf{G}\times\mathbf{H}) + \mathbf{F}\cdot\left(\frac{d\mathbf{G}}{dt}\times\mathbf{H}\right)+\mathbf{F}\cdot\left(\mathbf{G}\times\frac{d\mathbf{H}}{dt}\right)$

(6) $\frac{d}{dt}(\mathbf{F}\times(\mathbf{G}\times\mathbf{H})) = \frac{d\mathbf{F}}{dt}\times(\mathbf{G}\times\mathbf{H}) + \mathbf{F}\times\frac{d\mathbf{G}}{dt}\times\mathbf{H} + \mathbf{F}\times\left(\mathbf{G}\times\frac{d\mathbf{H}}{dt}\right)$

(note that the order of factors should be maintained). If $\mathbf{F}$ is a consant vector them $\frac{d\mathbf{F}(t)}{dt} = 0$

If $\mathbf{F} = \mathbf{F}(t, s)$ then $\frac{\partial \mathbf{F}}{\partial t} = \lim_{\delta t\to 0}\frac{\mathbf{F}(t+\delta t, s)-\mathbf{F}(t, s)}{\delta t}$;

$\frac{\partial \mathbf{F}}{\partial s} = \lim_{\delta s\to 0}\frac{\mathbf{F}(t, s+\delta s)-\mathbf{F}(t, s)}{\delta s}$ (Similarly) $\frac{\partial^2\mathbf{F}}{\partial t^2}, \frac{\partial^2\mathbf{F}}{\partial s\,\partial t}, \frac{\partial^2\mathbf{F}}{\partial s^2}$ are defined.

$d\mathbf{F} = \frac{\partial\mathbf{F}}{\partial t}dt + \frac{\partial\mathbf{F}}{\partial s}ds$. The necessary and sufficient condition for $\mathbf{F}(t)$ to have consant magnitude is $\mathbf{F}\cdot\frac{d\mathbf{F}}{dt} = 0$.

3.7 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Define a vector valued function.

b) A particle moves so that its position vector is given by $\mathbf{r} = \text{cow}\ \omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$ where ω is a constant. Show that (i) the velocity $\mathbf{V}$ of the particle is perpendicular to $\mathbf{r}$, (ii) the acceleration $\mathbf{a}$ is directed towards the origin and has magnitude propotional to the distance form the origin, (iii) $\mathbf{r}\times\mathbf{V}$ is a constant vector.

(ii) a) Explain the laws followed by the differentiation of addition and products of vectors.

b) If $\mathbf{a} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\,\mathbf{k}$, $\mathbf{b} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$, find $\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\}$ at $t = 0$.

(iii) a) Explain the partial differentiation and total differentiation of a vector function.

b) Find the first partial derivative of $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$.

II. *Briefly answer the following questions.*

i) A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time variable. Determine its velocity and accleration vectors and also the magnitudes of acceleration and velocity at $t = 0$.

ii) Find the angle between the tangents to the curve $\mathbf{r} = t^2\,\mathbf{i} + 2t\mathbf{j} - t^3\,\mathbf{k}$ at the points $t = \pm 1$.

iii) Show that if $\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t$, where $\mathbf{a}, \mathbf{b}, \omega$ are constants, then $\frac{d^2\mathbf{F}}{dt^2} = -\omega^2\,\mathbf{r}$ and

$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = -\omega\,\mathbf{a} \times \mathbf{b}$.

iv) Find a unit tangent vector to any point on the curve $x = a\cos\omega t$, $y = a\sin\omega t$, $z = bt$, where a, b, ω are constants.

v) Find the first partial derivative of $\mathbf{F} = x\mathbf{i} + 2y\,\mathbf{j}$.

vi) Show that $\mathbf{F}(t) = \mathbf{b}e^{-\lambda t} + \mathbf{c}\,e^{-\lambda t}$ satisfies the equation $\frac{d^2\mathbf{F}}{dt^2} - \lambda^2\,\mathbf{F} = 0$ (**b** and **c** are constant vectors).

vii) If $\mathbf{F} = x^2yz\,\mathbf{i} - 2xz^3\,\mathbf{j} + xz^2\,\mathbf{k}$ and $\mathbf{G} = 2z\,\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, find $\frac{\partial^2}{\partial x\,\partial y}(\mathbf{F} \times \mathbf{G})$ at $(1, 0, -2)$.

Answers

I. (iib) $7\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$; (iii b) $2x\,\mathbf{i} + 2y\mathbf{j}$; $-2y\,\mathbf{i} + 2x\,\mathbf{j}$

II. (i) $-\mathbf{i} + 6\mathbf{k}, \sqrt{37}$; $\mathbf{i} - 18\mathbf{j}$; $\sqrt{325}$

(ii) $\cos^{-1}\left(\frac{9}{17}\right)$ (iv) $\frac{a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j} + b\mathbf{k}}{\sqrt{a^2\omega^2 + b^2}}$

(v) $\frac{\partial\mathbf{F}}{\partial x} = \mathbf{i}$, $\frac{\partial\mathbf{F}}{\partial y} = 2\mathbf{j}$. (vii) $-4\mathbf{j} - 8\mathbf{j}$.

3.8 Answers to Self Assessment Questions

1. If you consider the position vector of a point on a circle of radius a you will have a vector $\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}$ with constant magnitude 'a' and direction changes as you change θ or position of points, and $\frac{d\mathbf{r}}{d\theta} = -a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j}$; $\mathbf{r} \cdot \frac{d\mathbf{r}}{d\theta} = -a^2\cos\theta\sin\theta + a^2\cos\theta\sin\theta = 0$.

2. If you consider a vector parallel to a line, then its direction is constant. Let **a** and **b** denote the position vectors of two fixed points on a line then $(\mathbf{b} - \mathbf{a})$ gives the direction of the line. Then $t(\mathbf{b} - \mathbf{a})$, where t is a scalar, represents any vector in that direction for $t > 0$. Let $\mathbf{r} = t(\mathbf{b} - \mathbf{a})$. Its direction is constant. Now $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = t(\mathbf{b} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a}) = 0$

3. $\frac{d\mathbf{F}}{dt} = 2\mathbf{i} + 6t\mathbf{j} + 12t^2\,\mathbf{k}$. The components of $\frac{d\mathbf{F}}{dt}$ are 2, $6t$, and $12t^2$.

UNIT-4 : DIVERGENCE AND CURL (Differential Operators)

4.0 Contents

4.1 Aims and Objectives

After going through this unit you will be able to :

i) operate the operators of gradient, divergence and curl on the given functions,

ii) obtain the divergence, gradient and curl of various products.

4.2 Introduction

Given a real function ϕ of a real variable t, we shall define a vector function called the gradient of ϕ denoted by $\nabla \phi$. We associate with a vector function **F** (t), a scalar function, known as divergence of **F** (t) denoted by ∇ . F. Again, associated with **F** (t) we define a vector function, known as curl of **F** (t) and is denoted by $\nabla \times$ F. Expansion for gradient, divergence and crul of various products are obtained.

4.3 The Vector Differential Operator Del (∇)

Let ϕ be a differentiable function of three variables x, y and z. We usually denote it by $\phi(x, y, z)$ to remind us of the number of variables. Then we define

$$\nabla\phi = \frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j} + \frac{\partial \phi}{\partial z}\,\mathbf{k}$$

and call it gradient of ϕ. Obviously $\nabla \phi$ is a vector function of x, y, z. Simbolically we write

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k} = \Sigma\, \mathbf{i}\, \frac{\partial}{\partial x}$$

so that ∇ operated on ϕ is $\nabla \phi$ · ∇ is called 'del' operator.

Remark 1 : $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in ∇ can be treated as three components of the vector operator ∇ and so ∇ can be considered a symbolic vector by itself.

Remark 2 : If c be a constant, then

$$\nabla [c\,\phi] = \Sigma\, \mathbf{i}\, \frac{\partial}{\partial x}(c\,\phi) = c\, \Sigma\, \mathbf{i}\, \frac{\partial \phi}{\partial x} = c\,(\nabla\phi)$$

Remark 3 : If ϕ and ψ are two scalar functions

$$\text{then } \nabla [\phi + \psi] = \nabla\phi + \nabla\psi$$

$$\text{For, } \nabla[\phi + \psi] = \Sigma\, \mathbf{i}\, \frac{\partial}{\partial x}(\phi + \psi) = \Sigma\, \mathbf{i} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial x} \right)$$

$$= \Sigma\, \mathbf{i}\, \frac{\partial \phi}{\partial x} + \Sigma\, \mathbf{i}\, \frac{\partial \psi}{\partial x}$$

$$= \nabla\phi + \nabla\psi$$

4.4 Divergence of a Vector Function

Let $\mathbf{F}(x, y, z)$ be a continously differentiable vector function, then we define

$$\nabla \cdot \mathbf{F} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \mathbf{F}$$

and call it divergence of $\mathbf{F}$. Obviously $\nabla . \mathbf{F}$ is a scalar function of x, y, z. Symbolically we write

$$\nabla \cdot = \mathbf{i} \cdot \frac{\partial}{\partial x} + \mathbf{j} \cdot \frac{\partial}{\partial y} + \mathbf{k} \cdot \frac{\partial}{\partial z} = \Sigma\, \mathbf{i} \cdot \frac{\partial}{\partial x}$$

so that $\nabla \cdot$ operated on $\mathbf{F}$ is $\nabla \cdot \mathbf{F}$

Remark 1 : If c is a constant, then $\nabla \cdot [c\,\mathbf{F}] = c\, \nabla \cdot \mathbf{F}$.

Remark 2 : If $\mathbf{F} = F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k}$, then

$$\text{Div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k} \right)$$

$$= \frac{\partial F_1}{\partial k} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

4.5 Curl of a Vector Function

Let $\mathbf{F}(x, y, z)$ be a differentiable vector function then we define

$$\nabla \times \mathbf{F} = \mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{F}}{\partial z}$$

and call it curl of $\mathbf{F}$. Symbolically we write

$$\nabla \times = \mathbf{i} \times \frac{\partial}{\partial x} + \mathbf{j} \times \frac{\partial}{\partial y} + \mathbf{k} \times \frac{\partial}{\partial z} = \Sigma\, \mathbf{i} \times \frac{\partial}{\partial x}$$

so that $\nabla \times$ operated on $\mathbf{F}$ is $\nabla \times \mathbf{F}$.

Remark 1 : If $\mathbf{F} = F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k}$, then

$$\text{curl } \mathbf{F} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \left(F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k} \right)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad \text{(symbolically)}$$

Remark 2 : If c is a constant, $\nabla \times (c\, \mathbf{F}) = c\, (\nabla \times \mathbf{F})$.

4.6 The Operator $\mathbf{a} \cdot \nabla$

The operator $\mathbf{a} \cdot \nabla$ is defined by the equation

$$\mathbf{a} \cdot \nabla = \mathbf{a} \cdot \mathbf{i} \frac{\partial}{\partial x} + \mathbf{a} \cdot \mathbf{j} \frac{\partial}{\partial y} + \mathbf{a} \cdot \mathbf{k} \frac{\partial}{\partial z} = \Sigma\, \mathbf{a} \cdot \mathbf{i} \frac{\partial}{\partial x}.$$

Let $\mathbf{F}(x, y, z)$ be a differentiable vector function, then

$$(\mathbf{a} \cdot \nabla)\, \mathbf{F}(x, y, z) = \Sigma\, (\mathbf{a} \cdot \mathbf{i}) \frac{\partial \mathbf{F}}{\partial x}$$

Let $\phi(x, y, z)$ be a differentiable scalar function of x, y, z then

$$(\mathbf{a} \cdot \nabla)\, \phi(x, y, z) = \Sigma\, (\mathbf{a} \cdot \mathbf{i}) \frac{\partial \phi}{\partial x}$$

$$\text{If } \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}, \text{ then}$$

$$\mathbf{a} \cdot \nabla = \Sigma\, \mathbf{a} \cdot \mathbf{i} \frac{\partial}{\partial x} = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$$

and therefore
$$(\mathbf{a} \cdot \nabla)\, \mathbf{F} = a_1 \frac{\partial \mathbf{F}}{\partial x} + a_2 \frac{\partial \mathbf{F}}{\partial y} + a_3 \frac{\partial \mathbf{F}}{\partial z}$$

and
$$(\mathbf{a} \cdot \nabla)\, \phi = a_1 \frac{\partial \phi}{\partial x} + a_2 \frac{\partial \phi}{\partial y} + a_3 \frac{\partial \phi}{\partial z}$$

Remark : The operator when it operates on a vector yields a vector and when it operates on a scalar gives a scalar.

4.7 Divergence, gradient and curl of various products

We shall now find the gradient, divergence and curl of various products.

If $\mathbf{F}(x, y, z)$, $\mathbf{G}(x, y, z)$ be two vector functions then $\mathbf{F} \cdot \mathbf{G}$ is a scalar function. Hence its gradient can be found out.

$\mathbf{F} \times \mathbf{G}$ is a vector function. Hence its divergence and curl can be found out.

If ϕ is a scalar function and $\mathbf{F}$ a vector function, then $\phi\mathbf{F}$ is a vector function and so its divergence and curl can be found out. We shall now proceed to establish the following results.

1. $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times \text{curl}\, \mathbf{G} + \mathbf{G} \times \text{curl}\, \mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$

Proof :

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{i} \frac{\partial}{\partial x}(\mathbf{F} \cdot \mathbf{G}) + \mathbf{j} \frac{\partial}{\partial y}(\mathbf{F} \cdot \mathbf{G}) + \mathbf{k} \frac{\partial}{\partial z}(\mathbf{F} \cdot \mathbf{G})$$

$$= \mathbf{i}\left(\frac{\partial \mathbf{F}}{\partial x} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x}\right) + \mathbf{j}\left(\frac{\partial \mathbf{F}}{\partial y} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial y}\right) + \mathbf{k}\left(\frac{\partial \mathbf{F}}{\partial z} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial z}\right)$$

$$= \Sigma\, \mathbf{i}\left(\mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x}\right) + \Sigma\, \mathbf{i}\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x}\right)$$

Now
$$\mathbf{G} \times \text{curl}\, \mathbf{F} = \mathbf{G} \times \Sigma\, \mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x} = \Sigma\, \mathbf{G} \times \left(\mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x}\right)$$

$$= \Sigma \left\{\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{i} - (\mathbf{G} \cdot \mathbf{i}) \frac{\partial \mathbf{F}}{\partial x}\right\}$$

$$= \Sigma\, \mathbf{i}\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x}\right) - \Sigma\left(\mathbf{G} \cdot \mathbf{i} \frac{\partial}{\partial x}\right)\mathbf{F}$$

$$\therefore \Sigma\, \mathbf{i}\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x}\right) = \mathbf{G} \times \text{curl}\, \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F}$$

Similarly
$$\Sigma\, \mathbf{i}\left(\mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x}\right) = \mathbf{F} \times \text{curl}\, \mathbf{G} + (\mathbf{F} \cdot \nabla)\mathbf{G}$$

Hence
$$\nabla(\mathbf{F} \cdot \mathbf{G}) = \Sigma\, \mathbf{i}\left(\mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x}\right) + \Sigma\, \mathbf{i}\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x}\right)$$

$$= \mathbf{F} \times \text{curl}\, \mathbf{G} + \mathbf{G} \times \text{curl}\, \mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$$

2. Div $(\phi \mathbf{F}) = \nabla \cdot (\phi \mathbf{F}) = \phi \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \operatorname{grad} \phi = \phi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \phi$

Proof :

$$\nabla \cdot (\phi \mathbf{F}) = \mathbf{i} \cdot \frac{\partial}{\partial x}(\phi \mathbf{F}) + \mathbf{j} \cdot \frac{\partial}{\partial y}(\phi \mathbf{F}) + \mathbf{k} \cdot \frac{\partial}{\partial z}(\phi \mathbf{F})$$

$$= \mathbf{i} \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{F} + \phi \frac{\partial \mathbf{F}}{\partial x}\right) + \mathbf{j} \cdot \left(\frac{\partial \phi}{\partial y}\mathbf{F} + \phi \frac{\partial \mathbf{F}}{\partial y}\right) + \mathbf{k} \cdot \left(\frac{\partial \phi}{\partial z}\mathbf{F} + \phi \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \phi \left(\mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{F}}{\partial z}\right) + \left(\mathbf{i} \frac{\partial \phi}{\partial x} \cdot \mathbf{F} + \mathbf{j} \frac{\partial \phi}{\partial y} \cdot \mathbf{F} + \mathbf{k} \frac{\partial \phi}{\partial z} \cdot \mathbf{F}\right)$$

$$= \phi \left(\Sigma\, \mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x}\right) + \left(\Sigma\, \mathbf{i} \frac{\partial \phi}{\partial x}\right) \cdot \mathbf{F}$$

$$= \phi \nabla \cdot \mathbf{F} + (\nabla \phi) \cdot \mathbf{F} = \phi (\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot (\nabla \phi)$$

3. curl $(\phi \mathbf{F}) = \nabla \times (\phi \mathbf{F}) = \phi \operatorname{curl} \mathbf{F} + \operatorname{grad} \phi \times \mathbf{F}$

$= \phi \nabla \times \mathbf{F} + \nabla \phi \times \mathbf{F}$

Proof : $\nabla \times (\phi \mathbf{F}) = \Sigma\, \mathbf{i} \times \frac{\partial}{\partial x}(\phi \mathbf{F}) = \Sigma\, \mathbf{i} \times \left(\frac{\partial \phi}{\partial x}\mathbf{F} + \phi \frac{\partial \mathbf{F}}{\partial x}\right)$

$$= \Sigma\, \mathbf{i} \times \left(\frac{\partial \phi}{\partial x}\mathbf{F}\right) + \Sigma\, \mathbf{i} \times \left(\phi \frac{\partial \mathbf{F}}{\partial x}\right)$$

$$= \Sigma \left(\mathbf{i} \frac{\partial \phi}{\partial x} \times \mathbf{F}\right) + \Sigma\, \phi \left(\mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x}\right)$$

$$= \nabla \phi \times \mathbf{F} + \phi \nabla \times \mathbf{F}$$

4. Div $(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$

Proof : Div $(\mathbf{F} \times \mathbf{G}) = \Sigma\, \mathbf{i} \cdot \frac{\partial}{\partial x}(\mathbf{F} \times \mathbf{G})$

$$= \Sigma\, \mathbf{i} \cdot \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial x}\right)$$

$$= \Sigma\, \mathbf{i} \cdot \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G}\right) - \Sigma\, \mathbf{i} \cdot \left(\frac{\partial \mathbf{G}}{\partial x} \times \mathbf{F}\right) \quad (\because \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a})$$

$$= \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x}\right) \cdot \mathbf{G} - \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{G}}{\partial x}\right) \cdot \mathbf{F}$$

($\because$ dot and cross can be interchanged in a triple scalar product)

$$= \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$$

5. Curl $(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

Proof : curl $(\mathbf{F} \times \mathbf{G}) = \Sigma\, \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{F} \times \mathbf{G})$

$$= \Sigma\, \mathbf{i} \times \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial x}\right)$$

$$= \Sigma\, \mathbf{i} \times \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G}\right) - \Sigma\, \mathbf{i} \times \left(\frac{\partial \mathbf{G}}{\partial x} \times \mathbf{F}\right) \qquad \ldots (1)$$

Using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$, we have

$$\mathbf{i} \times \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G}\right) = (\mathbf{i} \cdot \mathbf{G}) \frac{\partial \mathbf{F}}{\partial x} - \left(\mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{G}$$

$$= \left(\mathbf{G} \cdot \mathbf{i} \frac{\partial}{\partial x}\right)\mathbf{F} - \left(\mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{G}$$

$$\therefore \Sigma\, \mathbf{i} \times \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G}\right) = \mathbf{G} \cdot \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right)\mathbf{F}$$

$$- \left(\mathbf{i} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{F}}{\partial z} \right) \mathbf{G}$$

$$= (\mathbf{G} \cdot \nabla) \mathbf{F} - \mathbf{G} \text{ div } \mathbf{F}$$

Similarly, $- \sum \mathbf{i} \times \left(\frac{\partial \mathbf{G}}{\partial x} \times \mathbf{F} \right) = -(\mathbf{F} \cdot \nabla) \mathbf{G} + \mathbf{F} \text{ div } \mathbf{G}$

$\therefore$ (1) becomes curl $(\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla) \mathbf{F} - \mathbf{G} \text{ div } \mathbf{F} + \mathbf{F} \text{ div } \mathbf{G} - (\mathbf{F} \cdot \nabla) \mathbf{G}$

$$= \mathbf{F} \text{ div } \mathbf{G} - \mathbf{G} \text{ div } \mathbf{F} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}$$

4.8 Second Order Differential Functions

We assume that all the functions occuring, hereafter, are sufficiently often differentiable. As $\nabla\phi$ is a vector function, its divergence and curl are defined. It is known that $\nabla \cdot \mathbf{F}$ is a scalar function and so $\nabla (\nabla \cdot \mathbf{F})$ is defined. In the same way, since $\nabla \times \mathbf{F}$ is a vector quantity, we can find its divergence and curl, namely $\nabla \cdot (\nabla \times \mathbf{F})$ and $\nabla \times (\nabla \times \mathbf{F})$. Thus $\nabla \cdot (\nabla \phi)$, $\nabla \times (\nabla \phi)$, $\nabla \cdot (\nabla \times \mathbf{F})$, $\nabla (\nabla \cdot \mathbf{F})$, and $\nabla \times (\nabla \times \mathbf{F})$ are expressions where ∇ appears two times. We shall now proceed to find their values.

1. Div grade $\phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

Proof: We have $\nabla \cdot (\nabla \phi) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \right)$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$$

We write this as $\nabla^2 \phi$ where, symbolically,

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is known as the Laplacian operator.

Thus $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$.

2. Curl grad $\phi = \nabla \times (\nabla \phi) = 0$

Proof: We have curl grad $\phi = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \left(\mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \right)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 \phi}{\partial y \, \partial z} - \frac{\partial^2 \phi}{\partial z \, \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 \phi}{\partial z \, \partial x} - \frac{\partial^2 \phi}{\partial x \, \partial z} \right) \mathbf{j}$$

$$+ \left(\frac{\partial^2 \phi}{\partial x \, \partial y} - \frac{\partial^2 \phi}{\partial y \, \partial x} \right) \mathbf{k} = \mathbf{0}$$

3. Div curl $\mathbf{F} = 0$

Proof: Let $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$, so that

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Since $\text{div}\left(F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$, we have

$$\text{Div curl } \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$

4. The two functions curl curl F and grad div F i.e., $\nabla \times (\nabla \times \mathbf{F})$ and $\nabla(\nabla \cdot \mathbf{F})$ are related by $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

Proof: Let $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$. Then we have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

$$\text{Curl curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) & \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) & \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right]\mathbf{k}$$

$$= \mathbf{i}\,\frac{\partial}{\partial x}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] + \mathbf{j}\,\frac{\partial}{\partial y}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]$$

$$+ \mathbf{k}\,\frac{\partial}{\partial z}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] - \mathbf{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \mathbf{j}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right] - \mathbf{k}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]$$

$$- \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left[F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}\right]$$

$$= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

SAQ : State whether the following statements are true or false. Support your answers with reasons.

1. div ϕ is defined where ϕ is a scalar point function.
2. div **F** is a scalar quantity where **F** is a vector function.
3. Curl **F** is an operator where **F** is a vector point function.
4. div $(\nabla\phi)$ is defined, where ϕ is a scalar point function.
5. Curl (div v) is defined where v is a vector quantity.
6. ∇ . (curl v) is defined where v is a vector quantity.
7. ∇ (div v) is defined where v is a vector quantity.
8. div **v** $\times$ curl **v** can be defined.
9. If $\nabla^2\phi = 0$ then ϕ is called a harmonic function.
10. (grad ϕ) is irrotational.
11. Curl ϕ is a solenoidal vector.

Examples

Ex. 1: Show that $\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$, where r is the position vector of any point (x, y, z) in three dimensional space.

i.e., $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$

Sol : We have $\nabla\left(\frac{1}{r}\right)$

$$= \mathbf{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \mathbf{i}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y} + \mathbf{k}\left(\frac{1}{r}\right)\frac{\partial r}{\partial z}$$

From $\quad r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$, we get

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}} 2x = \frac{x}{\sqrt{\left(x^2 + y^2 + z^2\right)}} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$

$$\nabla\left(\frac{1}{r}\right) = \Sigma\,\mathbf{i}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}$$

$$= \mathbf{i}\,\frac{1}{r^2}\,\frac{x}{r} - \mathbf{j}\,\frac{1}{r^2}\,\frac{y}{r} - \mathbf{k}\,\frac{1}{r^2}\,\frac{z}{r}$$

$$= -\frac{1}{r^3}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = -\frac{\mathbf{r}}{r^3}.$$

$$\therefore \mathbf{a}\cdot\nabla\left(\frac{1}{r}\right) = \mathbf{a}\cdot\left(\frac{-\mathbf{r}}{r^3}\right) = -\frac{(\mathbf{a}\cdot\mathbf{r})}{r^3}$$

Ex. 2: Show that (i) div **r** = 3 and (ii) curl **r** = **0** where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Sol : (i) We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\therefore \text{div } \mathbf{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3.$$

(ii)
$$\text{curl } \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

Ex. 3 : Prove that div grad $r^m = m(m+1)\,r^{m-2}$

where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}| = r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$

Sol : Noting that $\frac{\partial r}{\partial x} = \frac{x}{r}$ etc, we have

$$\nabla r^m = \Sigma\,\mathbf{i}\,\frac{\partial}{\partial x}\,r^m = \Sigma\,\mathbf{i}\,\frac{\partial r}{\partial x}\,\frac{\partial}{\partial r}\,r^m$$

$$= \mathbf{i}\,mr^{m-1}\,\frac{x}{r} + \mathbf{j}\,mr^{m-1}\,\frac{y}{r} + \mathbf{k}\,mr^{m-1}\,\frac{z}{r}$$

$$= mr^{m-2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

$$= mr^{m-2}\,\mathbf{r} \qquad \ldots (1)$$

$$\therefore \text{div grad } r^m = \nabla\cdot\left[mr^{m-2}\,\mathbf{r}\right]$$

$$= \nabla\left(mr^{m-2}\right)\cdot\mathbf{r} + mr^{m-2}\,\text{div }\mathbf{r}$$

$$(\because \nabla\cdot(\phi\,\mathbf{F}) = \nabla\phi\cdot\mathbf{F} + \phi\,\text{div }\mathbf{F})$$

$$= m\left(\nabla\,r^{m-2}\right)\cdot\mathbf{r} + 3\,mr^{m-2} \quad (\because \text{div }\mathbf{r} = 3)$$

$$= m(m-2)\,r^{m-4}\,\mathbf{r}\cdot\mathbf{r} + 3\,mr^{m-2}$$

(From (1) Replacing m by $m-2$)

$$= m(m-2)\,r^{m-2} + 3\,mr^{m-2}$$

$$= m(m+1)\,r^{m-2}.$$

Ex. 4 : If **a** and **b** are constant vectors, show that

(i) grad $(\mathbf{r}\cdot\mathbf{a}) = \mathbf{a}$

(ii) grad $[\mathbf{r}\ \mathbf{a}\ \mathbf{b}] = \mathbf{a}\times\mathbf{b}$

(iii) $\text{div}\,(\mathbf{r}\times\mathbf{a}) = 0$

and (iv) $\text{Curl}\,(\mathbf{r}\times\mathbf{a}) = -2\mathbf{a}$,

where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Sol : (i) $\text{grad}\,(\mathbf{r}\cdot\mathbf{a}) = \Sigma\,\mathbf{i}\,\dfrac{\partial}{\partial x}(\mathbf{r}\cdot\mathbf{a})$

$$= \Sigma\,\mathbf{i}\left(\frac{\partial \mathbf{r}}{\partial x}\cdot\mathbf{a} + \mathbf{r}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)$$

$$= \Sigma\,\mathbf{i}\left(\frac{\partial \mathbf{r}}{\partial x}\cdot\mathbf{a}\right) \quad [\because \mathbf{a} \text{ is a constant vector}]$$

$$= \Sigma\,\mathbf{i}\,(\mathbf{i}\cdot\mathbf{a}) \quad \left[\because \frac{\partial \mathbf{r}}{\partial x} = \frac{\partial}{\partial x}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{i} \text{ etc.}\right]$$

$$= \mathbf{i}\,(a_1) + \mathbf{j}\,(a_2) + \mathbf{k}\,(a_3) \text{ where } \mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$$

$$= \mathbf{a}$$

(ii) $\text{Grad}\,[\mathbf{r}\;\mathbf{a}\;\mathbf{b}] = \text{grad}\,[\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})]$

$$= \text{grad}\,(\mathbf{r}\cdot\mathbf{c}) \text{ where } \mathbf{a}\times\mathbf{b} = \mathbf{c}, \text{ a constant vector.}$$

$$= \mathbf{c} \text{ by (i)}$$

$$= \mathbf{a}\times\mathbf{b}$$

(iii) $\text{div}\,(\mathbf{r}\times\mathbf{a}) = \Sigma\,\mathbf{i}\cdot\dfrac{\partial}{\partial x}(\mathbf{r}\times\mathbf{a})$

$$= \Sigma\,\mathbf{i}\cdot\left(\frac{\partial \mathbf{r}}{\partial x}\times\mathbf{a} + \mathbf{r}\times\frac{\partial \mathbf{a}}{\partial x}\right)$$

$$= \Sigma\,\mathbf{i}\cdot\left(\frac{\partial \mathbf{r}}{\partial x}\times\mathbf{a}\right) \left(\because \frac{\partial \mathbf{a}}{\partial x} = 0\right)$$

$$= \Sigma\,\mathbf{i}\cdot(\mathbf{i}\times\mathbf{a}) \left(\because \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}\right)$$

$$= 0,$$

since a triple scalar product is zero when two vectors in it are equal.

(iv) $\text{Curl}\,(\mathbf{r}\times\mathbf{a}) = \Sigma\,\mathbf{i}\times\dfrac{\partial}{\partial x}(\mathbf{r}\times\mathbf{a})$

$$= \Sigma\,\mathbf{i}\times\left(\frac{\partial \mathbf{r}}{\partial x}\times\mathbf{a} + \mathbf{r}\times\frac{\partial \mathbf{a}}{\partial x}\right)$$

$$= \Sigma\,\mathbf{i}\times\left(\frac{\partial \mathbf{r}}{\partial x}\times\mathbf{a}\right)$$

$$= \Sigma\,\mathbf{i}\times(\mathbf{i}\times\mathbf{a})$$

$$= \Sigma\,\{(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i} - (\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}\}$$

$$= \Sigma\,a_1\,\mathbf{i} - \Sigma\,(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}$$

$$= \mathbf{a} - 3\,\mathbf{a} = -2\,\mathbf{a} \quad [\text{Note that } \Sigma\,(\mathbf{i}\cdot\mathbf{i}) = 3]$$

Ex. 5 : Prove that $\text{curl}\,\{f(r)\,\mathbf{r}\} = \mathbf{0}$ where $f(r)$ is differentiable.

Sol. : $\text{curl}\,\{f(r)\,\mathbf{r}\} = \Sigma\,\mathbf{i}\times\dfrac{\partial}{\partial x}[f(r)\,\mathbf{r}]$

$$= \Sigma\,\mathbf{i}\times\left\{f(r)\,\frac{\partial \mathbf{r}}{\partial x} + \mathbf{r}\,f'(r)\,\frac{\partial r}{\partial x}\right\}$$

$$= \Sigma\,\mathbf{i}\times\left\{f(r)\,\mathbf{i} + \mathbf{r}\,f'(r)\,\frac{x}{r}\right\}$$

$$= f(r)\ \Sigma\, \mathbf{i} \times \mathbf{i} + \frac{f'(r)}{r}\ \Sigma\ x\,(\mathbf{i} \times \mathbf{r})$$

$$= \frac{f'(r)}{r}\,(x\,\mathbf{i} \times \mathbf{r} + y\,\mathbf{j} \times \mathbf{r} + z\,\mathbf{k} \times \mathbf{r}) \quad (\because\ \mathbf{i} \times \mathbf{i} = 0)$$

$$= \frac{f'(r)}{r}\,[(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \times \mathbf{r}]$$

$$= \frac{f'(r)}{r}\,(\mathbf{r} \times \mathbf{r}) = 0$$

4.9 Summary

Let ϕ be a scalar function of three variables x, y and z. Then

$\nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$ is called the gradient of ϕ. It is denoted by grad ϕ.

$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is called the del operator. Note that ϕ is a scalar point function, where as $\nabla \phi$ is a vector function. If $\mathbf{F}(x, y, z)$ is continuously differentiable vector function, then

$\nabla \cdot \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \mathbf{F}$ is called divergence of $\mathbf{F}$. It is denoted by div $\mathbf{F}$ also. It is a scalar quantity. Curl $\mathbf{F} = \mathbf{i} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{F}}{\partial z}$ is denoted by $\nabla \times \mathbf{F}$. It is a vector quantity.

If $\mathbf{a}$ is a vector,

$\mathbf{a} \cdot \nabla = \mathbf{a} \cdot \mathbf{i}\frac{\partial}{\partial x} + \mathbf{a} \cdot \mathbf{j}\frac{\partial}{\partial y} + \mathbf{a} \cdot \mathbf{k}\frac{\partial}{\partial z} = \Sigma\, \mathbf{a} \cdot \mathbf{i}\frac{\partial}{\partial x}$, is an operator.

If $\mathbf{F}(x, y, z)$, $\mathbf{G}(x, y, z)$ be two functions then, $\mathbf{F} \cdot \mathbf{G}$ is a scalar function, $\mathbf{F} \times \mathbf{G}$ is a vector function. The following identities have been proved..

(i) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times \text{curl}\,\mathbf{G} + \mathbf{G} \times \text{curl}\,\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$.

(ii) $\text{Div}\,(\phi\mathbf{F}) = \phi\ \text{div}\,\mathbf{F} + \mathbf{F} \cdot \text{grad}\,\phi$.

(iii) $\text{curl}\,(\phi\mathbf{F}) = \phi\ \text{curl}\,\mathbf{F} + \text{grad}\,\phi \times \mathbf{F}$.

(iv) $\text{Div}\,(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\,\mathbf{F} - \mathbf{F} \cdot \text{curl}\,\mathbf{G}$.

(v) $\text{Curl}\,(\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,\text{div}\,\mathbf{G} - \mathbf{G}\,\text{div}\,\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$.

The second order differential operator and functions

(i) $\text{Div} \cdot \text{grad}\,\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

(ii) $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplace operator

(iii) $\text{curl}\,(\text{grad}\,\phi) = 0$

(iv) curl curl $\mathbf{F}$ and grad div $\mathbf{F}$ are related by the relation.

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$

4.10 Sample Examination Questions

I. Answer the following questions in detail.

(i) a) Difine the operators gradient, divergence and curl.

b) Show that

(i) $\text{div}\,(\mathbf{F} + \mathbf{G}) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$

(ii) $\text{curl}\,(\mathbf{F} + \mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$

(iii) $\text{grad}\,(\phi\,\psi) = \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi$

and (iv) $\text{grad}\left(\frac{\phi}{\psi}\right) = \frac{\psi\,\text{grad}\,\phi - \phi\,\text{grad}\,\psi}{\psi^2}$

(ii) a) Prove that

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,\nabla\,.\,\mathbf{G} - \mathbf{G}\,\nabla\,.\,\mathbf{F} + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}.$

b) If **a** , **b** are constant vectors, prove that

(i) $\text{grad}\,(\mathbf{a}\,.\,\mathbf{F}) = (\mathbf{a}\,.\,\nabla)\,\mathbf{F} + \mathbf{a}\,\text{curl}\,\mathbf{F}$

(ii) $\text{div}\,(\mathbf{a} \times \mathbf{F}) = -\,\mathbf{a}\,.\,\text{curl}\,\mathbf{F}$

(iii) $\text{div}\,[\mathbf{r} \times (\mathbf{a} \times \mathbf{r})] = 2\mathbf{r}\,.\,\mathbf{a}$

(iv) $\text{curl}\,[\mathbf{r} \times (\mathbf{a} \times \mathbf{r})] = 3\mathbf{r} \times \mathbf{a}$

(v) $\text{grad}\,[(\mathbf{r} \times \mathbf{a})\,.\,(\mathbf{r} \times \mathbf{b})] = (\mathbf{b} \times \mathbf{r}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{r}) \times \mathbf{b}$

and (vi) $\text{div}\,[(\mathbf{r} \times \mathbf{a}) \times (\mathbf{r} \times \mathbf{b})] = 2\mathbf{a}\,.\,(\mathbf{b} \times \mathbf{r}) - 2\mathbf{b}\,.\,(\mathbf{a} \times \mathbf{r})$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(iii) a) Define the Laplacian operator and prove that

$\nabla \times (\nabla \times \mathbf{F}) = \nabla\,(\nabla\,.\,\mathbf{F}) - \nabla^2\,\mathbf{F}.$

b) Evaluate (i) $\nabla^2\,(\log r)$, (ii) $\nabla\,.\,(r^3\,\mathbf{r})$ and

(iii) $\nabla^2\left(\nabla\,.\,\frac{\mathbf{r}}{r^2}\right)$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\,\mathbf{r}\,| = r$

II. Briefly answer the following

(i) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\,\mathbf{r}\,| = r$, prove the following

a) $\nabla\,(r) = \frac{\mathbf{r}}{r}$

b) $\nabla^2\left(\frac{x}{r^3}\right) = 0$

c) $\nabla^2\left(\frac{1}{r}\right) = 0$

(ii) If $\mathbf{F} = x^2\,y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find $\nabla \times (\nabla \times \mathbf{F})$

(iii) If $\mathbf{F} = (3x^2\,y - z)\,\mathbf{i} + (xz^3 + y^4)\,\mathbf{j} - 2x^3\,z^2\,\mathbf{k}$, find $\nabla\,(\nabla\,.\,\mathbf{F})$ at the point (2, – 1, 0)

(iv) Prove that $\text{curl}\,(\mathbf{a}\,.\,\mathbf{r})\,\mathbf{a} = \mathbf{0}$

(v) Show that $\text{curl}\,\{\mathbf{a}\,f\,(r)\} = \frac{f'(r)}{r}\,\mathbf{r} \times \mathbf{a}$, where $f\,(r)$ is a differentiable function.

(vi) If $\mathbf{a} = g\,\text{grad}\,f$, where f and g are scalar fields, show that $\mathbf{a}\,.\,\text{curl}\,\mathbf{a} = 0$.

Answers

I. (iii) (i) $\frac{1}{r^2}$, (ii) $6r^3$ (iii) $\frac{2}{r^4}$

II. (ii) $(2x + 2)\,\mathbf{j}$ (iii) $-6\mathbf{i} + 24\mathbf{j} - 32\mathbf{k}$

4.11 Answers to self assessment Questions.

(1) No. divergence is defined only for vector point functions.

(2) Yes. By definition div **f** is a scalar quantity.

(3) No. curl **F** is a vector quantity, curl is an operator.

(4) Yes. div $(\nabla \phi)$ is defined, because $\nabla \phi$ is a vector quantity and div is defined for vectors.

(5) No, curl (div v) is not defined, because div v is a scalar.

(6) Yes, ∇ . (curl v) is defined.

(7) Yes, ∇ (div v) is defined.

(8) No, since div **v** is a scalar quantity cross product of a scalar with a vector is not defined.

(9) Yes, A function ϕ which satisfies the Laplacian equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$ is called a harmonic function.

(10) Yes, because curl (grad ϕ) = **0**.

(11) Yes, because div (curl q) = 0.

UNIT–5 : LINE, SURFACE AND VOLUME INTEGRALS

5.0 Contents

5.1 Aims and Objectives

After going through this unit, you will be able to :

i) integrate a vector valued function,

ii) transform a line integral into a surface integral using Stokes theorem and transform a surface integral into a volume integral using Gauss divergence theorem.

5.2 Introduction

In this unit, we shall discuss line integrals, surface integrals and volume integrals and then proceed to prove the important theorems on vector integration, namely, Green's theorem, Gauss Divergence Theorem and Stoke's Theorem. These theorems are of immence importance in hydrodynamics, mechanics, electromagnetic theory etc. The physical significance and applications of the theorems will be given in unit–6.

5.3 Integration

Let $\mathbf{F}(t)$ be a vector function of a real variable t. A vector function of t, say, $\mathbf{f}(t)$ is said to be an integral of $\mathbf{F}(t)$, if the derivative of $\mathbf{f}(t)$ is $\mathbf{F}(t)$. Thus, symbolically,

$$\text{if } \frac{d}{dt}\mathbf{f}(t) = \mathbf{F}(t), \text{ then } \int \mathbf{F}(t)\, dt = \mathbf{f}(t).$$

We then say that $\mathbf{F}$ is integrable. We can also say that a vector function $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ is integrable if F_x, F_y, F_z are integrable.

Remark 1: If $\mathbf{c}$ be an arbitrary constant, we have $\frac{d}{dt}[\mathbf{f}(t) + \mathbf{c}] = \frac{d}{dt}\mathbf{f}(t) + \frac{d\mathbf{c}}{dt} = \mathbf{F}(t)$ so that

$$\int \mathbf{F}(t)\, dt = \mathbf{f}(t) + \mathbf{c}.$$

Thus the integral $\mathbf{f}(t)$ is indefinite to the extent of an arbitrary additive constant $\mathbf{c}$. For this reason $\mathbf{f}(t)$ is called the indefinite integral and $\mathbf{c}$ is called the constant of integration. The value of $\mathbf{c}$ is determined by the help of a given initial or geometrical condition.

Remark 2: Let $\mathbf{F}(t) = F_1(t)\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k}$ be an integrable function. Then the integral of $\mathbf{F}(t)$ is given by

$$\int \mathbf{F}(t)\,dt = \mathbf{i}\int F_1(t)\,dt + \mathbf{j}\int F_2(t)\,dt + \mathbf{k}\int F_3(t)\,dt.$$

5.3.1 Definite Integral

If $\int \mathbf{F}(t)\,dt = \mathbf{f}(t)$ then $\mathbf{f}(b) - \mathbf{f}(a)$ is said to be the definite integral of $\mathbf{F}(t)$ between the limits $t = a$ and $t = b$ and is denoted by the symbol.

$\int_a^b \mathbf{F}(t)\,dt$. Thus we have $\int_a^b \mathbf{F}(t)\,dt = \mathbf{f}(b) - \mathbf{f}(a)$.

5.3.2 Standard Results

Various results on integration may be derived from the corresponding results on classical differentiation and integration obtained in unit 3. Thus we have

(i) $\frac{d}{dt}[\mathbf{r}(t)\,.\,\mathbf{s}(t)] = \frac{d\mathbf{r}}{dt}.\mathbf{s} + \mathbf{r}\,.\,\frac{d\mathbf{s}}{dt}$.

$$\therefore \int \left(\frac{d\mathbf{r}}{dt}.\,\mathbf{s} + \mathbf{r}\,.\,\frac{d\mathbf{s}}{dt}\right) dt = \mathbf{r}(t)\,.\,\mathbf{s}(t) + c \qquad \text{... (1)}$$

(ii) In particular, if $\mathbf{s}(t) = \mathbf{r}(t)$, then

$\int \left(2\mathbf{r}\,.\,\frac{d\mathbf{r}}{dt}\right) dt = r^2(t) + c$, Here r^2 stands for $|\mathbf{r}|^2$

(iii) Since $\frac{d}{dt}\left[\frac{d\mathbf{r}}{dt}\right]^2 = 2\left(\frac{d\mathbf{r}}{dt}\right)\cdot\left(\frac{d^2\mathbf{r}}{dt^2}\right)$

We have $\int 2\left(\frac{d\mathbf{r}}{dt}.\,\frac{d^2\mathbf{r}}{dt^2}\right) dt = 2\left(\frac{d\mathbf{r}}{dt}\right)^2 + c = \frac{d\mathbf{r}}{dt}\,.\,\frac{d\mathbf{r}}{dt} + c.$

(iv) Since $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}$

we have $\int \left(\frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}\right) dt = \mathbf{r}(t) \times \mathbf{s}(t) + \mathbf{c}$

(v) If $\mathbf{a}$ be a constant vector, since

$\frac{d}{dt}[\mathbf{a} \times \mathbf{r}] = \mathbf{a} \times \frac{d\mathbf{r}}{dt}$

We have $\int \left(\mathbf{a} \times \frac{d\mathbf{r}}{dt}\right) dt = \mathbf{a} \times \mathbf{r}(t) + \mathbf{c}.$

Remark : The constant of integration has the same nature as the function that is being integrated. Thus in the first three of the above examples the constant of integration is a scalar, where as in the last two examples it is a vector.

5.4 Line Integral

Let x, y, z be functions of t defined on an interval $[a, b]$. The end points of $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ said to constitute a curve C. Consider a continous vector function $\mathbf{F}(\mathbf{r})$ on a curve C. Divide C into n parts at the points. $A = P_0, P_1, \ldots P_{i-1}, P_i, \ldots P_n = B$.

Let their position vectors be $\mathbf{r}_0, \mathbf{r}_1, \ldots \mathbf{r}_{i-1}, \mathbf{r}_i \ldots \mathbf{r}_n$. Let $\mathbf{u}_i$ be the position vector of any point on the arc $P_{i-1}\,P_i$.

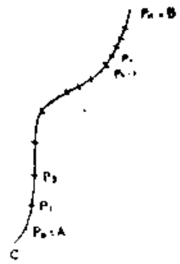

Fig. 1

Now consider the sum $S = \sum_{i=0}^{n} \mathbf{F}(\mathbf{u}_i) \cdot \delta\mathbf{r}_i$

where $\delta\mathbf{r}_i = \mathbf{r}_i - \mathbf{r}_{i-1}$. The limit of this sum as $n \to \infty$ in such a way that $|\delta\mathbf{r}_i| \to 0$, provided it exists, is called the tangential line integral of $\mathbf{F}(\mathbf{r})$ along C and is symbolically written as

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \text{ or } \int \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt.$$

The student can identify this limit in the sense of Riemann integration when he learns it.

If $\mathbf{F}(\mathbf{r}) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$

$$\text{then} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}) \cdot [\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz]$$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \Sigma \int_a^b F_1[x(t), y(t), z(t)]\,\frac{dx}{dt}\,dt$$

The integral above can be taken as the definition of the integral.

5.5 Surface Integral

The set of all points $[x, y, z(x, y)]$ when x, y vary on a two dimensional domain L is called a surface. Consider a continuous vector function $\mathbf{F}(\mathbf{r})$ on a surface S. Divide S into a finite number of sub surfaces. Let $\delta\mathbf{S}$ be a vector whose magnitude is the surface area of δS and whose direction is along the normal to the surface at P.

Set up the sum $\Sigma\,\mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{S}$ where the summation extends over all the sub-surfaces. The limit of this sum as the number of subsurfaces increase infinitely and the area of each sub-surface tends to zero, is called the Normal Surface Integral of $\mathbf{F}(\mathbf{r})$ over S and is denoted by

$$\int \mathbf{F} \cdot d\mathbf{S} \text{ or } \int_S \mathbf{F} \cdot \mathbf{n}\,dS$$

where **n** is a unit outward drawn normal at P to S. The student can view the limit of the sum as follows.

Remarks : Let $\mathbf{n} = \cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma \mathbf{k}$, where $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of **n** at some point on the surface.

Put $\delta \mathbf{S} = \mathbf{n}\, \delta S = (\cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k})\, \delta S$

it follows that $\delta S \cos\alpha$, $\delta S \cos\beta$, $\delta S \cos\gamma$ are the orthogonal projections of the elementary area δS on the yz, zx and xy planes respectively. Then p·t

$$\delta x\, \delta y = \delta S \cos\gamma = \delta S\, |\mathbf{n} \cdot \mathbf{k}|$$

so that $\delta S = \dfrac{\delta x\, \delta y}{|\mathbf{n} \cdot \mathbf{k}|}$

Similarly $\delta S = \dfrac{\delta y\, \delta z}{|\mathbf{n} \cdot \mathbf{i}|}$ and $\delta S = \dfrac{\delta z\, \delta x}{|\mathbf{n} \cdot \mathbf{j}|}$

Now define $\displaystyle\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ as the double integral

$$\iint_R \mathbf{F} \cdot \mathbf{n} \frac{dx\, dy}{|\mathbf{n} \cdot \mathbf{k}|}$$

where R is the orthogonal projection of S on xy - plane.

5.6 Volume Integral

The following is to be understood in a way analogous to the surface integral. Consider a continuous vector function **F** (**r**) defined on a region V enclosed by a surface S. Divide V into a finite number of sub regions $V_1, V_2, \ldots V_n$.

Let δV_i be the volume of the sub region V_i enclosing any point whose position vector is $\mathbf{r}_i$. Set up the sum $\displaystyle\sum_{i=1}^{n} \mathbf{F}(\mathbf{r}_i)\, \delta V_i$. The limit of this sum as $n \to \infty$ in such a way that $\delta V_i \to 0$ is called the volume integral of **F** (**r**) over V and is symbolically written as $\displaystyle\int_V \mathbf{F}\, dV$.

The student, familiar with triple integrals can identify the limit as follows.

If $\mathbf{F}(\mathbf{r}) = F_1(x, y, z)\, \mathbf{i} + F_2(x, y, z)\, \mathbf{j} + F_3(x, y, z)\, \mathbf{k}$

then
$$\int_V \mathbf{F}\, dV = \mathbf{i} \iiint_V F_1\, dx\, dy\, dz + \mathbf{j} \iiint_V F_2\, dx\, dy\, dz + \mathbf{k} \iiint_V F_3\, dx\, dy\, dz$$

5.7 Green's Theorem For Plane

If $F(x, y)$, $G(x, y)$, $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial x}$ be continuous in a region R of the xy – plane bounded by a simple closed curve C, then

$$\int_C (F\,dx + G\,dy) = \int\int_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx\,dy$$

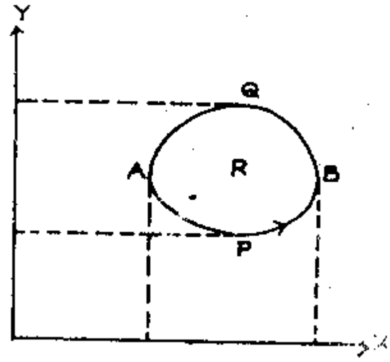

Fig. 2

Suppose that the region R is such that any line parallel to coordinate axes meets C in atmost two points.

(If not we can divide the region into several regions in each of which this property holds, see fig.3). Let the region be included between the lines $x = a$, $x = b$, $y = c$ and $y = d$.

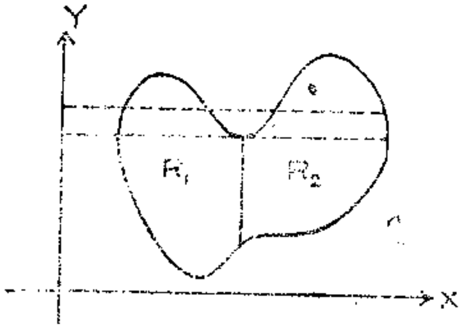

Fig.3

Let the equations to the curves APB and BQA be $y = F_1(x)$ and $y = F_2(x)$ respectively.

We have

$$\int\int_R \frac{\partial F}{\partial y}\,dx\,dy = \int_{x=a}^{b} \int_{y=F_1(x)}^{F_2(x)} \frac{\partial F}{\partial y}\,dx\,dy$$

$$= \int_a^b [F(x, y)]_{y=F_1(x)}^{y=F_2(x)}\,dx$$

$$= \int_{x=a}^{b} \left[F\{x, F_2(x)\} - F\{x, F_1(x)\} \right] dx$$

$$= -\int_b^a F\{x, F_2(x)\}\,dx - \int_a^b F\{x, F_1(x)\}\,dx$$

$$= -\left[\int_{BQA} F(x, y)\,dx + \int_{APB} F(x, y)\,dx \right]$$

$$= -\left[\int_C F(x, y)\,dx \right] = -\int F\,dx \qquad \ldots (1)$$

Let the equations to the curves QAP and PBQ be $x = G_1(y)$ and $x = G_2(y)$ respectively.

We have $$\int\int_R \frac{\partial G}{\partial y}\, dx\, dy = \int_{y=c}^{a} \int_{x=G_1(y)}^{G_2(y)} \frac{\partial G}{\partial x}\, dx\, dy$$

$$= \int_{x=c}^{a} [G(x, y)]_{x=G_1(y)}^{G_2(y)} dy$$

$$= \int_c^d [G\{G_2(y), y\}\, dy - G\{G_1(y), y\}]\, dy$$

$$= \int_c^d G\{G_2(y), y\}\, dy + \int_d^c G\{G_1(y), y\}\, dy$$

$$= \int_{PBQ} G(x, y)\, dx + \int_{QAP} G(x, y)\, dy$$

$$= \int_C G(x, y)\, dy$$

From (1) and (2), we get

$$\int (F dx + G dy) = \int\int_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx\, dy$$

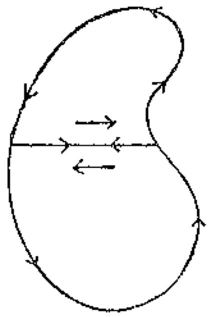

Fig. 4

Remark : This result can be extended to regions which may be divided into a finite number of sub regions such that the boundary of each is cut by a line parallel to an axis atmost in two points. Applying Green's theorem to each of these sub regions and adding the results, the surface integral combine into an integral over the whole region; the line integral over the common boundaries cancel [for each is covered twice in opposite directions], where as the remaining line integrals combine into the line integral over the curve C.

5.8 Stoke's Theorem

If S is an open surface bounded by a closed curve C and $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ be any continuously differentiable vector point functions,

then $$\oint_C \mathbf{F} \cdot d\,\mathbf{r} = \int_S \text{Curl } \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the unit normal at any point.

Proof : Consider a surface S. Let its projections on the coordinate planes yz, zx, xy are regions bounded by closed curves as shown in the figure 5. Let S be represented by $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$, where f, g, h are single valued continuous differentiable functions. Let $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ be a vector field defined over the surface. Now we have to show that

$$\iint_S (\nabla \times \mathbf{F}) \,.\, \mathbf{n}\, dS = \iint_S \left[\nabla \times (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k})\right] . \,\mathbf{n}\, dS$$

$$= \oint_C \mathbf{F} \,.\, d\mathbf{r}$$

where C is the boundary of S as shown in the figure.

$$\text{Now since } \nabla \times F_1\,\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \frac{\partial F_1}{\partial z}\mathbf{j} - \frac{\partial F_1}{\partial y}\mathbf{k}$$

we have $\left[\nabla \times (F_1\,\mathbf{i})\right] . \,\mathbf{n}\, dS = \left(\dfrac{\partial F_1}{\partial z}\,\mathbf{n}\,.\,\mathbf{j} - \dfrac{\partial F_1}{\partial y}\,\mathbf{n}\,.\,\mathbf{k}\right) dS.$ ... (1)

If $z = f(x, y)$ is taken as the equation of S, then the position vector to any point of S is

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$

so that $\dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \dfrac{\partial f}{\partial y}\mathbf{k}$. But $\dfrac{\partial \mathbf{r}}{\partial y}$ is a vector tangent to S and hence perpendicular to $\mathbf{n}$, so that

$$\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial y} = 0 = \mathbf{n} \cdot \mathbf{j} + \frac{\partial z}{\partial y}\,\mathbf{n} \cdot \mathbf{k} \quad \text{or} \quad \mathbf{n} \cdot \mathbf{j} = -\frac{\partial z}{\partial y}\,\mathbf{n} \cdot \mathbf{k}$$

Substituting in (1) we obtain

$$\left[\nabla \times (F_1\mathbf{i})\right] . \,\mathbf{n}\, dS = -\left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}\right) \mathbf{n}\,.\,\mathbf{k}\, dS \quad (2)$$

Now on $S = F_1(x, y, z) = F_1(x, y, f(x, y)] = F(x, y)$,

hence $\dfrac{\partial F_1}{\partial y} + \dfrac{\partial F_1}{\partial z}\dfrac{\partial z}{\partial y} = \dfrac{\partial F}{\partial y}$ and hence (2) becomes

$$\left[\nabla \times (F_1\mathbf{i})\right] . \,\mathbf{n}\, dS = -\frac{\partial F}{\partial y}\,\mathbf{n}\,.\,\mathbf{k}\, dS = -\frac{\partial F}{\partial y}\, dx\, dy$$

$$\text{Then } \iint_S \left[\nabla \times (F_1\mathbf{i})\right] . \,\mathbf{n}\, dS = \iint_S -\frac{\partial F}{\partial y}\, dx\; d$$

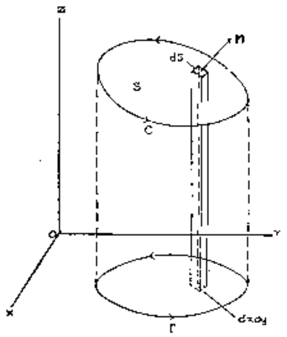

Fig. 5

where R is the projection of S on the xy - plane. By Green's theorem for the plane the last integral equals $\oint_\Gamma F\,dx$ where Γ is the boundary of R. Since at each point (x, y) of Γ the value of F is the same as the value of F_1 at each point (x, y, z) of C and since dx is same for both the curves we must have

$$\oint_{\Gamma} F\, dx = \oint_{C} F_1\, dx \text{ so that } \iint_{S} \left[\nabla \times \left(F_1 \mathbf{i}\right)\right] . \mathbf{n}\, dS = \oint_{C} F_1 dx.$$

Similarly taking projections on the other coordinate planes and/by a similar argument we get

$$\iint_{S} (\nabla \times F) . \mathbf{n}\, dS = \oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

The theorem is also valid for S which may not satisfy the restrictions imposed above. In such cases subdivide the surface each of which satisfies these conditions and extend the proof of this theorem to each of them.

5.9 Gauss Divergence Theorem

The normal surface integral of a vector function $\mathbf{F}$ over the boundary of a closed region is equal to the volume integral of div $\mathbf{F}$ taken over the region.

Proof : Consider a closed surface S which is such that any line parallel to coordinate axes cuts S in at most two points. Let the equation of S be $z = f(x, y)$. Let S_1 and S_2 be the lower and upper portions of the surface S. Assume the equations of S_1 and S_2 to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. We shall denote the projection of the surface on the xy plane by R.

Let $\mathbf{F} = F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k}$

$$\text{Then} \iiint_{V} \frac{\partial F_3}{\partial z} dV = \iiint_{V} \frac{\partial F_3}{\partial z} dz\, dy\, dx$$

$$= \iint_{R} \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z} dz \right] dy\, dx$$

$$= \iint_{R} \left[F_3 (x, y, z)\right]_{z=f_1}^{f_2} dy\, dx$$

$$= \iint_{R} \left[F_3 (x, y, f_2) - F_3 (x, y, f_1)\right] dy\, dx \qquad (1)$$

Let the outward drawn normal $\mathbf{n}_2$ to S_2 makes an acute angle γ_2 with $\mathbf{k}$, (see fig. 6)

we have $dy\, dx = \cos \gamma_2\, dS_2 = \mathbf{k} . \mathbf{n}\, dS_2$.

Again, let the outward drawn normal $\mathbf{n}_1$ to S_1 make an obtuse angle γ_1 with $\mathbf{k}$, then $dy\, dx = -\cos \gamma_1\, dS_1 = -\mathbf{k} . \mathbf{n}_1\, dS_1$.

$$\text{Then} \iint_{R} F_3 (x, y, f_2)\, dy\, dx = \iint_{S_2} F_3\, \mathbf{k} . \mathbf{n}_2\, dS_2 \qquad (2)$$

$$\iint_R F_3(x, y, f_1)\, dy\, dx = -\iint_{S_1} F_3\, \mathbf{k} \cdot \mathbf{n}_1\, dS_1 \qquad (3)$$

Using (2) and (3), (1) reduces to

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_{S_2} F_3\, \mathbf{k} \cdot \mathbf{n}_2\, dS_2 + \iint_{S_1} F_3\, \mathbf{k} \cdot \mathbf{n}_1\, dS_1$$

$$= \iint_S F_3\, \mathbf{k} \cdot \mathbf{n}\, dS \qquad (4)$$

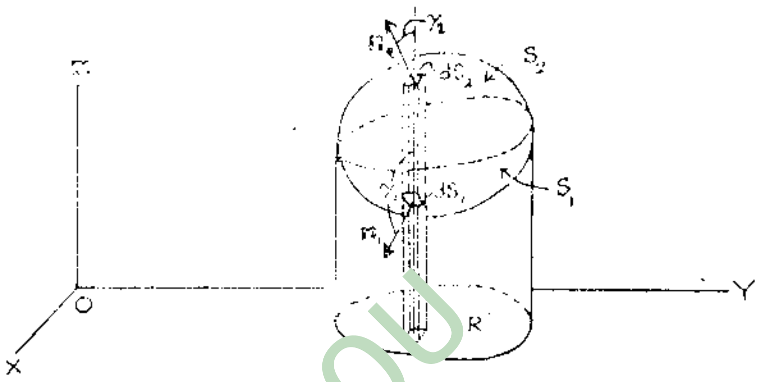

Fig. 6

Similarly, by projecting S on to the yz – plane and zx – planes, we get

$$\iiint_V \frac{\partial F_1}{\partial x} dV = \iint_S F_1\, \mathbf{i} \cdot \mathbf{n}\, dS \qquad (5)$$

$$\iiint_V \frac{\partial F_2}{\partial y} dV = \iint_S F_2\, \mathbf{j} \cdot \mathbf{n}\, dS \qquad (6)$$

Adding (4), (5) and (6) we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$$

$$= \iint_S \left(F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k}\right) \cdot \mathbf{n}\, dS$$

or $$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

The theorem can be extended to surfaces which are such that the lines parallel to coordinate axes meet them in more than two points.

Examples

Ex. 1 : Evaluate $\int_C \mathbf{f} \cdot d\mathbf{r}$ where

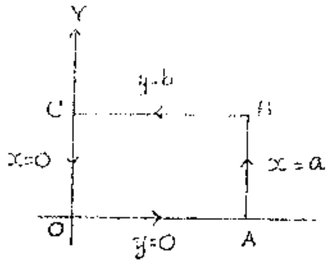

Fig. 7

$\mathbf{f} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$ and C is the rectangle bounded by $y = 0, x = a, y = b$ and $x = 0$.

Sol. : If $\mathbf{r}$ is the position vector of any point (in the xy plane in this problem)

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$$

$$\therefore \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

$$\therefore \int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \left\{ (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j} \right\} \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C \left[(x^2 + y^2)\,dx - 2xy\,dy \right]$$

The curve C consists of four straight lines OA, AB, BC and CO as shown in the figure.

$$\therefore \int_C \mathbf{f} \cdot d\mathbf{r} = \int_{OA} \mathbf{f} \cdot d\mathbf{r} + \int_{AB} \mathbf{f} \cdot d\mathbf{r}$$

$$+ \int_{BC} \mathbf{f} \cdot d\mathbf{r} + \int_{CO} \mathbf{f} \cdot d\mathbf{r} \qquad (1)$$

on OA, $y = 0$; $\therefore$ $dy = 0$ and x varies from 0 to a.

$$\therefore \int_{OA} \mathbf{f} \cdot d\mathbf{r} = \int_{x=0}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad (2)$$

on OB, $x = a$, $\therefore$ $dx = 0$ and y varies from 0 to b

$$\therefore \int_{OB} \mathbf{f} \cdot d\mathbf{r} = \int_{y=0}^{b} -2ay\,dy = \left[-ay^2\right]_0^b = -ab^2 \qquad (3)$$

on BC, $y = b$, $\therefore$ $dy = 0$ and x varies from a to 0

$$\therefore \int_{BC} \mathbf{f} \cdot d\mathbf{r} = \int_{x=a}^{0} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_a^0 = -\frac{a^3}{3} - a^2 b \qquad (4)$$

on CO, $x = 0$, $dx = 0$, y varies from b to 0

$$\int_{CO} \mathbf{f} \cdot d\,\mathbf{r} = \int_{y=b}^{0} \left\{ [0^2 + y^2]\, 0 - 2 \times 0 \,.\, y \right\} dy = 0 \qquad (5)$$

Substituting the values of (2), (3), (4) & (5) in (1), we get

$$\int_C \mathbf{f} \cdot d\,\mathbf{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2\,ab^2.$$

Ex. 2 : Evaluate $\int_S (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}) \,.\, d\mathbf{S}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Sol. : Obviously over this curved surface region the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is itself the normal vector. In fact

$$\mathbf{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} \;;\; \text{Here } \phi = (x^2 + y^2 + z^2 - 1)$$

$$\therefore\; \mathbf{n} = \frac{2\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\therefore\; \mathbf{f} \cdot \mathbf{n} = (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

$$= xyz + xyz + xyz = 3\,xyz.$$

$$\mathbf{k} \cdot \mathbf{n} = \mathbf{k} \cdot [x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}] = z.$$

$$\therefore\; \int \mathbf{f} \cdot \mathbf{n}\, dS = \iint_R \mathbf{f} \cdot \mathbf{n} \frac{dx\,dy}{\mathbf{n} \cdot \mathbf{k}}$$

$$= \iint \frac{3x\,yz\,dx\,dy}{z}$$

Over first quadrant

$$= 3 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy\; dx\; dy$$

$$= 3 \int_{x=0}^{1} x \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx.$$

$$= \frac{3}{2} \int_0^1 x\,(1 - x^2)\; dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

Ex. 3 : Verify Green's theorem in the plane for

$$\int_C (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy$$

around the boundary C of the region $y^2 = 8x$ and $x = 2$.

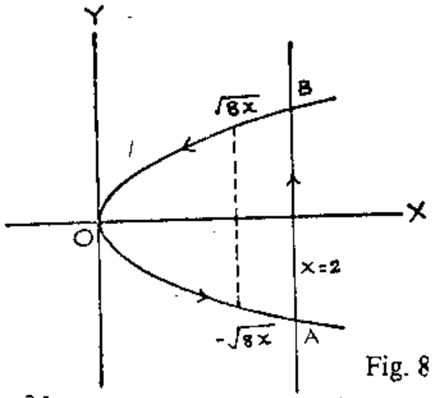

Fig. 8

Sol : By Green's Theorem

$$\int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$\text{L. H. S.} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{2} \int_{-\sqrt{8x}}^{\sqrt{8x}} \{2xy - [-2x]\}\,dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\,dx\,dy.$$

$$= \int_{x=0}^{2} [xy^2 + 2xy]_{y=-\sqrt{8x}}^{\sqrt{8x}}\,dx$$

$$= 4 \int_{x=0}^{2} x\sqrt{8x}\,dx = 4\sqrt{8} \int_{x=0}^{2} x\sqrt{x}\,dx.$$

$$= 4\sqrt{8} \int_{x=0}^{2} x^{3/2}\,dx = 8\sqrt{2}\left[\frac{x^{5/2}}{5/2}\right]_0^2$$

$$= \frac{16\sqrt{2}}{5}\,2^{5/2} = \frac{16}{5}\sqrt{2}\,.\,2^2\sqrt{2} = \frac{128}{5}.$$

We now evaluate the line integral

$$\int_C (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy,$$

where C consists of the line AB and the arc BOA.

$$\therefore \int_C (x^2 - 2\,xy)\,dx + (x^2\,y + 3)\,dy$$

$$= \int_{AB} [x^2 - 2\,xy]\,dx + (x^2\,y + 3)\,dy$$

$$+ \int_{BOA} [x^2 - 2\,xy]\,dx + (x^2\,y + 3)\,dy \qquad (1)$$

On AB, $x = 2$, $dx = 0$ and y varies from -4 to 4.

$$\int_{AB} = \int_{y=-4}^{4} [4 - 4y]\,0 + [4y + 3]\,dy$$

$$= \int_{y=-4}^{4} (4y + 3)\,dy$$

$$= \left[2y^2 + 3y\right]_{-4}^{4} = 24.$$

On BOA, $y^2 = 8x$

$$\therefore x = y^2/8\,;\; dx = \frac{2y\,dy}{8} = \frac{y\,dy}{4}.$$

$$\therefore \int_{BOA} = \int_{y=4}^{-4} \left[\frac{y^4}{64} - \frac{2y^2}{8}\cdot y\right]\frac{y\,dy}{4} + \left[\frac{y^4}{64}\cdot y + 3\right]dy$$

$$= \int_{y=4}^{-4} \left[\frac{5y^5}{4\times 64} - \frac{y^4}{16} + 3\right]dy$$

$$= \left[\frac{5y^6}{6\times 4\times 64} - \frac{y^5}{5\times 16} + 3y\right]_{4}^{-4}$$

$$= \frac{2\,[4]^5}{5\times 16} - 24 = \frac{128}{5} - 24 \qquad (2)$$

From (1) and (2)

$$\int_C (x^2 - 2xy)\,dx + (x^2y + 3)\,dy = \int_{AB} + \int_{BOA}$$

$$= 24 + \frac{128}{5} - 24 = \frac{128}{5};$$

Ex. 4 : Verify Stoke's theorem for vector field $\mathbf{f} = (2x - y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2z\,\mathbf{k}$ over the upper half surface $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy – plane.

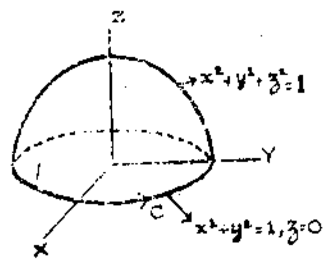

Fig. 9

Sol. : By Stokes theorem, we have

$$\text{Curl } \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= [-2yz + 2yz]\,\mathbf{i} + \mathbf{j}\,(0 - 0) + \mathbf{k}\,(0 + 1) = \mathbf{k}.$$

$$\int_S \text{Curl } \mathbf{f}\,.\,\mathbf{n}\,dS = \int_S \mathbf{k}\,.\,\mathbf{n}\,dS$$

$$= \iint_R \mathbf{k}\,.\,\mathbf{n}\,\frac{dx\,dy}{|\mathbf{k}\,.\,\mathbf{n}|}$$

(where R, the projection of S upon the xy plane, is the region enclosed by the circle C ; $x^2 + y^2 = 1$, $z = 0$)

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$$= \int_{x=-1}^{1} [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= 2\int_{x=-1}^{1} \sqrt{1-x^2}\,dx\,.$$

$$= \left[\sin^{-1} x + x\sqrt{1-x^2}\right]_{-1}^{1} = \pi.$$

We find $\displaystyle\int \mathbf{f}\,.\,d\mathbf{r} = \int_C [\,(2x-y)\,dx - yz^2\,dy - y^2 z\,dz\,]$

along the circle C. Since $z = 0$ on C

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C (2x - y)\, dx.$$

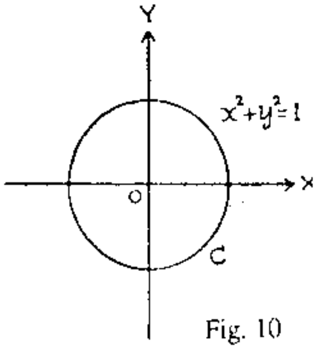

Fig. 10

Put $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$.

$$\int_C (2x - y)\, dx = \int_{\theta=0}^{2\pi} (2\cos - \sin\theta)(-\sin\theta)\, d\theta.$$

$$= \int_{\theta=0}^{2\pi} (\sin^2\theta - \sin 2\theta)\, d\theta.$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{1-\cos 2\theta}{2} - \sin 2\theta\right] d\theta$$

$$= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} + \frac{\cos 2\theta}{2}\right]_0^{2\pi} = \pi.$$

Thus $\displaystyle\int_C \mathbf{f} \cdot d\mathbf{r} = \pi = \int_S \text{curl } \mathbf{f} \cdot \mathbf{n}\, dS$

Ex. 5 : Verify divergence theorem for

$\mathbf{f} = (x^2 - yz)\mathbf{i} + (y^2 - zx)\,j + (z^2 - xy)\,k$ taken over the rectangular parallelopiped $0 \le x \le a;\ 0 \le y \le b;\ 0 \le z \le c.$

Sol: $\text{div } \mathbf{f} = \dfrac{\partial}{\partial x}(x^2 - yz)$

$$+ \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z$$

$$= 2(x + y + z)$$

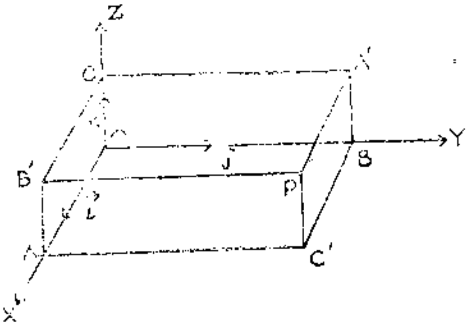

fig.11

$$\int_V \text{div} f\ dV \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} (x + y + z)\, dx\, dy\, dz$$

$$= 2\int_0^c dz \int_0^b dy \left[\frac{x^2}{2} + yx + zx\right]_0^a$$

$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \left(\frac{a^2}{2} + ay + az\right) dy\, dz$$

$$= 2\int_0^c \left[\frac{a^2y}{2} + \frac{ay^2}{2} + ayz\right]_0^b dz$$

$$= 2\int_0^c \left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right] dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc\,(a+b+c).$$

$$\int_S \mathbf{f}\cdot\mathbf{n}\,dS = \int_{S_1} \mathbf{f}\cdot\mathbf{n}\,dS + \int_{S_2} \mathbf{f}\cdot\mathbf{n}\,dS + \ldots + \int_{S_3} \mathbf{f}\cdot\mathbf{n}\,dS$$

Where $S_1, S_2, S_3, S_4, S_5, S_6$, are respectively the faces $OAC'B$, $CB'PA'$, $OBA'C$, $AC'PB'$ $OCB'A$, $BA'PC'$ of the parallelopiped.

On S_1, $z = 0$, $\mathbf{n} = -\mathbf{k}$, $0 \le x \le a$; $0 \le y \le b$.

$$\int_{S_1} \mathbf{f}\cdot\mathbf{n}\,dS = \int_{S_1} \left[x^2\mathbf{i} + y^2\mathbf{j} + (0 - xy)\,\mathbf{k}\right]\cdot(-\mathbf{k})\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} xy\,dx\,dy = \frac{a^2b^2}{4}$$

on S_2, $z = c$, $\mathbf{n} = \mathbf{k}$

$$\int_{S_2} \mathbf{f}\cdot\mathbf{n}\,dS = \int_{S_2} \left\{(x^2 - cy)\,\mathbf{i} + (y^2 - xc)\,\mathbf{j} + (c^2 - xy)\,\mathbf{k}\right\}\cdot\mathbf{k}\;dx\;dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} (c^2 - xy)\,dx\,dy = \int_{x=0}^{a}\left[c^2y - \frac{xy^2}{2}\right]_1^b dx$$

$$= \int_{x=0}^{a}\left[c^2b - \frac{xb^2}{2}\right] dx = \left[c^2b - \frac{xb^2}{2}\right]_0^a = abc^2 - \frac{a^2b^2}{4}$$

Similarly $\displaystyle\int_{S_3} \mathbf{f}\cdot\mathbf{n}\,dS = \frac{b^2c^2}{4}$, $\displaystyle\int_{S_4} \mathbf{f}\cdot\mathbf{n}\,dS = a^2bc - \frac{b^2c^2}{4}$

$$\int_{S_5} \mathbf{f} \cdot \mathbf{n}\, dS = \frac{c^2 a^2}{4}, \quad \int_{S_6} \mathbf{f} \cdot \mathbf{n}\, dS = ab^2 c - \frac{c^2 a^2}{4}.$$

$$\therefore \quad \int_{S} \mathbf{f} \cdot \mathbf{n}\, dS = abc\,(a + b + c).$$

Ex. 6 : Verify divergence theorem for $\mathbf{f} = 4\,xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ taken over the cube bounded by $x = 0$, $x = 1$; $y = 0$, $y = 1$; $z = 0$, $z = 1$.

Sol. : By divergence theorem we have

$$\int_{S} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{V} \operatorname{div} \mathbf{f}\, dV$$

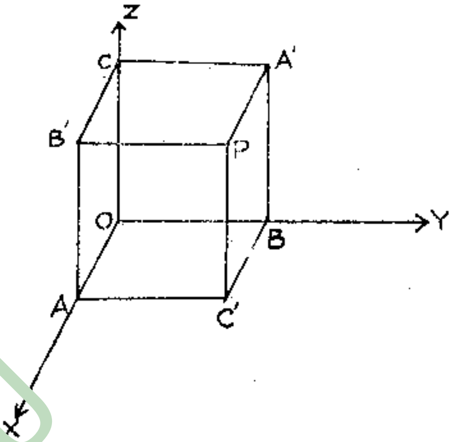

Fig . 12

To evaluate $\int_{S} \mathbf{f} \cdot \mathbf{n}\, dS$, note that S consists of six planes

$OAC'B$, $CB'PA$; $OBA'C$, $AC'PB'$, $OAB'C$, $BC'PA'$. Let them be denoted by $S_1, S_2, S_3, S_4, S_5,$ and S_6 respectively.

On S_1, $\mathbf{n} = -\mathbf{k}$ and $z = 0$, $0 \le x \le 1$, $0 \le y \le 1$

$$\therefore \quad \int_{S_1} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{S_1} (-y^2\,\mathbf{j}) \cdot (-\mathbf{k})\, dx\, dy = 0.$$

On S_2, $\mathbf{n} = \mathbf{k}$ $z = 1$, $0 \le x \le 1$, $0 \le y \le 1$

$$\int_{S_2} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{S_2} (4\,x\,\mathbf{i} - y^2\mathbf{j} + y\,\mathbf{k}) \cdot \mathbf{k}\;\; dx\; dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} y\, dx\, dy = \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}\,[x]_0^1 = \frac{1}{2}$$

On S_3, $\mathbf{n} = -\mathbf{i}$, $x = 0$, $0 \le y \le 1$; $0 \le z \le 1$.

$$\int_{S_3} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{S_3} (-y^2\mathbf{j} + yz\,\mathbf{k}) \cdot (-\mathbf{i})\, dS = 0$$

On S_4, $\mathbf{n} = \mathbf{i}$; $x = 1$, $0 \le y \le 1$; $0 \le z \le 1$.

$$\int_{S_4} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{S_4} \left[4z\,\mathbf{i} - y^2\mathbf{j} + yz\,\mathbf{k}\right] \cdot \mathbf{i}\, dS$$

$$= 4 \int_{y=0}^{1} \int_{x=0}^{1} z\, dy\, dz = 4 \cdot \frac{1}{2} = 2.$$

On S_5, $\mathbf{n} = -\mathbf{j}$, $y = 0$

$$\int_{S_5} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{S_5} [4xz\mathbf{i}] \cdot [-\mathbf{j}]\, dS = 0$$

On S_6, $\mathbf{n} = \mathbf{j}$, $y = 1$

$$\therefore \int_{S_6} \mathbf{f} \cdot \mathbf{n}\, dS = \int_{S_6} [4\, xz\, \mathbf{i} - \mathbf{j} + z\, \mathbf{k}] \cdot \mathbf{j}\, dS$$

$$= \int_{y=0}^{1} \int_{z=0}^{1} -\, dx\, dz = -1$$

$$\therefore \int_{S} \mathbf{f} \cdot \mathbf{n}\, dS = \frac{1}{2} + 2 - 1 = \frac{3}{2}.$$

To evaluate $\int_V \text{div}\, \mathbf{f}\, dV$

$\text{div}\, \mathbf{f} = 4z - 2y + y = 4z - y$

$$\int_V \text{div}\, \mathbf{f}\, dV = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} [4z - y]\, dx\, dy\, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \left[2z^2 - yz\right]_0^1 dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} (2 - y)\, dx\, dy$$

$$= \int_{x=0}^{1} \left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_{x=0}^{1} \left(2 - \frac{1}{2}\right) dx = \frac{3}{2}$$

Hence Verified.

5.10 Summary

We studied the integration of vector valued functions. If $\frac{d\,(\mathbf{f}\,(t))}{dt} = \mathbf{F}\,(t)$, then

$$\int \mathbf{F}\,(t)\,dt = \mathbf{f}\,(t) \text{ and for a definite integral } \int_a^b \mathbf{F}\,(t)\,dt = \mathbf{f}\,(b) - \mathbf{f}\,(a).$$

Greens theorem for a plane transforms a line integral over a closed (plane) curve into a double integral over a (plane) region enclosed by the curve.

Stokes theorem transforms a line integral over a closed curve in space into a surface integral over a surface enclosed by the curve.

Gauss divergence theorem transorms a surface integral into a volume integral.

5.11 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Explain vector integration.

b) Evaluate $\oint_C (3x + 4y)\,dx + (2x - 3y)\,dy$ where C, a circle of radius 2 with centre at the origin of the xy - plane is traversed in the positive sense.

(ii) a) How do you define surface and volume integrals.

b) State and prove the green's theorem in a plane.

(iii) a) State and prove Stokes theorem.

b) Verify Stoke's theorem for $\mathbf{F} = (y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}$ where S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above xy - plane.

(iv) a) State and prove the Gauss divergence theorem.

b) Verify the divergence theorem for $\mathbf{a} = 2x^2 y\,\mathbf{i} - y\,\mathbf{j} + 4xz^2\,\mathbf{k}$ taken over in the region in the first octant bounded by $x^2 + z^2 = 9$ and $x = 2$.

II. *Briefly answer the following questions.*

(i) Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy - y^2)\,dx + (3x^2 - 2xy)\,dy$ along the cycloid $x = \theta - \sin\theta$ $y = 1 - \cos\theta$.

(ii) Find the area bounded by an arc of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$, $a > 0$ and the x - axis.

(iii) Evaluate $\int\int_S \mathbf{F}.\,\mathbf{n}\,dS$, where $\mathbf{F} = 2xy\,\mathbf{i} + yz^2\,\mathbf{j} + xz\,\mathbf{k}$ and S is

a) the surface of the parallelopiped bounded by

$x = 0, y = 0, z = 0, x = 2, y = 1$ and $z = 3$.

b) The surface of the region bounded by $x = 0, y = 0, y = 3, z = 0$ and $x + 2z = 6$.

(iv) Evaluate $\int_S \mathbf{r} \cdot \mathbf{n}\, dS$ where s is the sphere of radius 2, with centre at (0, 0, 0).

(v) If **H** = Curl **A** prove that $\int\int_S (\mathbf{H} \cdot \mathbf{n})\, dS = 0$ for any closed surface s

(vi) Prove that $\int\int\int_V \frac{dV}{r^2} = \int\int_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^2}\, dS$

(vii) Verify Stoke's theorem for $\mathbf{F} = xz\,\mathbf{i} - y\,\mathbf{j} + x^2y\,\mathbf{k}$ where S is the surface of the region bounded by $x = 0, y = 0, z = 0, 2x + y + 2z = 8$ which is not included in the xz plane.

Answers

I. (ib) -8π, (iiib) Common value $= -4$ (ivb) 180.

II. (i) $6\pi^2 - 4\pi$, (ii) $3\pi a^2$, (iii) a) 30, b) $\frac{351}{2}$

(iv) 32π (vii) Common Value $= \frac{32}{3}$.

UNIT-6 : APPLICATIONS

6.1 Aims and Objectives

After going through this unit you will be able to apply the results of Vector analysis to various branches of physics and geometry.

6.2 Introduction

Units 3–5 mainly deal with the calculus of vectors. We shall, now, obtain some interesting and useful results which help us to have a clear insight into the calculus of vectors by the following applications to geometry and physics. Vector differentiation plays important role in differential geometry and mechanics while divergence theorem and Stoke's theorem have important applications in several branches of physics, namely 'hydrodynamics' electromagnetic theory etc.

We have already mentioned, in unit–3 that the derivative of a vector function of a real variable represents, geometrically the tangent at a point to a curve. We have also illustrated that the derivatives of vector functions represent velocity and acceleration in physics. We shall now discuss the applications of vector differentiation to geometry and mechanics.

6.3 Applications to Geometry : (Serret–Frenet Formulae for a Curve)

We know that a curve is represented by an equation of the form $\mathbf{r} = \mathbf{F}(t)$ where r is the position vector of a point on the curve. Associated with each point on a curve, there is a set of three mutually perpendicular lines called tangent, principal normal and binormal.

We have already seen (in unit–3) that when $Q \to P, \frac{d\mathbf{r}}{dt}$ is parallel to the tangent at P. If we take the arc length, s, as the parameter, we have

$$\left|\frac{d\mathbf{r}}{ds}\right| = \underset{Q\to P}{\text{Lt}} \frac{|\overrightarrow{PQ}|}{PQ} = \underset{Q\to P}{\text{Lt}} = \frac{PQ}{PQ} = 1.$$

$\therefore \frac{dr}{ds}$ is a unit vector. If we denote the unit vector along the tangent at P by $\mathbf{t}$, then

$$\frac{dr}{ds} = \mathbf{t}.$$

The vector $\mathbf{t}$ is called the unit tangent vector. Also, since $\mathbf{t} \equiv \frac{d\mathbf{r}}{ds}$ is a differentiable function and since $\mathbf{t}\cdot\mathbf{t} = 1$, we can differentiate w.r.t. s and get $\mathbf{t}\cdot\frac{d\mathbf{t}}{ds} = 0$.

$$\therefore \frac{d\mathbf{t}}{ds} \perp \mathbf{t}.$$

If $\mathbf{n}$ is a unit vector perpendicular to t in the direction of $\frac{d\,\mathbf{t}}{ds}$, we have that $\frac{d\,\mathbf{t}}{ds}$ and $\mathbf{n}$ are parallel. We now define the unit binormal $\mathbf{b}$ by the equation $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ such that the triad of vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is right handed. Since $\frac{d\,\mathbf{t}}{ds}$ and $\mathbf{n}$ are parallel, we have a real number $k > 0$ such that

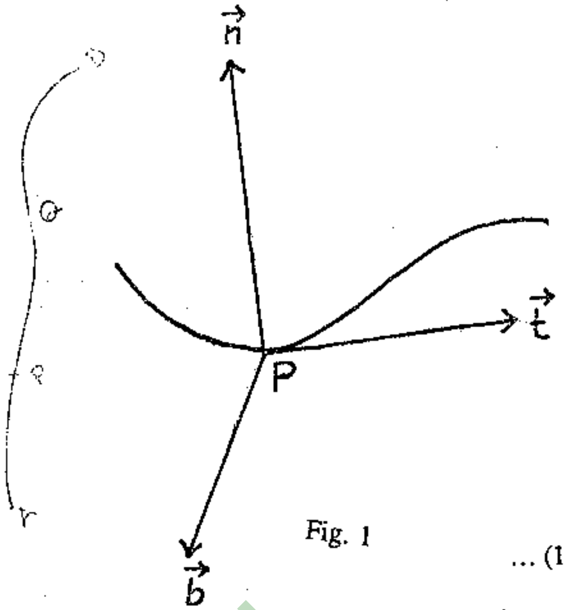

Fig. 1

$$\frac{d\,\mathbf{t}}{ds} = k\mathbf{n} \qquad \ldots (1)$$

where $\mathbf{n}$ is called the unit principal normal and k the curvature.

Also since $\mathbf{n}$ is a unit vector $\frac{d\,\mathbf{n}}{ds} \perp \mathbf{n}$. $\qquad \ldots (2)$

$\therefore \frac{d\,\mathbf{n}}{ds}$ lies in the plane of $\mathbf{t}$ and $\mathbf{b}$.

$\therefore$ For real numbers λ and τ, we have

$$\frac{d\,\mathbf{n}}{ds} = \lambda\,\mathbf{t} + \tau\mathbf{b} \qquad \ldots (3)$$

Differentiating $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ w.r.t. s, we get

$$\frac{d\,\mathbf{b}}{ds} = \frac{d\,\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\,\mathbf{n}}{ds}$$

$$= k\,\mathbf{n} \times \mathbf{n} + \mathbf{t} \times (\lambda t + \tau\mathbf{b})$$

$$= \tau\,(-n) \qquad \ldots (4)$$

Differentiating $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ w.r.t. s we get

$$\frac{d\,\mathbf{n}}{ds} = \frac{d\,\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\,\mathbf{t}}{ds}$$

$$= -\tau\,\mathbf{n} \times \mathbf{t} + \mathbf{b} \times k\,\mathbf{n}$$

$$= \tau\,\mathbf{b} - k\,\mathbf{t}. \qquad \ldots (5)$$

From (3) and (5) we have $\lambda = -k$.

Equations (1), (4) and (5) namely

$$\frac{d\,\mathbf{t}}{ds} = k\,\mathbf{n}$$

$$\frac{d\,\mathbf{n}}{ds} = \tau\mathbf{b} - kt$$

$$\frac{d\,\mathbf{b}}{ds} = \tau\mathbf{n}$$

are called Serret–Frenet Formulae for a curve in space. Here τ is known as torsion. (For further details the student is advised to see any book on vector calculus).

6.4 Applications to Mechanics

We shall obtain the tangential and normal components of velocity and accelerations, in the case of motion of a body in space, as our next application of vector differentiation. (In a similar way we can obtain the components of velocity and acceleraton when the body describes a plane curve). Let the moving axes in space be parallel to the unit tangent, unit principal normal and unit binormal **t, n, b** to the trajectory of the particle at the time.

$$\text{Then we know velocity } \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\frac{d\mathbf{r}}{ds} = \frac{ds}{dt}.$$

$$\text{and acceleration } \mathbf{a} = \frac{d\mathbf{v}}{ds} = \frac{d^2s}{dt^2}\mathbf{t} + \frac{ds}{dt}\cdot\frac{d\mathbf{t}}{dt}$$

$$= \frac{d^2s}{dt^2}t + \frac{ds}{dt}\cdot\frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \frac{d^2s}{dt^2}\mathbf{t} + \left(\frac{ds}{dt}\right)^2 k\,\mathbf{n} + 0\mathbf{b} \text{ [using (1)]}$$

Thus $\frac{d^2s}{dt^2}, k\left(\frac{ds}{dt}\right)^2, 0$ are the components of acceleration along **t, n, b.**

6.5 Physical Interpretations

If to each point of space a scalar function ϕ is associated, we say that a scalar field is given. If to each point of space a vector function **F** is assigned, then a vector field is said to be given.

Solenoidal Field

A vector field **F** is called a solenoidal field if div **F** vanishes.

Irrotational Field

A vector field **F** is an irrotational field or conservative field if curl **F** vanishes.

Physical Interpretation of Line Integral, Volume Integral and Surface Integral

If **F** represents a force acting on a particle moving along the curve C, then the line integral

$\int_C \mathbf{F}\cdot d\,\mathbf{r}$ represents the work done by the force. Also the line integral $\oint_C \mathbf{F}\cdot d\,\mathbf{r}$ of a continuous vector function **F** along a closed curve C is called the circulation **F** along C.

We have defined in Unit–4 the surface Integral of a vector function **F** over a closed surface S as

$$\int_S \mathbf{F}\cdot\mathbf{n}\,dS$$

This is physically interpreted as Flux of **F** acroos S.

Also a vector function **F** is said to be solenoidal in a region if its flux across every closed surface in the region is zero.

Physical Interpretation of Gauss Divergence Theorem in Hydrodynamics Application

Let the vector fuction **q** denote the velocity vector of an incompressible fluid (a fluid having constant density) of unit density and let S denote any closed surface drawn in the fluid. By Gauss divergence theorem

$$\int_S \mathbf{q} \cdot \mathbf{n}\, dS = \int_V \operatorname{div} \mathbf{q}\, dV \qquad \ldots (1)$$

where $\mathbf{q} \cdot \mathbf{n}\, dS$ denotes the amount of fluid that flows out in unit time through the elements dS. Hence the left hand side of (1) denotes the amount of fluid flowing across the surface S in unit time from the inside to the outside. This amount may be positve or negative. Now the total amount flowing outward must be continuously supplied so that in the inside region we must have sources producing fluid. div $\mathbf{q}$ denotes the amount of fluid per unit time per unit volume that goes through any point. Hence the right hand side of (1) denotes the amount of fluid per unit time supplied by the sources within S. Thus the equality (1) is evident.

6.6 Application to Fluid Dynamics

A fluid of density ρ moves with velocity $\mathbf{v}$ then

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \text{ where } \mathbf{J} = \rho \mathbf{v}.$$

Proof : Consider an arbitrary surfaces S enclosing a volume V of the fluid. At any time the mass of the fluid within V is

$$M = \iiint_V \rho\, dV$$

The time rate of increase of this mass is $\frac{\partial M}{\partial t}$

$$= \frac{\partial}{\partial t} \iiint_V \rho\, dV = \iiint_V \frac{\partial \rho}{\partial t}\, dV$$

The mass of fluid per unit volume leaving V (i.e., the flux) is $\iint_S \rho \mathbf{v} \cdot \mathbf{n}\, dS$

and the time rate of increase in mass is therefore

$$-\iint_S \rho \mathbf{v} \cdot \mathbf{n}\, dS = -\iiint_V \nabla \cdot (\rho \mathbf{v})\, dV$$

by divergence theorem. Then

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \nabla \cdot (\rho \mathbf{v})\, dV$$

i.e., $\iiint_V \left[\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} \right] dV = 0$

since V is arbitrary, $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$ where $\mathbf{J} = \rho \mathbf{v}$.

This equation is called continuity equation.

We can obtain equation of continuity in electromagnetic theory in a similar way.

6.7 Application in Electostatics

6.7.1 Application of Gauss divergence Theorem.

Gauss divergence theorem is applied in deriving the Laplace's equation and Poisson equation in electrostatics.

Coulomb's law in electrostatics states that if two electric charges e_1, e_2 are placed in vacuum at a distance r apart, the force between them is proportional to $\frac{e_1 e_2}{r^2}$.

i.e., $\mathbf{f} = c \frac{e_1 e_2}{r^2} \frac{\mathbf{r}}{r} = c \frac{e_1 e_2}{r^3} \mathbf{r}$, c is a constant or this can be shown to be

$$= \frac{e_1 e_2}{r^3} \mathbf{r}.$$

Also if we define electrostatic potential ϕ as

$$\phi = \sum_{i=1}^{n} \frac{e_i}{r_i}$$

$$\text{then } \nabla \phi = \sum_{i=1}^{n} \frac{e_i}{r_i^3} \mathbf{r}_i = \mathbf{E},$$

where **E** is electrostatic field intensity. Gauss theorem in electrostatics is that the flux of electrostatic intensity **E** out of any closed surface S is given by

$$\int_S \mathbf{E} \cdot d\,\mathbf{S} = 4\pi \int_V \rho \, dV$$

where ρ is charge density.

Now applying divergence theorem

$$\int_S \mathbf{E} \cdot d\,\mathbf{S} = \int_V \operatorname{div} \mathbf{E} \, dV = 4\pi \int_V \rho \, dV$$

$$\text{i.e., } \int_V (\operatorname{div} \mathbf{E} - 4\pi\rho) \, dV = 0$$

Since this is applicable to all volumes, however small,

$\operatorname{div} \mathbf{E} - 4\pi\rho = 0$ i.e., $\operatorname{div} \mathbf{E} = 4\pi\rho = \nabla \cdot \mathbf{E}$

or $\nabla \cdot (\nabla\phi) = -4\pi\rho$ or $\nabla^2\phi = -4\pi\rho$ which is Poisson's equation. If $\rho = 0$, $\nabla^2\phi = 0$ which is Laplace's equation. Hence we have illustrated the application of divergence theorem.

6.7.2 Application of Stoke's Theorem

The following simple physical situation illustrates the application of Stoke's Theorem in Hydrodynamics.

If C be any closed curve drawn in a fluid at time t, then we know that $\int_C \mathbf{q} \cdot d\,\mathbf{r}$ is called the circulation round the curve ($\mathbf{q}$ is the velocity of the fluid). Now we will prove the following result using Stokes Theorem.

If the motion in a simply connected region is irrotational (i.e., curl $\mathbf{q} = \mathbf{0}$) at any instant, then the circulation around any closed curve at the instant is also zero. For, we have by Stoke's theorem, circulation

$$\Gamma = \int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{q} \cdot \mathbf{n}\, dS$$

where S denotes a surface enclosed by C. The motion being irrotational at the instant, curl $\mathbf{q} = 0$ at that time at every point of the fluid. Hence $\Gamma = 0$.

Thus the above physical examples are enough to illustrate the physical applications and interpretations of Gauss divergence theorem and Stokes theorem. The importance of these theorems will be more clear if we see the following illutrative examples.

Examples

Ex. 1: Show that the kinetic energy of a body of homogeneous liquid within a closed surface S moving irrotationally is

$$\frac{1}{2}\rho \int_S \phi\, \nabla\phi \cdot \mathbf{n}\, dS \quad \phi \text{ being the velocity potential.}$$

Sol: We have by Gauss divergence theorem,

$$\frac{1}{2}\rho \int_S \phi\, \nabla\phi \cdot \mathbf{n}\, dS = \frac{1}{2}\rho \int_V \nabla . (\phi\nabla\phi)\, dV$$

$$= \frac{1}{2}\rho \int_V [\nabla\phi \cdot \nabla\phi + \phi\,(\nabla \cdot \nabla\phi)]\, dV$$

$$= \frac{1}{2}\rho \int_V (\nabla\phi)^2\, dV + \frac{1}{2}\rho \int_{-V} \phi\nabla^2\phi\, dV$$

$$= \int_V \frac{1}{2}\rho\, \mathbf{q}^2\, dV, \text{ for } \nabla^2\phi = 0, -\nabla\phi = \mathbf{q}$$

$$= \text{Kinetic energy}$$

Ex. 2: Show that the necessary and sufficient condition for a vector function $\mathbf{F}$ to be irrotational in a simply connected region is that curl $\mathbf{F} = \mathbf{0}$ at every point of the region.

Sol.: Let $\mathbf{F}$ be irrotational, then $\mathbf{F} = \nabla\phi$, where ϕ is a scalar (we assume this result, the proof left to the reader).

$$\therefore \text{Curl } \mathbf{F} = \text{Curl } (\nabla\phi) = \nabla \times \nabla\phi = \mathbf{0}.$$

Again, let curl $\mathbf{F} = \mathbf{0}$, we have by Stoke's theorem

$$\int_C \mathbf{F} \cdot d\,\mathbf{r} = \int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = 0.$$

Hence $\mathbf{F}$ is irrotational.

6.8 Summary

We have seen various applications of vector analysis. In geometry, if 's' denote the length of an arc of the curve from a fixed point then the derivatives of the unit triad of vectors along the tangent, normal and binormal with respect to 's' denote the Serret–Frenet formula.

$\int_C \mathbf{F} \cdot \mathbf{dr}$ represents the work done by the force $\mathbf{F}$ acting on a particle moving along the curve C.

If $\mathbf{F}$ is a continuous vector function then $\int_C \mathbf{F} \cdot d\,\mathbf{r}$ along a closed curve C is called the circulation of $\mathbf{F}$ along C.

Gauss divergence theorem is applied in deriving Laplaces and Poisson's equation in electrostatics.

Application of Stoke's theorem is known in fluid dynamics. .

6.9 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Obtain Serret-Frenet formulae.

b) Find the tangential and normal accelerations of a particle moving in a plane curve.

(ii) a) If ρ is the density and v is the velocityof a fluid, show that $\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0$.

b) Show that the vector function $\mathbf{F} = \left(x^2 - yz\right)\mathbf{i} + \left(y^2 - zx\right)\mathbf{j} + \left(z^2 - xy\right)\mathbf{k}$ is irrotational.

II. *Briefly answer the following questions.*

i) A particle moves along the curve $\mathbf{r} = \left(t^3 - 4t\right)\mathbf{i} + \left(t^2 + 4t\right)\mathbf{j} + \left(8t^2 - 3t^3\right)\mathbf{k}$, where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time $t = 2$.

ii) If $\mathbf{F}$ is a solenoidal field, show that $\int_V \mathbf{F} \cdot \mathbf{r}\, dV = \frac{1}{2} \oint_S r^2\, \mathbf{F} \cdot dS$.

iii) Prove, by Stokes theorem that Curl (grad ϕ) = 0.

iv) Find the condition that $\mathbf{q} = (\alpha x + \beta y)\,\mathbf{i} + (8x + 8y)\,\mathbf{j}$ may be velocity of incompressible fluid.

v) A fluid motion is given by $\mathbf{v} = (y + z)\,\mathbf{i} + (z + x)\,\mathbf{j} + (x + y)\,\mathbf{k}$. Is the motion irrotational?

iv) Show that for an incompressible fluid moving irrotationally, div grad $\phi = 0$, where ϕ is the velocity potential.

Answers

I. (ib) $\frac{dr}{dt}, r\frac{d\theta}{dt}, \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

II. (i) $16, 2\sqrt{73}$ (iv) $\alpha = -8$.

 Author of the Units 1–6: Dr.D. Radha Krishna Reddy

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Ed : P.S. Rao

BLOCK-2 : THREE DIMENSIONAL GEOMETRY; PLANE AND STRAIGHT LINE

Introduction

Analytical geometry provides a general method for solving geometric problems (by the method of co-ordinates). The rapid growth of science and technology in the seventeenth and eighteenth centuries necessitated the development of this branch of Mathematics. The foundations of analytical geometry were provided by the famous French philosopher and mathematician Rene' Descartes (1596 - 1650). We have Euclidian geometry which is mostly qualitative in nature, but does not go deep into the quantitative nature of complicated figures like ellipse, helix, hyperbola etc., To deal with such figures a qualitative analysis is needed. Coordinate geometry helps us in calculating quantitative aspects of such figures, e.g., locating foci of conicoids, axis of conics and conicolds etc. The Russian mathematician of genius; N.I. Lobachevsky (1792 - 1856) brought about a genuine revolution in geometry. His ideas were so new that his contemporaries did not appreciate them and only comparitively recently was his theory generally recognised and developed further. The analytical geometry that is being presented here has no bearing with this new geometry.

The two Blocks 2 and 3 deal with plane and straight line and the sphere respectively. The units of these Blocks, also briefly introduced unit wise, aim at providing a fairly complete exposition of the properties of the plane, the straight line and the sphere.

UNIT-7 : DIRECTION COSINES AND RATIOS OF A LINE

7.0 Contents

7.1 Aims and Objectives

After going through unit you should be able to :

i) identify the direction cosines and direction ratios of a line,

ii) find the angle between two lines in terms of their direction cosines

iii) Obtain the projection of a point and also a line on a given line.

7.2 Introduction

In plane the position of a point is determined by two numbers x, y which are signed distances from two mutually perpendicular straight lines called X, Y axes. The position of a point in space is, however, determined by the theree numbers x, y, z. The distance between two points, division of the join of two points, direction cosines of a line, angle between two lines in terms of their directions cosines are also dealt with.

7.3 Rectangular Cartesian Co-ordinates

Let $X'OX$, $Y'OY$ be two perpendicular straight lines. Through O, their point of intersection call 1 the origin, draw a line $Z'OZ$ perpendicular to the XOY plane; so that we have three mutually perpendicular straight lines $X'OX$, $Y'OY$, $Z'OZ$ or simply $X'X$, $Y'Y$, $Z'Z$ known as rectangular co-ordinate axes (Fig. 1).

The Plane XOY containing the lines $X'OX$ and $Y'OY$ may be imagined as the plane of the paper; the line OZ pointing above and perpendicular to it and OZ' below it. Note that viewed from Z the rotation from OX to OY appears anti clock-wise. Such a system of lines OX, OY, OZ is called right-handed. A left-handed system is obtained by inter- changing Z and Z'. The positive directions of the axes are indicated by arrow heads. These three axes, taken in pairs, detemine three planes XOY, YOZ and ZOX or briefly XY, YZ, ZX planes mutually at right angles;

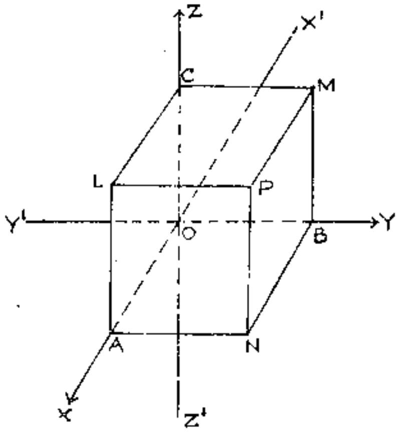


Fig . 1

known as *retangular co-ordinate planes*. Through any point P in space draw three planes parallel to YZ, ZX, XY planes to meet the axes in A, B, C (Fig. 1). If $OA = x$, $OB = y$, $OC = z$, then x, y, z are called the rectangular cartesian co-ordinates of P and to denote this fact we write $P(x, y, z)$.

The co-ordinate planes divide space into eight parts called octants and the signs of the co-ordinates of a point determine the octant in which it lies. Table, given below shows the signs for the eight octants.

Octant	$OXYZ$	$OX'YZ$	$OX'Y'Z$	$OXY'Z$	$OXYZ'$	$OX'YZ'$	$OX'Y'Z'$	$OXY'Z'$
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

In vector algebra we speak of **i**, **j**, **k** as unit vectors chosen along the directions OX, OY, OZ of a right-handed system respectively. This notation is used throughout.

Cor : Distance of $P(x, y, z)$ from the origin is given by $\sqrt{x^2 + y^2 + z^2}$.

From fig. 1, $OP^2 = ON^2 + NP^2$. But $ON^2 = OA^2 + AN^2$.

Since $AN = OB$, $NP = OC$, we have $OP^2 = OA^2 + OB^2 + OC^2 = x^2 + y^2 + z^2$ or

$$OP = \sqrt{x^2 + y^2 + z^2}.$$

This is also obtained by taking the magnitude of the vector $\vec{OP}$ i.e. $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

7.4 Change of Origin

Let $X'OX$, $Y'OY$, $Z'OZ$, $\alpha'\,\omega\,\alpha$, $\beta'\,\omega\,\beta$, $\gamma\,\omega\,\gamma$ be two sets of parallel axes (fig. 2). Let $P(x, y, z)$ be referred to the first and $P(\zeta, \eta, \tau)$ referred to the second set. Let $\omega(a, b, c)$ be referred to the first set.

NM is the line of intersection of the planes $\beta\,\omega\,\gamma$, XOY and the plane through P parallel to $\beta\,\omega\,\gamma$ cuts $\alpha\,\omega\,\beta$ in GH and XOY in KL as in fig. 2.

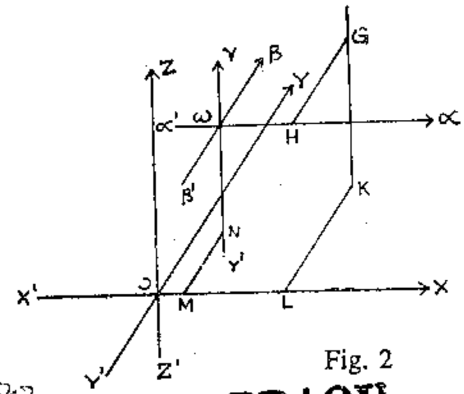


Fig. 2

Then $OL = OM + ML = OM + \omega H$. Therefore $x = a + \xi$. Similarly, $y = b + \eta, z = c + \tau$; whence

$$\xi = x - a, \eta = y - b, \tau = z - c \qquad \text{... (1)}$$

The reader may verify the truth of the result by taking P in any one of the octants.

This can also be seen using vectors. We know by vector addition $\vec{\omega P} = \vec{OP} - \vec{O\omega}$. In this we substitute $\vec{OP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; $\vec{\omega P} = \xi\,\mathbf{i} + \eta\,\mathbf{j} + \tau\,\mathbf{k}$ and $\vec{OA} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$. Now equating the coefficients of **i**, **j**, **k** on both the sides we get the result (1).

Cor : Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

If we change the origin to P, then the co-ordinates of Q become $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Now , by corollary of 7.3., we get the distance PQ as indicated. This can also be obtained by taking the magnitude of the vector $\vec{PQ}$ i.e., $\vec{OQ} - \vec{OP}$ or $(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$.

7.5 Section formulae

In order to find the co-ordinates of a point $R(x, y, z)$ dividing the join of two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in a given ratio $\lambda : 1$, draw planes through P, Q, R parallel to the plane YZ to meet OX in P', Q', R' (fig. 3).

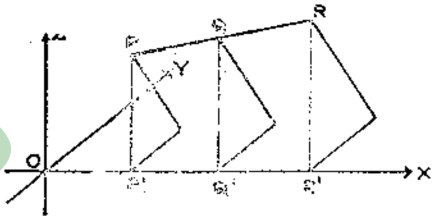


Fig. 3

Since, from elementary solid geometry, three parallel planes divide any two straight lines proportionally, we have

$$\frac{P'R'}{P'Q'} = \frac{PR}{PQ} = \frac{\lambda}{\lambda + 1}.$$

Therefore, $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{\lambda}{\lambda + 1}$. Solving we get $x = \dfrac{\lambda x_2 + x_1}{\lambda + 1}$.

Similarly, $y = \dfrac{\lambda y_2 + y_1}{\lambda + 1}$, $z = \dfrac{\lambda z_2 + z_1}{\lambda + 1}$.

These are the co-ordinates of R for all real values of λ except when $\lambda = -1$. If λ is positive, R lies between P and Q, if negative, R is on the same side of both P and Q.

Speaking in terms of vectors, $\vec{PR} = \lambda \vec{RQ}$ or $\vec{OR} - \vec{OP} = \lambda\left(\vec{OQ} - \vec{OR}\right)$.

Hence $OR = \dfrac{\lambda \vec{OQ} + \vec{OP}}{\lambda + 1}$. Substituting in this

$\vec{OR} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\vec{OP} = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$, $\vec{OQ} = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$ and equating the components of like vectors on both the sides, we get the co-ordinates of R.

Cor. : The mid-point of PQ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

This follows by taking $\lambda = 1$.

7.6 Angle between two non-intersecting (Skew) lines

Def. : This is defined as the angle θ $(0 \leq \theta \leq \pi)$ between two intersecting lines drawn from any point parallel to each of the given lines (Fig. 4).

Thus the angle between two non-interesecting lines AB and CD is equal to angle POQ between their parallels through O.

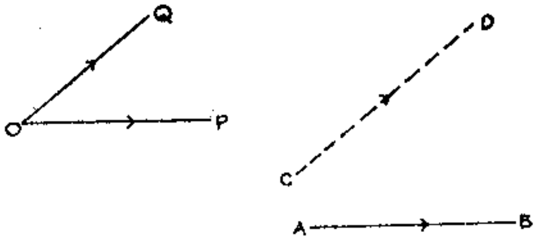


Fig. 4

7.7 Direction Cosines of a Line

Def. : If α, β, γ be the angles which any directed line, say AB of fig. 5 makes with the positive directions of the X, Y, Z axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the given line. They are usually denoted by l, m, n.

Obviously $-l, -m, -n$ are the direction cosines of the drected line BA.

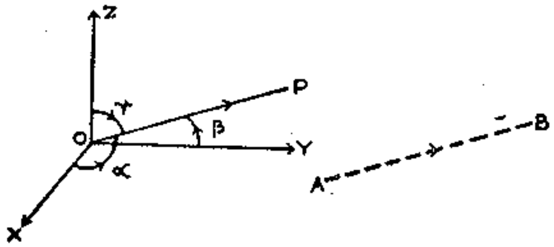


Fig. 5

(i) If O be the origin and (x, y, z) the co-ordinates of a point P and l, m, n are the direction cosines of OP and OP has length r, then

(a) $x = lr, \; y = mr, \; z = nr$;

(b) $l^2 + m^2 + n^2 = 1$.

Proof : (a) Through P draw PL perpendicular to Y – axis (Fig. 6). From the right - angled triangle OLP, we have $OL = OP \cos\beta$ i.e., $y = mr$. Similarly $x = lr, z = nr$.

(b) Since $r^2 = x^2 + y^2 + z^2 = r^2 \; (l^2+m^2+n^2)$

We get $l^2 + m^2 + n^2 = 1$.

We can also establish (a) and (b) using "scalar product" of vectors.

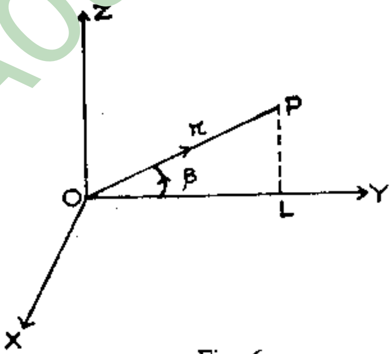


Fig. 6

(a) Since $\overrightarrow{OP} = \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, we get $\mathbf{r}\,.\,\mathbf{i} = x, \mathbf{r}\,.\,\mathbf{j} = y, \mathbf{r}\,.\,\mathbf{k} = z$. But $\mathbf{r}\,.\,\mathbf{i} = r \cos\alpha = lr$; $\mathbf{r}\,.\,\mathbf{j} = r\cos\beta = mr, \mathbf{r}\,.\,\mathbf{k} = r\cos\gamma = nr$. Hence $x = lr, \; y = mr, \; z = nr$.

(b) Using these relations, we get $\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$ which is a unit vector, Hence $l^2 + m^2 + n^2 = 1$.

(ii) *Direction Ratios* : If a, b, c be three numbers proportional to the direction cosines l, m, n of a line, we have

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore $l = \pm \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$,

$$m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the same sign, positive or negative is to be taken throughout.

Def. : Three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line.

7.8 Projection on a straight line

(i) Projection of a point

Let P be the foot of the perpendicular (fig. 7) from a given pioint A on a given line BC. Then P is called the orthogonal projection (or simply projection) of the point A on the line BC. This is also given by the point of interesction of the plane through the given point, perpendicular to the given line.

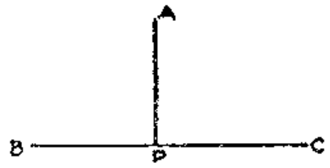


Fig. 7

(ii) Projection of a segment of a line

The projection of a directed segment AB of a line on any line CD is the directed segment $A'B'$, of CD where A', B' are the projections of A, B respectively on the line CD. Clearly A' and B' are the points of intersection of line CD and the planes Π_1, Π_2 respectively through A, B perpendicular to CD.

Theorem : The projection of a given segment AB of a line on any line CD is $AB \cos\theta$ where θ is the angle between AB and CD.

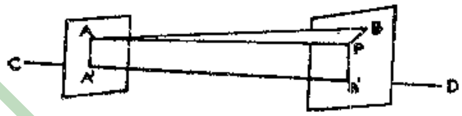


Fig. 8

Proof : Through A draw a line AP parallel to CD to meet the plane through B perpendicular to CD at P (fig. 8). Since AP is parallel to CD, angle $PAB = \theta$. Since CD is parpendicular to the plane Π_2. AP which is parallel to CD is also perpendicular to Π_2. Therefore angle APB is 90°. Clearly $AA'B'P$ is a rectangle so that we have $AP = A'B'$. Since $AP = AB \cos\theta$, we have $A'B' = AB \cos\theta$.

Cor. : The direction cosines of a line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

Let L, M be the feet of the perpendiculars (fig. 9) drawn from P, Q to the Y – axis respectively so that $OL = y_1$, $OM = y_2$. Projection of PQ on y–axis $= LM = OM - OL = y_2 - y_1$. If l, m, n be the direction cosines of the directed line segment, then its projection on Y – axis $= PQ.\,m$. Therefore $m\ PQ = y_2 - y_1$. Similarly $l\,.\,PQ = x_2 - x_1$, $n.PQ = z_2 - z_1$.

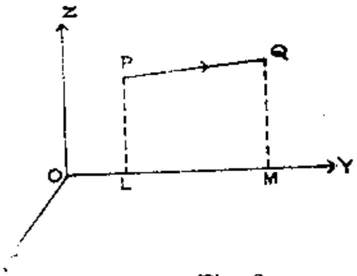


Fig. 9

Hence $\dfrac{x_2 - x_1}{l} = \dfrac{y_2 - y_1}{m} = \dfrac{z_2 - z_1}{n} = PQ$, which is the required result.

The above relation also follows, using vectors.

Thus, $\vec{PQ} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$

$= (l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k})\,PQ.$

7.8.1 Projection of a broken line (consisting of continuous segments)

If P_1, P_2, P_3 be any three points in space, then the sum of the projections of directed segments $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$ on any line AB is equal to the projection of directed segment $\overrightarrow{P_1P_3}$ on the same line.

Let Q_1, Q_2, Q_3 be the projections of the points P_1, P_2, P_3 on the line AB, then $\overrightarrow{Q_1Q_2}, \overrightarrow{Q_2Q_3}$, $\overrightarrow{Q_3Q_1}$ are the projections of $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$ and $\overrightarrow{P_3P_1}$ respectively. From figs. (i) and (ii) of 10, it is evident that

$\overrightarrow{Q_1Q_2} + \overrightarrow{Q_2Q_3} = \overrightarrow{Q_1Q_3}$. Hence the result.

Note that we are dealing with vectors when we speak of directed line segments.

This result can evidently be extended to any number of points as follows.

If $P_1, P_2 \ldots P_n$ be any number of points in space, then the sum of the projections of $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$, ... $\overrightarrow{P_{n-1}P_n}$ on any line AB is equal to the projection of $\overrightarrow{P_1P_n}$ on the same line.

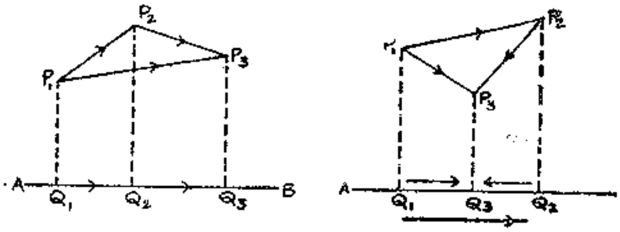


Fig. 10

Cor. : The projection of the join of two points $P\ (x_1, y_1, z_1)$ and $Q\ (x_2, y_2, z_2)$ on a line whose direction cosines are l, m, n is

$$l\ (x_2 - x_1) + m\ (y_2 - y_1) + n\ (z_2 - z_1).$$

Through P and Q draw planes parallel to the co-ordinate planes, forming a parallelopiped whose diagonal is PQ (fig. 11).

Since the edges PL, LT, TQ are parallel to OX, OY, OZ respectively we have

$PL = x_2 - x_1$, $LT = y_2 - y_1$, $TQ = z_2 - z_1$

The projections of PL, LT, TQ on the line having direction cosines l, m, n are $(x_2 - x_1)\, l$, $(y_2 - y_1)\, m$, $(z_2 - z_1)\, n$. But projection of PQ on the given line = sum of the projections of PL, LT, TQ on that line i.e., $l\ (x_2 - x_1) + m\ (y_2 - y_1) + n\ (z_2 - z_1)$.

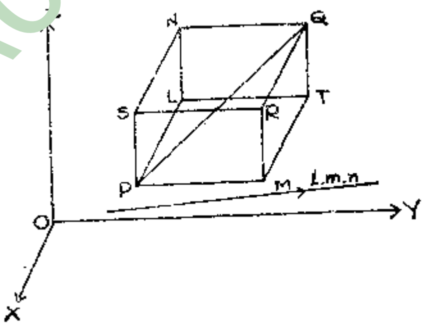


Fig. 11

In terms of vectors, the projection of PQ on a line having l, m, n as its direction cosines is $\overrightarrow{PQ} \cdot \hat{n}$ where $\hat{n} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$. But $\overrightarrow{PQ} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$.
Therefore the projection of PQ on the given line $= l\ (x_2 - x_1) + m\ (y_2 - y_1) + n\ (z_2 - z_1)$.

7.9 Angle between two lines in terms of their direction cosines

If OP and OQ have direction cosines l, m, n ; and l', m', n' respectively and θ is the angle that OP makes with OQ, then $\cos\theta = ll' + mm' + nn'$.

Let P have the co-ordinates (x, y, z) and $OP = r$ (fig. 12). The projection of OP on OQ = the sum of the projections of OM, MN, NP on OQ i. e., $r \cos\theta = xl' + ym' + zn'$. Since $x = lr$, $y = mr$, $z = nr$, we get $\cos\theta = ll' + mm' + nn'$.

The reader is advised to verify that the result is true in whatever octants the points P and Q may lie.

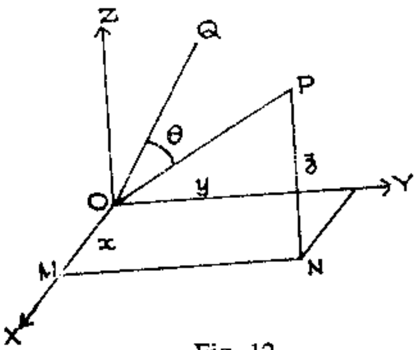


Fig. 12

Cor. 1 : We have the famous identity due to Lagrange :

$$(l^2 + m^2 + n^2)\,(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$

$$= (mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2.$$

Since $l^2 + m^2 + n^2 = 1,\ l'^2 + m'^2 + n'^2 = 1,$ we write

$$\sin^2\theta = 1 - \cos^2\theta = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$

$$= (mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2.$$

Cor. 2 : If θ is the angle between two lines whose direction ratios are $a, b, c\,;\ a', b', c'$, then

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{a'^2 + b'^2 + c'^2}},$$

$$\sin\theta = \pm\frac{\sqrt{(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{a'^2 + b'^2 + c'^2}}$$

Cor. 3 : If the lines are at right angles, then $\theta = \pi/2$. Hence a necessary and sufficient condition for the perpendicularity of the lines is

$$ll' + mm' + nn' = 0 \text{ or } aa' + bb' + cc' = 0.$$

Cor. 4 : If the lines are parallel, then either $\theta = 0$ or π. Hence from the expression of $\sin\theta$, we get a necessary and sufficient condition for the parallelism of two lines :

$$lm' - ml' = mn' - nm' = nl' - ln' = 0$$

$$\text{Or } \frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'},\ \text{Or } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

In terms of the vectors $\overset{\wedge}{\overrightarrow{OP}} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k},\ \overset{\wedge}{\overrightarrow{OQ}} = l'\hat{\mathbf{i}} + m'\hat{\mathbf{j}} + n'\hat{\mathbf{k}}$. Taking their scalar product and noting that both of them are unit vectors we obtain $\cos\theta = ll' + mm' + nn'$, from which all the corollaries 1 – 4 follow.

Examples

Ex. 1 : If $A\ (4, 3, 2),\ B\ (5, 4, 6),\ C\ (-1, -1, 5)$ are the corners of a triangle ; find the co-ordinates of the point in which the bisector of the angle 'A' meets the side BC.

Sol. : Let AD be the bisector of angle A (fig. 13) so that

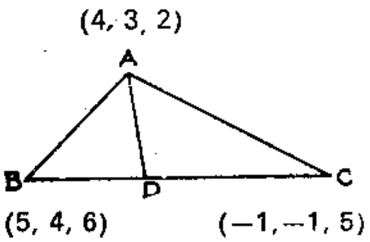


Fig. 13

$$\frac{BD}{DC} = \frac{AB}{AC}$$

But $AB = \sqrt{(5-4)^2 + (4-3)^2 + (6-2)^2} = 3\sqrt{2}$

and $AC = \sqrt{(-1-4)^2 + (-1-3)^2 + (5-2)^2} = 5\sqrt{2}$

Therefore $BD : DC :: 3 : 3$

The co-ordinates of D are

$$\left[\frac{3\times(-1) + 5\times 5}{3+5},\ \frac{3\times(-1) + 5\times 4}{3+5},\ \frac{3\times 5 + 5\times 6}{3+5}\right]$$

i. e., $\left(\frac{22}{8}, \frac{17}{8}, \frac{45}{8}\right)$

Ex. 2 : Find the co-ordinates of the foot of the perpendicular from A (1, 1, 1) on the line joining B (1, 4, 6) and C (5, 4, 4).

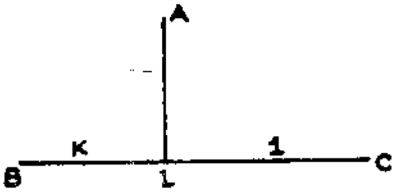


Fig. 14

Sol. : Let L, the foot of the perpendicular from A on BC (fig. 14) divide it in the ratio $K : 1$ so that

$$L\left[\frac{5K+1}{K+1}, \frac{4K+4}{K+1}, \frac{4K+6}{K+1}\right].$$

The direction ratios of AL are $\left[\frac{5K+1}{K+1}-1, \frac{4K+4}{K+1}-1, \frac{4K+6}{K+1}-1\right]$ and those of BC are (4, 0, –2). Since AL is $\perp r$ to BC, we require

$$\left(\frac{5K+1}{K+1}-1\right)\times 4+\left(\frac{4K+4}{K+1}-1\right)\times 0+\left(\frac{4K+6}{K+1}-1\right)\times(-2)=0.$$

which gives $K = 1$. Thus $L \equiv (3, 4, 5)$.

Ex. 3 : Find the direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to (1, –2, –2), (0, 2, 1).

Sol. : If l, m, n be direction cosines of the line perpendicular to the given lines, we have

$l \,.\, 1 + m \,.\, (-2) + n \,.\, (-2) = 0$, i. e., $l - 2m - 2n = 0$

$l \,.\, 0 + m.\ (2) + n.\ (1) = 0$, i. e., $0 \,.\, l + 2 \,.\, m + n = 0$,

These give

$\frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$ i. e., the direction ratios are (2, –1, 2).

Hence from 7.7 (ii),

$$l = \frac{2}{\sqrt{(2)^2+(-1)^2+(2)^2}} = \frac{2}{3},\ m = -\frac{1}{3},\ n = \frac{2}{3}.$$

Ex. 4 : A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube; Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Sol. : Take O a corner of the cube as the origin and the three edges OA, OB, OC through it as its axes.

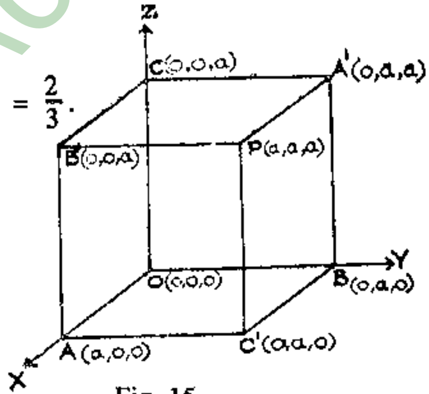


Fig. 15

Let $OA = OB = OC = a$. Then the co-ordinates of the corners are as shown in fig. 15. The four diagonals are OP, AA', BB', CC'. Clearly the direction cosines of OP are $\frac{a-0}{\sqrt{3a^2}}, \frac{a-0}{\sqrt{3a^2}}, \frac{a-0}{\sqrt{3a^2}}$ i. e., $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. Similarly direction cosines of AA' are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; those of BB' are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; and of CC' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OP, AA', BB', CC' respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n), \quad \cos\beta = \frac{1}{\sqrt{3}}(-l+m+n),$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l-m+n), \quad \cos\delta = \frac{1}{\sqrt{3}}(l+m-n).$$

Squaring and adding, we get

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{1}{3} [(l + m + n)^2 + (-l + m + n)^2 + (l - m + n)^2 + (l + m - n)^2] = \frac{4}{3} (l^2 + m^2 + n^2) = \frac{4}{3}$. Since $l^2 + m^2 + n^2 = 1$.

Ex. 5: Prove that the two lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are

(i) perpendicular if and only if $a^{-1} + b^{-1} + c^{-1} = 0, abc \neq 0$

(ii) parallel if and only if $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0$.

where a, b, c are positive.

Sol : Eliminating n between the given relations, we get

$$ar^2 - (c - a - b) r + b = 0 \quad \ldots (1)$$

where $r = l/m$. Similar quadratic equations in m/n, n/l can be obtained on eliminating l and m respecrively from the given relations.

Let $l_1, m_1, n_1; l_2, m_2, n_2$ be the direction cosines of the two lines in question. The product of the roots of r or l/m is b/a. Therefore $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$. Similarly from the other two quadratic equations, we get $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{c}{b}$; $\frac{n_1}{l_1} \cdot \frac{n_2}{l_2} = \frac{a}{c}$.

$$\text{Therefore, } al_1l_2 = bm_1m_2 = cn_1n_2$$

$$\text{or } \frac{l_1 l_2}{1/a} = \frac{m_1m_2}{1/b} = \frac{n_1n_2}{1/c} \quad \ldots(2)$$

(i) Two lines will be perpendicular if and only if (See Cor. of 7.9) $l_1 l_2 + m_1m_2 + n_1n_2 = 0$. Using (2), we get $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ or $a^{-1} + b^{-1} + c^{-1} = 0$ as the desired condition.

(ii) Two lines will be parallel if and only if (See Cor. 4 of 7.9) $l_1/l_2 = m_1/m_2 = n_1/n_2$ i.e., $l_1/m_1 = l_2/m_2$, $m_1/n_1 = m_2/n_2$ and $n_1/l_1 = n_2/l_2$. The relation $l_1/m_1 = l_2/m_2$ holds if and only if the roots of equation (1) are equal i.e., its discriminant is zero.

i.e., $(c - a - b)^2 = 4ab$

or if, $c - a - b = \pm 2\sqrt{ab}$

or if, $c = a + b \pm 2\sqrt{ab} = (\sqrt{a} \pm \sqrt{b})^2$

or if, $\pm\sqrt{c} = \sqrt{a} \pm \sqrt{b}$

or if, $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0$.

It is easy to verify that the above relation ensures $m_1/n_1 = m_2/n_2$ and $n_1/l_1 = n_2/l_2$.

SAQ1 : Define angle between two skew lines.

SAQ 2: Define the projection of a line segment on another line.

7.10 Summary

Position of a point in space is represented by three numbers called coordinates. Distance of any point P (x, y, z) from the origin 'O' is $\sqrt{x^2 + y^2 + z^2}$. A line in space is determined by its direction cosines l_1, m_1, n_1. Projection of a line joining $A (x_1, y_1, z_1)$, $B (x_2, y_2, z_2)$ on a line with direction cosines l, m, n is $= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$. If θ is the angle between the lines with d.c's l_1, m_1, n_1 ; l_2, m_2, n_2 then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

7.11 Sample Examination Questions

I. *Answer the following questions in detail.*

(i) (a) Define direction cosines and direction ratios of a line.

(b) Find the direction cosines of a line perpendicular to two lines whose direction ratios are 1, 2, 3 and –2, 1, 4.

(ii) (a) Obtain the co-ordinates of a point dividing the join of two points in the ratio $\lambda:1$.

(b) Find the foot of the perpendicular from the origin to the line joining the points (–9, 4, 5) and (11, 0, –1).

II. *Briefly answer the following questions.*

(i) A (3, 2, 0), B (5, 3, 2), C (–9, 6, –3) are three points forming a triangle. AD, the bisector of the angle BAC meets BC at D. Find the co-ordinates of D.

(ii) Find the angle between any two diagonals of a cube.

(iii) Show that the lines drawn from the origin with the direction ratios (1, –1, 1), (2, –3,0) and (1, 0, 3) are coplanar.

(iv) Show that the co-ordinates of the centroid of the tetrahedran whose vertices are (x_r, y_r, z_r), $r = 1, 2, 3, 4$ are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

(v) Show that the pair of lines whose direction cosines are given by $3lm - 4ln + mn = 0$, $l + 2m + 3n = 0$ are perpendicular.

(vi) Show that the lines whose direction cosines given by the equations $l + m + n = 0$, $al^2 + bm^2 + cn^2 = 0$ are

(i) Perpendicular, if $a + b + c = 0$

(ii) parallel if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

Answers

I. (ib) $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$. (iib) (1,2, 2)

II. (i) (38/16, 57/16, 17/16) (ii) $\cos^{-1}\frac{1}{3}$.

7.12 Answers to Self Assesment Questions

(i) Angle between two skew lines is defined to be the angle betwen the planes passing through these lines and $\perp$r to the planes in which they are lying or the angle between their supporting directions when they are shifted to the origin.

(ii) Suppose AB is a line segment. Draw perpendiculars from A and B upon the lines on which projection is required. If these feet of the perpendiculars are L, M then LM denotes the projection of the line segment AB upon the line l.

Unit–8 : EQUATION OF A PLANE

8.0 Contents

8.1 Aims and objectives

After going through this unit you will be able to :

(i) obtain the equation of a plane in any of the standard forms,
(ii) determine the angle between two given planes,
(iii) determine the length of the perpendicular from a given point to a given plane,
(iv) obtain the equations of the plane bisecting the angles between two given planes.

8.2 Introduction

In this Unit we study the general equation of a plane, its particular forms and the systems of planes. An expression for the angle between two planes in terms of direction ratios of their normals has beeen obtained. Using the formula for the perpendicular distance from a point to a plane, the equations to the planes bisecting the angles between two given planes have been written down.

8.3 Definition of a plane

Though the concept of a plane is assumed to be known, yet it is desirable to characterise a plane as follows.

Let OA and OB be two intersecting lines.The set of all points P such that $\overrightarrow{OP} = \lambda\overrightarrow{OA} + \mu\,\overrightarrow{OB}$ constitutes a plane.

Given a point P on a plane, there is a line passing through P which is perpendicular to all the lines lying in the plane passing through P. This line is called the normal to the plane at P. A plane is completely specified by giving a point and a line passing through the point. Actually that is the plane passing through P with the given line as its normal at P. If OP and OQ are two lines in a plane $\overrightarrow{OP} \times \overrightarrow{OQ}$ is a vector along the normal to the plane at O.

An equation of the plane is a relation satisfied by the co-ordinates (x, y, z) of every point on it.

8.4 Equation of a Plane

8.4.1 Normal Form

A triangle ABC determine a plane. We call it the plane ABC. Let O be outside it. Draw OK perpendicular to the plane. Let $OK = p$ (a positive number) and direction consines of $\overrightarrow{OK}$ be l, m, n .

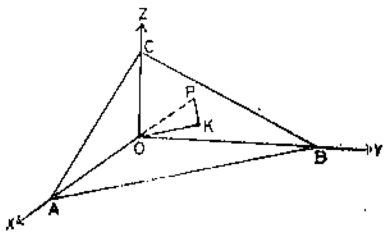


Fig. 1

Let $P(x, y, z)$ be any point on the plane. Join OP and PK. Then OK is projection of OP on OK itself. Then by cor. of 7.8, we get

$$p = l(x-0) + m(y-0) + n(z-0)$$

$$\text{or } lx + my + nz = p \qquad \ldots (1)$$

which is known as the equation of the plane in the normal form.

In fig. 1, since OKP is a right angled triangle with right angle at K, $OK = OP \cos\theta$ where θ is the angle between $\overrightarrow{OK}$ and $\overrightarrow{OP}$ (acute). In terms of vectors eq. (1) can be written as $\mathbf{r} \cdot \hat{\mathbf{n}} = p$ where $\mathbf{r}$ is the position vector of any point in the plane and $\hat{\mathbf{n}}$, a unit vector along OK.

8.4.2 General equation

Eq. (1) shows that the genral equation of a plane is of the form

$$ax + by + cz + d = 0 \qquad \ldots (2)$$

where a, b, c are not all zero.

Conversely it is easy to see that every equation of the first degree in x, y, z can be put in the form (1) with p positive.

Let Q be a fixed point in the plane. If P is a variable point in the plane then PQ is perpendicular to the normal to the plane. Therefore the vector equation of the plane is given by

$$\mathbf{n} \cdot \left(\mathbf{r} - \overrightarrow{OQ}\right) = 0,$$

where $\mathbf{r} = \overrightarrow{OP}$.

Cor. 1. Equation of any plane through the point $\left(x_1, y_1, z_1\right)$ is of the form

$$a\left(x - x_1\right) + b\left(y - y_1\right) + c\left(z - z_1\right) = 0 \qquad \ldots (3)$$

Proof: Eq. (3) is of the first degree in x, y, z and hence represents a plane. Evidently $\left(x_1, y_1, z_1\right)$ is a point on the plane.

In particular the equation of a plane passing through origin is $ax + by + cz = 0$.

Cor. 2. The equation $\lambda S_1 + \mu S_2 = 0$ represents a system of planes passing through the line of intersection of the planes.

$$S_1 \equiv ax + by + cz + d = 0$$

$$S_2 \equiv a_1 x + b_1 y + c_1 z + d_1 = 0$$

λ and μ are parameters, not both zero for

(i) $\lambda S_1 + \mu S_2 = 0$, being of the first degree in x, y, z represents a plane.

(ii) the equation $\lambda S_1 + \mu S_2 = 0$ is evidently satsified by the co-ordinates of the points which satisfy eqs. $S_1 = 0$ and $S_2 = 0$ whatever values λ and μ may take.

Usually it is assumed that the equation of any plane passing through the line of intersection of the planes $S_1 = 0$ and $S_2 = 0$ is of the form $\lambda S_1 + \mu S_2 = 0$. This needs a proof as given here under.

Let Π_1 and Π_2 be the planes represented by $S_1 = 0$ and $S_2 = 0$ respectively. Their line of intersection is evidently perpendicular to both the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ of the two planes. Hence $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the line of intersection of the two planes. Let Π_3 be a plane given $S_3 \equiv a_2 x + b_2 y + c_2 z + d_2 = 0$ passing through the intersection of Π_1 and Π_2 and $\mathbf{n}_3$ be its normal. Evidently $\mathbf{n}_3$ is perpendicular to $\left(\mathbf{n}_1 \times \mathbf{n}_2\right)$. Therefore $\mathbf{n}_3$ is a linear combination of $\mathbf{n}_1$ and $\mathbf{n}_2$. Hence $\mathbf{n}_3 = \lambda \mathbf{n}_1 + \mu \mathbf{n}_2$ for some scalar λ and μ. Taking the dot product of this with $\mathbf{r}$ both sides we get $S_3 = \lambda S_1 + \mu S_2$, the required result.

Usually in solving problems it is the converse that is used. Also we need a plane passing through intersection of two distinct planes and different from either of them. The given planes, being distinct, neither λ nor μ is zero. Hence the equation of the desired plane can be put in the form $S_1 + \lambda S_2 = 0$.

The equation $ax + by + cz + k = 0$ represents a system of planes perpendicular to the line with direction ratios a, b, c; k being the parameter.

8.4.3 Tranformation to the normal form

It is easily seen that the normal form of the equation $ax + by + cz + d = 0$.

$$\frac{a}{\sqrt{\Sigma a^2}} x + \frac{b}{\sqrt{\Sigma a^2}} y + \frac{c}{\sqrt{\Sigma a^2}} z = -\frac{d}{\sqrt{\Sigma a^2}}, \text{ if } d \text{ be negative}$$

$$-\frac{a}{\sqrt{\Sigma a^2}} x - \frac{b}{\sqrt{\Sigma a^2}} y - \frac{c}{\sqrt{\Sigma a^2}} z = \frac{d}{\sqrt{\Sigma a^2}}, \text{ if } d \text{ be positive}$$

where Σa^2 stands for $a^2 + b^2 + c^2$.

8.4.4. Angle between two planes

Def. The acute angle between two planes is defined as the angle between their normals.

Note that between the planes there is another angle which is supplement of the above.

Let the two planes be given by

$$ax + by + cz + d = 0$$

$$\text{and } a'x + b'y + c'z + d' = 0$$

Then the direction ratios of their normals are a, b, c and a', b', c'.

The angle θ between the planes i.e., angle between their normals (See cor. 2 of 7.9) is given by

$$\cos\theta = \pm\frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{a'^2 + b'^2 + c'^2}}$$

The planes will be perpendicular if $aa' + bb' + cc' = 0$ and parallel if $a/a' = b/b' = c/c'$.

8.4.5 Intercept form

We shall find the equation of a plane in terms of the intercepts a, b, c which it makes on the axes.

Let the required equation of the plane be

$$\alpha x + \beta y + \gamma z + \delta = 0 \qquad \ldots (1)$$

The plane cuts the axes A, B, C (Fig.2) such that $OA = a$, $OB = b$ and $OC = c$ i.e., it passes through the points $A\ (a, 0, 0)$, $B\ (0, b, 0)$, $C\ (0, 0, c)$. Therefore $\alpha a + \delta = 0$, $\beta b + \delta = 0$, $\gamma c + \delta = 0$, when $\alpha = -\delta/a$, $\beta = -\delta/b$, $\gamma = -\delta/c$. Substituting these values of α, β, γ in (1), we get

$$-\frac{\delta}{a} x - \frac{\delta}{b} y - \frac{\delta}{c} z + \delta = 0$$

$$\text{or } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

knwon as the equation of the plane in intercept form.

If a plane does not have at least one of the intercepts, following are the possibilities.

(1) It does not intersect x–axis and y–axis, in which case it is parallel to the XY plane and therefore the equation of the plane is $z = c$ i.e., $z/c = 1$.

(2) It intersects the XY plane. In this case the line of intersection may be parallel to one of the axes or it intersects both the axes. If the line of intersection is parallel to X-axis (say), then its equation is of the form $y/b = 1$ or $y/b + z/c = 1$ according as the plane is parallel to the Z-axis or not. In case the line of intersection is parallel to the Y-axis,then the equation of the plane is of the form $x/a = 1$ or $x/a + z/c = 1$.

Finally if the plane intersects both X and Y axes but not the Z axis, then its equation is of the form $x/a + y/b = 1$.

8.4.6 Plane through three points

We shall derive the equation of the plane passing through the three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \ldots (1)$$

As the given points lie on the plane, we have

$$ax_1 + by_1 + cz_1 + d_1 = 0, \quad \ldots (2)$$

$$ax_2 + by_2 + cz_2 + d_2 = 0, \quad \ldots (3)$$

$$ax_3 + by_3 + cz_3 + d_3 = 0, \quad \ldots (4)$$

In order to have a non-trivial solution for a, b, c, d (a, b, c, d are not all zero), it is necessary and sufficient that the coefficient determinant

$$\Delta = \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

is zero for every x, y, z on plane (1). Therefore $\Delta = 0$ is the required equation of the plane.

Theorem : The vector equation of a plane that passes through three points whose position vectors are **a**, **b** and **c** is

$$\mathbf{r} = (1 - s - t)\,\mathbf{a} + s\mathbf{b} + t\mathbf{c},$$

where **r** is the position vector of any point P on the plane, s and t being arbitrary scalars.

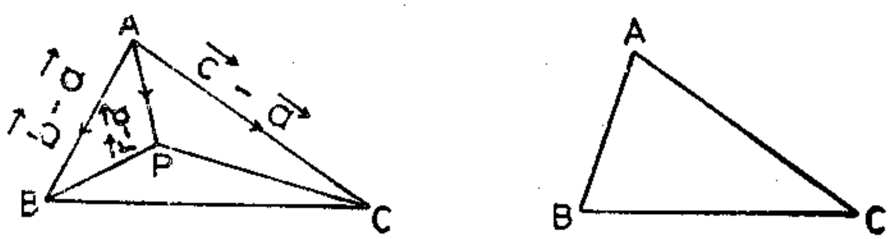
Proof : Let **r** be the position vector of any point in the plane but not on AB or AC. Then **r** – **a** is a linear combination of the two independent vectors **b** – **a** and **c** – **a**. Therefore we may write

$$\mathbf{r} - \mathbf{a} = s\,(\mathbf{b} - \mathbf{a}) + t\,(\mathbf{c} - \mathbf{a})$$

$$\text{or } \mathbf{r} = \mathbf{a} - s\mathbf{a} - t\mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

$$\text{or } \mathbf{r} = (1 - s - t)\,\mathbf{a} + s\,\mathbf{b} + t\,\mathbf{c}, \quad \ldots (5)$$

s, t being arbitrary (fig. 3 (i)).



(i) (ii)

Fig. 3

It can be easily verified that for any point P on AB or AC, the above equation still holds good (fig. 3). When P lies on AB, one can wirte $\overrightarrow{PA} = s\,\overrightarrow{BA}$ (see fig. 2 (ii)). In other words $\mathbf{r} = (1-s)\,\mathbf{a} + s\mathbf{b}$, which is simply the vector equation of the line joining the points A and B. This is also obtained by putting $t = 0$ in eq. (5). Similarly if P lies on AC we get $\mathbf{r} = (1-t)\,\mathbf{a} + t\,\mathbf{c}$ as the vector equation of the line joining the points A and C.

Cor. A necessary and sufficient condition for four points (x_r, y_r, z_r) ; $r = 1, 2, 3, 4$ in three dimensional space to be coplanar is

$$\Delta_1 = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Proof : If the four points are coplanar, then (x_4, y_4, z_4) lies on the plane $\Delta = 0$. Hence $\Delta_1 = 0$.

Convesely $\Delta_1 = 0$ ensures that one of the points lies on the planes passing through the other three points. There is no loss of generality in assuming that there are at least three of the points which form a plane.

Given four points A, B, C, D whose position vectors are $\mathbf{a}$, $\mathbf{b}$ $\mathbf{c}$, $\mathbf{d}$ respectively, then a necessary and sufficient condition for the coplanarity of the four points is

$$\mathbf{d} = (1 - s - t)\,\mathbf{a} + s\mathbf{b} + t\mathbf{b}. \qquad \text{... (6)}$$

Since D is a point in the plane containing the points A, B, C ; $\mathbf{d}$ satisfies the vector equation (5) and hence eq. (6) holds good. Conversely eq. (6) implies $\overrightarrow{AD}$ is expressible as a liner combination of the two independent vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In other words the four points A, B, C, D are coplanar.

Note: The volume of the tetrahedron formed by four non-complanar points A, B, C, D (fig. 4) is the absolute value of

$$\frac{1}{6}\left[\{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\}\cdot(\mathbf{d}-\mathbf{a})\right]$$

A necessary and sufficient condition for the coplanarity of the four points is also given by

$$\{(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\}\cdot(\mathbf{d}-\mathbf{a}) = 0$$

$$\text{or } (\mathbf{b}\times\mathbf{c})\cdot\mathbf{d} - (\mathbf{a}\times\mathbf{c})\cdot\mathbf{d}$$

$$+ (\mathbf{a}\times\mathbf{b})\cdot\mathbf{d} - (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 0. \qquad \text{... (7)}$$

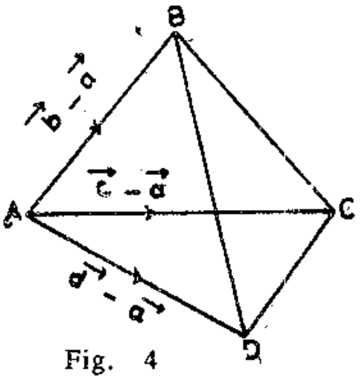


Fig. 4

If A, B, C, D are given by (x_r, y_r, z_r) ; $r = 1, 2, 3, 4$ respectively, eq. (7) can be written as

$$\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

$$\text{or } \Delta_1 = 0.$$

Note: The volume of a tetrahedron is the absolute value of $\frac{1}{6}\Delta_1$.

8.5 Two sides of a plane

Two points $A\left(x_1, y_1, z_1\right), B\left(x_2, y_2, z_2\right)$ lie on the same or different sides of the plane $ax + by + cz + d = 0$ according as the non-zero expressions $ax_1 + by_1 + cz_1 + d$, $ax_2 + by_2 + cz_2 + d$ are of the same or of different signs.

Proof: Let the line AB meet the given plane in a point P and let P divide AB in the same ratio $K:1$ so that K is positive or negative according as P divides AB internally or externally i.e., when A and B lie the opposite or the same side of the plane.

We know from 7.5 the coordinates of P are $\left(\dfrac{Kx_2 + x_1}{K+1}, \dfrac{Ky_2 + y_1}{K+1}, \dfrac{Kz_2 + z_1}{K+1}\right)$

Since P lies on the given plane, we have

$$a \cdot \frac{Kx_2 + x_1}{K+1} + b \cdot \frac{Ky_2 + y_1}{K+1} + c\, \frac{Kz_2 + z_1}{K+1} + d = 0$$

$$\text{or } K\left(ax_2 + by_2 + cz_2 + d\right) + \left(ax_1 + by_1 + cz_1 + d\right) = 0$$

$$\text{or } K = -\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$$

This shows that K is negative or positive according as $ax_1 + by_1 + cz_1 + d$, $ax_2 + by_2 + cz_2 + d$ are the same or of different signs. Thus the theorem is proved.

8.6 Length of the perpendicular from a point to a plane

The perpendicular distance of the point $P\left(x_1, y_1, z_1\right)$ from the plane $lx + my + nz = p$ is

$$\pm\left(lx_1 + my_1 + nz_1 + p\right).$$

Let L be the foot of the perpendicular from P to the plane $lx + my + nz = p$. Let this plane be designated by Π. PL (fig. 5) is then the required perpendicular distance. Let its direction cosines be l, m, n.

Let $Q\ (\alpha, \beta, \gamma)$ be a point on Π so that

$l\alpha + m\beta + n\gamma = p$. ... (1)

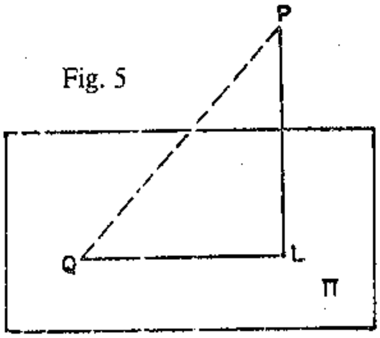
Therefore PL = Projection of PQ on PL.

$$= \pm\left[l\left(x_1 - \alpha\right) + m\left(y_1 - \beta\right) + n\left(z_1 - \gamma\right)\right]$$

$$= \pm\left[lx_1 + my_1 + nz_1 - p\right] \text{ by}$$

virtue of eq (1).

Fig. 5



Cor. The length of the perpendicular from $\left(x_1, y_1, z_1\right)$ to the plane $ax + by + cz + d = 0$ is

$$\pm\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}.$$

The normal form of the given equation of the plane (See 8.4.1) is

$$\frac{a}{\sqrt{\Sigma a^2}}x + \frac{b}{\sqrt{\Sigma a^2}}y + \frac{c}{\sqrt{\Sigma a^2}}z + \frac{d}{\sqrt{\Sigma a^2}} = 0.$$

Hence from the above article the required length of the perpendicular follows.

The vector equation of a plane through A, given by the position vector **a**, containing the directions given by the vector $\vec{\alpha}$ and $\vec{\beta}$ is $(\mathbf{r} - \mathbf{a}) \cdot (\vec{\alpha} \times \vec{\beta}) = 0$, where **r** is the position vector of any point on the plane. This is readily seen by noting that $\mathbf{r} - \mathbf{a}$, $\vec{\alpha}$ and $\vec{\beta}$ are coplanar and therefore their triple scalar product is zero.

Let B be a point outside the plane which contains a point A, two lines AC and AD represented by $\vec{\alpha}$ and $\vec{\beta}$ respectively. Let the position vecotrs of B and A be $\mathbf{b}$ and $\mathbf{a}$. Let L be the foot of the perpendicular from B to the plane. The three edges AB, AC, AD represented by $\mathbf{b} - \mathbf{a}, \alpha$ and β respectively form a parallelopiped whose volume is their scalar triple product $(\mathbf{b} - \mathbf{a}) \cdot (\vec{\alpha} \times \vec{\beta})$. Since $|\vec{\alpha} \times \vec{\beta}|$ represents the area of the parallelogram formed by the edges AC and AD, the perpendicular distance from B to the plane is obtained by dividing its volume by its base area. Thus

$$BL = \pm \frac{(\mathbf{b} - \mathbf{a}) \cdot (\vec{\alpha} \times \vec{\beta})}{|\vec{\alpha} \times \vec{\beta}|}$$

8.7 Bisectors of the angle between two planes

We derive the equations of the bisectors of the angles between the planes
$ax + by + cz + d + 0;\ a_1x + b_1y + c_1z + d_1 = 0.$

Let P be any point on any one of the planes bisecting the angles between the planes. Since the perpendiculars from P to the two planes are equal in magnitude, we get from cor. of 8.6.

$$\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{a_1x + b_1y + c_1z + d}{\sqrt{a_1^2 + b_1^2 \quad c_1^2}} \qquad \dots (1)$$

as the equations of the two bisecting planes. The joint equation of these planes can be written as

$$\frac{(ax + by + cz + d)^2}{a^2 + b^2 + c^2} = \frac{(ax_1 + b_1y + c_1z + d_1)^2}{a_1^2 + b_1^2 + c_1^2}.$$

which is a second degree equation in x, y, z.

Of the two bisecting planes given by (1), one bisects the acute and the other, the obtuse angle between the given planes. The bisector of the acute angle makes with either of the given planes an angle less than 45°, while the bisector of the obtuse angle makes with either of them an angle greater than 45°. This provides a test for distinguishing between the bisector plane of acute angle and that of the obtuse angle. Also of the bisecting planes given by (1), the one in which the constant appearing on either side of the equation has the same sign is also the plane bisecting that angle between the given planes which contains the origin. To prove this let us consider

$$\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \qquad \dots (2)$$

in which d and d_1 are both positive.

Since, by virute of the equality (2), the expression $ax + by + cz + d$ and $a_1x + b_1y + c_1z + d_1$ must have the same sign (denominators being both positive), the points (x, y, z) on the locus lie on the origin (fig. 6 (i)) or the non-origin (fig. 6 (ii)) side of both the planes, i.e., the points on the locus lie in the angle between the planes containing the origin.

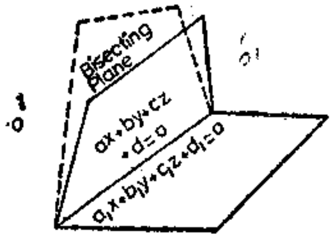


Fig. 6 (i)

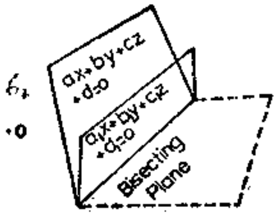


Fig. 6 (ii)

Thus the equation (2) represents the plane bisecting that angle between the planes which contains the origin.

The other one, namely,

$$\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = -\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

represents the plane bisecting the other angle between the given planes.

Examples

Ex. 1: Find the equation of a plane passing through the points P (2, 2, –1), Q (3, 4, 2), R (7, 0, 6).

Sol : The equation of a plane through P (2, 2, –1), is

$$a(x-2) + b(y-2) + c(z+1) = 0 \quad \dots (1)$$

This will pass through Q and R, if

$$a + 2b + 3c = 0$$

$$5a - 2b + 7c = 0$$

These give

$$\frac{a}{20} = \frac{b}{8} = \frac{c}{-12} \text{ or } \frac{a}{5} = \frac{b}{2} = \frac{c}{-3}.$$

Substituting these values of a, b, c in eq. (1), we have

$$5(x-2) + 2(y-2) - 3(z+1) = 0$$

or $$5x + 2y - 3z - 17 = 0$$

which is the required equation.

Note : Although the problem can be solved by §4.5, the reader would find it more convenient to follow the above method.

Ex. 2 : Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 9$.

Sol : Any plane through (2, 2, 1) is

$$a(x-2) + b(y-2) + c(z-1) = 0 \quad \dots (1)$$

This will pass through(9, 3, 6) if

$$a(9-2) + b(3-2) + c(6-1) = 0$$

i.e., $$7a + b + 5c = 0 \quad \dots (2)$$

The plane (1) will be perpendicular to the given plane if

$$2a + 6b + 6c = 0 \quad \dots (3)$$

From eqs (2) and (3) we have

$$\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \text{ or } \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}.$$

Substituting these values in eq. (1), we get

$$3(x-2) + 4(y-2) - 5(z-1) = 0$$

or $$3x + 4y - 5z = 9$$

which is the required equation.

Ex. 3: Find the equation of the plane which passes through the point (3, –3, 1) and is

(i) parallel to the plane $2x + 3y + 5z + 6 = 0$;

(ii) normal to the line joing the points (3, 2, –1) and (2, –1, 5);

(iii) perpendicular to the planes $7x + y + 2z = 6$ and $3x + 5y - 6z = 8$.

Sol: (i) Any plane parallel to the given plane

$$2x + 3y + 5z + k = 0$$

which passes through (3, –3, 1) if $k = -2$.

Hence the required plane is $2x + 3y + 5z - 2 = 0$.

(ii) Any plane through (3, –3, 1) is

$$a(x-3) + b(y+3) + c(z-1) = 0 \qquad \ldots (1)$$

The direction ratios of the normal to this plane are a, b, c. However these are given as those of the line joining the points (3, 2, –1) and (2, –1, 5) i.e., 1, 3, –6.

Hence $\frac{a}{1} = \frac{b}{3} = \frac{c}{-6}$

Then eq. (1) becomes

$$1(x-3) + 3(y+3) - 6(z-1) = 0$$

or $$x + 3y - 6z + 12 = 0,$$

which is the equation of the required plane.

(iii) Plane (1) should be perpendicular to the given planes.

Hence $$7a + b + 2c = 0,$$

$$3a + 5b - 6c = 0.$$

Solving these by cross multiplication,

$$\frac{a}{-16} = \frac{b}{48} = \frac{c}{32} \text{ or } \frac{a}{1} = \frac{b}{-3} = \frac{c}{-2}.$$

Hence eq. (1) takes the form

$$1(x-3) - 3(y+3) - 2(z-1) = 0$$

or $$x - 3y - 2z - 10 = 0$$

which is the required equation.

Ex. 4: Find the equation of the plane passing through the intersection of the planes

$$x + y + z = 6 \text{ and } 2x + 3y + 4z + 5 = 0$$

and the point (1, 1, 1).

Sol : From cor. 2 of 8.4.2 the plane

$$x + y + z - 6 + K(2x + 3y + 4z + 5) = 0 \qquad \ldots (1)$$

passes through the intersection of the given planes for all values of K.

Plane (1) passes through (1, 1, 1) if

$-2 + 14K = 0$ or $K = 3/14$.

Putting $K = 3/14$in eq. (1), we obtain

$$20x + 23y + 26z - 69 = 0$$

which is the equation of the required plane.

Ex. 5: A variable plane passes through the fixed point (a, b, c) and meets the co-ordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the co-ordinate planes is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

Sol: Let ABC be any plane (fig. 7) through the fixed point $H(a,b,c)$ such that $OA = x_1, OB = y_1, OC = z_1$.

Then from 8.4.5, its equation is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.

Since H lies on it, we get $$\frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1. \qquad \dots (1)$$

The planes through A, B, C parallel to the co-ordinate planes are $x = x_1, y = y_1, z = z_1$, which meet in $P(x_1, y_1, z_1)$.

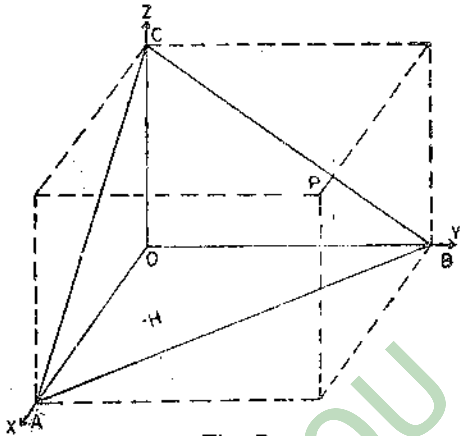


Fig. 7

Therefore changing x_1 to x, y_1 to y and z_1 to z in eq. (1) we get the equation of the locus of P as

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

Ex. 6: Find the equations of the planes bisecting the angles between the planes

$3x - 4y + 5z = 3$, $5x + 3y - 4z = 9$.

Also point out which of the planes bisects the acute angle.

Sol: The equations of the planes bisecting the angles between the given planes from 8.7 are

$$\frac{3x - 4y + 5z - 3}{\sqrt{[(3)^2 + (-4)^2 + (5)^2]}} = \pm \frac{5x + 3y - 4z - 9}{\sqrt{[(5)^2 + (3)^2 + (-4)^2]}}$$

or $2x + 7y - 9z - 6 = 0$ and $8x - y + z - 12 = 0$.

Let θ be the angle between the planes $2x + 7y - 9z - 6 = 0$ and $5x + 3y - 4z = 9$. Then from 8.4.4, we have

$$\cos\theta = \frac{2 \times 5 + 7 \times 3 + (-9) \times (-4)}{\sqrt{[(2)^2 + (7)^2 + (-9)^2]} \cdot \sqrt{[(5)^2 + (3)^2 + (-4)^2]}} = \frac{67}{5\sqrt{268}}$$

Note that angle between a bisecting plane and one of the given planes is not greater than 90°. Hence we have taken $\cos\theta$ to be positive.

$$\text{Now } \tan\theta = \frac{\sqrt{2211}}{67} \text{ which is less than 1,}$$

i.e., $\theta < 45°$.

Now θ is half the angle between the given planes, so that the plane $2x + 7y - 9z - 6 = 0$ bisects that angle between the planes i.e., $2\theta < 90°$ (acute).

Hence the plane $2x + 7y - 9z = 6$ bisects the acute angle.

SAQ : State whether the following statements are true or false. Support your answers with reasonings; give complete proof if necessary.

1. A first degree equation in three variables always represents a plane surface.
2. If $a_1 x + b_1 y + c_1 z + d_1 = 0$ is a plane then d.c's of its normal are a_1, b_1, c_1.
3. $x - y - z + 3 = 0$, $2x + y + z + 3 = 0$ represent equations of two mutually parallel planes.
4. $3x + 3y + 6z + 10 = 0$, $5x + 5y + 10z + 2 = 0$ represent equations of two mutually perpendicular planes.
5. Four points always lie in a plane.
6. It is not always possible to draw a plane from any three points in the space.
7. (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) are coplanar points.
8. $3x + 5y + 7z + a = 0$ represent a family of planes perpendicular to a fixed direction, where a is an arbitrary constant. If so what is the fixed direction?
9. $a(x-2) + b(y-3) + c(z-5) = 0$ represents a family of planes passing through a fixed point. If so what is the fixed point?

8.8 Summary

Obtained the equation of a plane in various forms. If $S_1 = 0$, $S_2 = 0$ are the equaions of two planes then $S_1 + \lambda S_2 = 0$ represents a family of planes through the line of intersection of the planes, where λ is a parameter. If θ is the angle between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ then the cosine of the angle between the planes is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{\Sigma a_1^2}\ \sqrt{\Sigma a_2^2}}$$

The vector equation of a plane through three points with position vectors **a, b, c** is

$$\mathbf{r} = (1 - s - t)\,a + sb + tc$$

The volume of a tetrahedron formed by four non-coplanar points **a, b, c, d** is given by

$$V = [\{(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})\} \cdot (\mathbf{d}-\mathbf{a})].$$

The length of the $\perp$r from a given point (x_1, y_1, z_1), to the plane $ax + by + cz + d = 0$ is

$$\pm\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}.$$

8.9 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Obtain the equation of a plane in the normal form.

b) Find the equaion of the plane through the three points (1, 1, 1), (1, –1, 1), (–7, –3, –5) and show that it is perpendicular to the XZ - plane.

(ii) a) Obtain the necessary and sufficient conditions for four points in space to be coplanar.

b) Find the equation of the plane through A (–1, 1, 1) and B (1, –1, 1) and perpendicular to the plane $x + 2y + 2z = 5$.

(iii) a) Obtain the equation of the plane in the intercept form.

b) A plane meets the co-ordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a, b, c). Show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

II. *Briefly answer the following questions.*

i) Obtain the equation of the plane through the point (–1, 3, 2) and perpendicular to the two planes $x + 2y + 2z = 5$; $3x + 3y + 2z = 8$.

ii) Find the equation of the plane through the point (2, 3, 4) and parallel to the plane $5x - 6y + 7z = 3$.

iii) Find the equation of the plane that passes through (3, –3, 1) and is normal to the line joining the points (3, 4, –1) and (2, –1, 5).

iv) Find the equation of the plane passing through the line of intersection of the planes $2x - y = 0$ and $3x - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$.

v) A variable plane which remains at a constant distnace p from the origin cuts the co-ordinate axes at A, B, C. Find the locus of centroid of

(i) the triangle ABC; (ii) the tetrahedron $OABC$.

vi) Show that the origin and the point (2, –4, 3) lie on different sides of the plane $x + 3y - 5z + 7 = 0$.

vii) Show that the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$ is $\frac{1}{6}$.

viii) Find the bisector of the obtuse angle between the planes $3x + 4y - 5z + 1 = 0$, $5x + 12y - 13z = 0$.

Answers

I. (ib) $3x - 4z + 1 = 0$, (iib) $2x + 2y - 3z + 3 = 0$

II. (i) $2x - 4y + 3z + 8 = 0$ (ii) $5x - 6y + 7z = 20$

(iii) $x + 5y - 6z + 19 = 0$ (iv) $28x - 17y + 9z = 0$

(v) $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$, $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$

(viii) $14x - 8y + 13 = 0$.

8.10 Answers to Self Assessment Questions

1. True $ax + by + cz + d = 0$ always represents a plane.
2. False. a_1, b_1, c_1 represents the direction ratios of the normal to the plane.
3. False. Since the inner product of the coefficients, $(2 \times 1 - 1 \times 1 - 1 \times 1 = 0)$, is equal to zero, the normals of the planes are mutually ⊥r and hence the planes are perpendicular to each other.
4. False. The coefficients in both the equations are in the same proportion, $\frac{3}{5} = \frac{3}{5} = \frac{6}{10}$, the planes are parallel.
5. False. For example, the vertices of a tetrahedraon are not in the same plane.
6. False. It is always possible to draw a plane through any three given points.
7. True. The fourth point lies on the plane through the first three points.
8. True. For different values of a, we get different planes having the same direction.
9. True. The equation represents a family of planes passing through a fixed point.

Unit-9 : EQUATION OF A STRAIGHT LINE

9.0 Contents

9.1 Aims and Objectives

After going through this unit, you will be able to :

i) obtain the equation of a straight line in the required form,

ii) obtain the equation of a plane containing two given lines,

(iii) test the conditions for the coplanarity of two given lines,

(iv) Obtain the general equation of a straight line intersecting two given straight lines.

9.2 Introduction

In this Unit we regard a straight line as the intersection of two planes and obtain its equation in unsymmetrical form. Also given the direction cosines of a line and a point on it, its equation can be expressed in symmetrical form. We explain how to convert the equation of a line from unsymmetrical to symmetrical form. Equation of a line in terms of two given points on it has been derived. The angle between a line and plane has been defined and an expression obtained. Conditions for (i) a line to lie in a plane and (ii) the coplanarity of two lines have been formulated. Equation of a plane containing two given lines has been obtained. Finally, it has been pointed out that given two lines, the equation of a line passing through their intersection can be derived using two parameters.

9.3 Equations of a straight line

9.3.1 General form

Two equations of the first degree in x, y, z

i.e., $$ax + by + cz + d = 0, \quad \ldots (1)$$

and $$a_1x + b_1y + c_1z + d_1 = 0, \quad \ldots (2)$$

taken together represents a straight line which is the line of intersection of two non-parallel planes (1) and (2). Any straight line can be put in this form by taking any two planes through it. Thus the general equations of a straight line are

$$ax + by + cz + d = 0 \text{ and } a_1x + b_1y + c_1z + d_1 = 0.$$

They are known as the equations in the unsymmetrical form.

9.3.2 Symmetrical form

Equation of the line AB passing through the point $A\,(x_1, y_1, z_1)$ and having direction cosines l, m, n are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Let $P\,(x, y, z)$ be any point on the given line AB (fig. 1) such that the distance $AP = r$.

Draw AL, PM perpendiculars to the Y-axis.

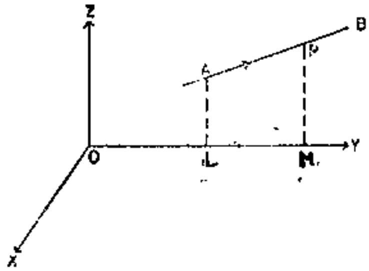


Fig. 1

Then $LM = OM - OL = y - y_1$ (with proper signs for projected segments as already explained in Unit 7). Also LM = projection of AP on the Y - axis = $r \cos \beta = rm$ where β is the angle which AP makes with OY. Therefore $y - y_1 = mr$. Similarly $z - z_1 = nr$, $x - x_1 = lr$. Hence the required equations are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r. \qquad \text{... (1)}$$

Cor. 1 : Any point on the line (1) has co-ordinates $(x_1 + lr, y_1 + mr, z_1 + nr)$.

Cor. 2 : The equations of the line through (x_1, y_1, z_1) and having direction ratios a, b, c are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{... (2)}$$

It is important to note that each member of eq. (2) is not equal to r, the distance of the point (x, y, z) from (x_1, y_1, z_1) since a, b, c are not direction cosines.

9.3.3 Line through two points

The equations of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

for the direction ratios of the line joining the given points are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

The vector equation of a line corresponding to eq. (1) is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{s}$, where $\mathbf{r}$ is the position vector of the point P, $\mathbf{a}$ the position vector of a given point A on the line, $\mathbf{s}$ a unit vector along AB and λ an arbitrary scalar.

9.3.4 Transformation from unsymmetrical to the symmetrical form

We transform the equations

$ax + by + cz + d = 0$, $a_1x + b_1y + c_1z + d_1 = 0$ of a line to the symmetrical form.

Let a', b', c' be direction ratios of the line. Since the line lies in both the planes (non-parallel) $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$,

it is perpendicular to the normals to both of them. As the direction ratios of the normals to the two planes are a, b, c and a_1, b_1, c_1 we have

$$aa' + bb' + cc' = 0,$$
$$a_1a' + b_1b' + c_1c' = 0.$$

Therefore, $\dfrac{a'}{bc_1-b_1c} = \dfrac{b'}{ca_1-c_1a} = \dfrac{c'}{ab_1-a_1b}$.

In order to find the co-ordinates of any one point on the line, we take for the sake of convenience. the point of intersection of the line with the plane $z = 0$. This point which is given by the equations

$ax + by + d = 0$ and $a_1x + b_1y + d_1 = 0$ is $\left(\dfrac{bd_1-b_1d}{ab_1-a_1b}, \dfrac{a_1d-ad_1}{ab_1-a_1b}, 0\right)$.

Thus, in the symmetrical form, the equations of the given line are

$$\frac{x - (bd_1-b_1d)/(ab_1-a_1b)}{bc_1-b_1c} = \frac{y - (a_1d-ad_1)/(ab_1-a_1b)}{ca_1-c_1a} = \frac{z-0}{ab_1-a_1b}.$$

9.4 Angle between a line and a plane

Def : Angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

Note that usually angle between two lines is taken to lie in $(0, \pi/2)$.

We find the angle between the line

$$\frac{x-x_1}{a'} = \frac{y-y_1}{b'} = \frac{z-z_1}{c'} \qquad \text{... (1)}$$

and the plane

$$ax + by + cz + d = 0 \qquad \text{... (2)}$$

Let θ be the required angle. Then the angle between the line and the normal to the plane is $\frac{\pi}{2} - \theta$. Since the direction cosines of the line and the normal are respectively proportional to a', b', c' and a, b, c we have

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{a'^2+b'^2+c'^2}}$$

Cor. The straight line (1) is parallel to the plane (2) if $\theta = 0$ i.e.,

$$aa' + bb' + cc' = 0,$$

a condition which is also evident from the fact that if a line be parallel to a plane, it is perpendicular to the normal to the plane.

9.5 Conditions for a line to lie in a plane

The necessary and sufficient conditions for a line L

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \qquad \text{... (1)}$$

to lie in a plane Π

$$ax + by + cz + d = 0 \qquad \text{... (2)}$$

are $$al + bm + cn = 0 \qquad \text{... (3)}$$

and $$ax_1 + by_1 + cz_1 + d = 0 \qquad \text{... (4)}$$

If the line L has to lie in the plane Π, then any point on the line (1) i.e., $(lr + x_1, mr + y_1, nr + z_1)$ should be a point on the plane (2). This requires

$a\,(lr + x_1) + b\,(mr + y_1) + c\,(nr + z_1) + d = 0$

or $(al + bm + cn)\,r + (ax_1 + by_1 + cz_1 + d) = 0 \qquad \text{... (5)}$

Since eq. (5) must hold good for all values of r, the coefficient of r and the constant in (5) should separately vanish. Hence we obtain (3) and (4).

Conversely (3) and (4) imply that we require a line : (i) it is perpendicular to the normal to the plane (ii) it passes through (x_1, y_1, z_1) a point in the plane. Evidently (1) is the required line.

Cor. General equation of the plane containing the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0,$

where $\quad al+bm+cn = 0$

9.6 Coplanar lines

We find the condition that two given lines should lie in a plane.

Let their equations be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad \dots (1)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \qquad \dots (2)$$

The equation to a plane containing the first line is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \qquad \dots (3)$$

where $\quad al_1+bm_1+cn_1 = 0. \qquad \dots (4)$

If it contains the line (2), then in addition to (4) we have

$$a(x_1-x_2)+b(y_1-y_2)+c(z_1-z_2)=0 \qquad \dots (5)$$

and $\quad al_2+bm_2+cn_2 = 0 \qquad \dots (6)$

Therefore eliminating a, b, c between (5), (4) and (6), we obtain the required condition.

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Solving (4) and (6), we get

$$\frac{a}{m_1n_2-m_2n_1}=\frac{b}{n_1l_2-n_2l_1}=\frac{c}{l_1m_2-l_2m_1}=k.$$

Assuming the lines (1) and (2) to be non-parallel and substituting the values a, b, c in (3), we get the equation to the plane containing the lines as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

9.7 Line intersecting two given lines

If $u_1 = 0 = v_1$ and $u_2 = 0 = v_2$ be two straight lines, then the general equations of a straight line intersecting both of them are

$u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$, where λ_1, λ_2 are any two constant numbers.

The line $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$ lies in the plane $u_1 + \lambda_1 v_1 = 0$ which again contains the line $u_1 = 0 = v_1$.

The two lines $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$; $u_1 = 0 = v_1$ are, therefore coplanar and hence they intersect.

Similarly, the same line intersects the line $u_2 = 0 = v_2$.

SAQ's : State whether the following statements are true or false. Support your answers with reasoning.

(i) Two planes always intersect in a line or they are parallel to each other.

(ii) Direction ratios of the line

$$ax + by + cz + d = 0,$$
$$a_1x + b_1y + c_1z + d_1 = 0$$

are $bc_1 - b_1c$, $ca_1 - c_1a$, $ab_1 - a_1b$.

(iii) Direction ratios of the line joining the points (4, 5, 6), (3, 4, 5) are 1, 1, 1.

(iv) Angle between the plane $2x + 3y + 4z + 1 = 0$ and a line with direction ratios 4, 5, 6 is $\cos^{-1} \dfrac{8 + 15 + 24}{\sqrt{29} \cdot \sqrt{77}}$.

Examples :

Ex. 1 : Find the equations of the line through the point (3, –1, 11) and perpendicular to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

Obtain the foot of the perpendicular also.

Sol : The co-ordinates of any point P on the given line (Cor. 1 of 9.3.2) are $2r$, $3r + 2$, $4r + 3$, where r is so chosen that P is the foot of the perpendicular from the point (3, –1, 11) to the plane. Then the line joining P to the point (3, –1, 11) is perpendicular to the given line. Hence

$2(2r - 3) + 3(3r + 2 + 1) + 4(4r + 3 - 11) = 0$ or $r = 1$.

Therefore, P has co-ordinates (2, 5, 7) and the equations of the required perpendicualr (see 9.3.1) are

$$\frac{x-3}{3-2} = \frac{y+1}{-1-5} = \frac{z-11}{11-7}$$

$$\text{or } \frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}.$$

Ex. 2 : Find the symmetrical form of the equations of the line $x + y + z + 1 = 0$, $4x + y - 2z + 2 = 0$.

Sol : To find a point on the line (9.3.4) put $z = 0$ in the given equations. Then

$$x + y + 1 = 0$$
$$4x + y + 2 = 0.$$

Solving, we get $x = -\frac{1}{3}$ and $y = -\frac{2}{3}$. Thus $\left(-\frac{1}{3}, -\frac{2}{3}, 0\right)$ is a point on the line.

Let a, b, c be the direction ratios of the line. Since it lies on both the given planes, it is perpendicular to their normals whose direction ratios are 1, 1, 1 and 4, 1, −2. Hence

$$a + b + c = 0,$$
$$4a + b - 2c = 0.$$

Solving, $\frac{a}{-3} = \frac{b}{6} = \frac{c}{-3}$ or $\frac{a}{-1} = \frac{b}{2} = \frac{c}{-1}$.

Therefore the equations of the line in the symmetrical form are

$$\frac{x + 1/3}{-1} = \frac{y + 2/3}{2} = \frac{z}{-1}$$

Ex. 3 : Show that the lines

$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$, $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar and find the equation of the plane containing them.

Sol : By 9.6, the condition for coplanarity of the two given lines is

$$\begin{vmatrix} -2 & -4 & 8 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0, \text{ satisfied.}$$

The equation of the plane containing them is

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0,$$

which, on expansion, gives $6x - 5y - z = 0$

Ex. 4 : Assuming the line $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$ as given, find the equations of the line of greatest slope in the plane $2x + y - 5z = 12$ and passing through the point (2, 3, −1).

Sol : [A plane perpendicular to the given line and the given plane have a common line of intersection unless both of them are parallel. Assuming them to be non-parallel, a line in the given plane but perpendicular to the common line of intersection, mentioned above, is called the line of greatest slope (fig. 2).

Let Π_1 be a plane perpendicular to the given line AB. Let Π_2 be the given plane and let both the planes Π_1 and Π_2 intersect along LM. A line CD, lying in plane Π_2 but perpendicular to LM is, by definition the line of greatest slope].

In this problem the line AB is given by $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$. The equation of the plane Π_1 normal to AB is $4x - 3y + 7z + k = 0$ where k is arbitrary. The equation of the Π_2 is $2x + y - 5z = 12$. Let the direction raios of LM, the line of intersection of planes Π_1 and Π_2 be a, b, c. Since LM is perpendicular to the normals of both the planes, we have

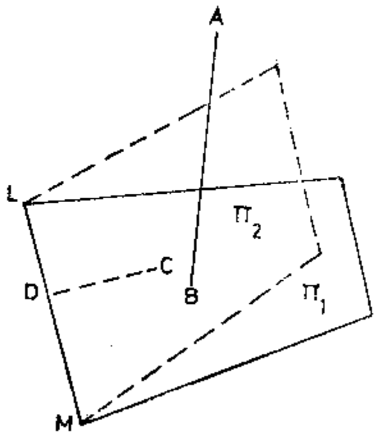


Fig. 2

$$4a - 3b + 7c = 0,$$
$$2a + b - 5c = 0.$$

Solving we get $\frac{a}{4} = \frac{b}{17} = \frac{c}{5}$.

Since CD, the line of greatest slope in the plane Π_2 is perpendicular to both *LM* and normal to the Π_2, its direction ratios a', b', c' are given by

$$4a' + 17b' + 5c' = 0,$$
$$2a_1' + b_1' - 5c_1' = 0.$$

Solving we get

$$\frac{a'}{3} = \frac{b'}{-1} = \frac{c'}{1}.$$

Among the many lines perpendicualr to *LM*, we so choose *CD* as to pass through the given point (2, 3, –1) of the Π_2. Hence the equations of the line of greatest slope through (2, 3, –1) and having direction ratios 3, –1, 1 are

$$\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z+1}{1}$$

Ex. 5 : Find the equations to the line that intersects the lines

$$2x + y - 1 = 0 = x - 2y + 3z;$$
$$3x - y + z + 2 = 0 = 4x + 5y - 2z - 3$$

and is parallel to the line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

Sol : The general equations of the line intersecting the two given lines is

$$2x + y - 1 + \lambda_1 (x - 2y + 3z) = 0,$$
$$3x - y + z + 2 + \lambda_2 (4x + 5y - 2z - 3) = 0$$

which will be parallel to the given line if λ_1, λ_2 be so chosen that the two planes representing it are separately parallel to the given line.

This requires

$$1 (2 + \lambda_1) + 2 (1 - 2\lambda_1) + 3 (3 \lambda_1) = 0 \text{ i.e., } \lambda_1 = -2/3.$$
$$1 (3 + 4 \lambda_2) + 2 (-1 + 5 \lambda_2) + 3 (1 - 2 \lambda_2) = 0 \text{ i.e., } \lambda_2 = -1/2.$$

The required equations of the line, therefore, are

$$4x + 7y - 6z - 3 = 0, \quad 2x - 7y + 4z + 7 = 0.$$

9.8 Summary

The General form of the equations of a line are

$$ax + by + cz + d = 0,$$
$$a_1x + b_1y + c_1z + d_1 = 0.$$

Symmetrical form of the equations of a line are $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$, where l, m, n denote the direction cosines of the line and (x_1, y_1, z_1) is a fixed point on the line.

The necessary and sufficient conditions for a given line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ to lie on a given plane $ax + by + cz + d = 0$ are

$$al + bm + cn = 0$$
$$ax_1 + by_1 + cz_1 = 0.$$

If two lines are given by $U_1 = 0 = V_1$, $U_2 = 0 = V_2$, then the equation of lines through the intersection of the two lines is given by

$U_1 + \lambda_1 V_1 = 0 = U_2 + \lambda_2 V_2$, where λ_1, λ_2 are parameters.

9.9 Sample Examination Questions

I. *Answer the following questions in detail.*

(i) (a) Obtain the equations of a straight line in symmetrical form.

(b) Obtain the symmetrical form of the equations $x - 2y + 3z = 4$, $2x - 3y + 4z = 5$.

(ii) (a) Obtain the necessary and sufficient condition for a line to lie in a plane.

(b) Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$, $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Find their point of intersection and the plane in which they lie.

(iii) (a) Obtain the condition for the coplanarity of two given lines.

(b) Show that the lines $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$, $3x + 2y + z - 2 = 0 = x - 3y + 2z - 13$ are coplanar and find the equation to the plane in which they lie.

II. *Briefly answer the following questions.*

(i) Find the image of the point P (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

(ii) Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\frac{1}{3}(x-2) = \frac{1}{4}(y+1) = \frac{1}{12}(z-2)$ and the plane $x - y + z = 5$.

(iii) Show that the line $\frac{x-3}{3} = \frac{2-y}{4} = \frac{z-1}{1}$ intersects the line $x + 2y + 3z = 2x + 4y + 3z + 3 = 0$. Find their point of intersection.

(iv) Find the angle between the lines.
$x - 2y + z = 0 = x + y - z$, $x + 2y + z = 0 = 8x + 12y + 5z$.

(v) Find the equations of the line that intersects the lines $2x + y - 4 = 0 = y + z$; $x + 3z = 4$; $2x + 5z = 8$ and passes through the point (2, –1, 1).

(vi) Find the equations of the line drawn through the point (–4, 3, 1), parallel to the plane $x + 2y - z = 5$ so as to intersect the line $-\frac{(x+1)}{3} = -\frac{(y-3)}{1} = \frac{(z-1)}{1}$. Find also the point of intersection.

Answers

I. (ib) $\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{1}$ (iib) $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$; $x - 2y + z = 0$.

(iiib) $21x - 19y + 22z - 125 = 0$.

II. (i) (–3, 5, 2) (ii) 13 (iv) $\cos^{-1}\left(8/\sqrt{406}\right)$ (v) $x + y + z = 2$, $x + 2z = 4$

(vi) $\frac{(x+4)}{3} = -\frac{(y-3)}{2} = \frac{(z-1)}{1}$; (2, 1, 3)

9.10 Answers to Self Assessment Questions

(i) True, points common to the planes lie on a line; otherwise the planes may not intersect at all.

(ii) True. The line lying in the given planes is $\perp r$ to the normals to be the planes.

(iii) Ture. (iv) True.

Unit–10 : SHORTEST LINE SEGMENT BETWEEN TWO SKEW LINES

10.0 Contents

10.1 Aims and Objectives

After going through this unit you will be able to :

(i) obtain the length and equations of the shortest line segments between two skew lines.

10.2 Introduction

In this Unit we explain the concept of shortest line segment between two skew lines. We derive its equations and obtain an expression for its length. An example has been worked out (i) by the general method as outlined in the theory (ii) by a method which determines the position of the shortest line segment between the two skew lines and is often useful in establishing a criterion for the two lines to intersect and (iii) lastly by a method based on the concept that the shortest line segment is the length of the perpendicular from a point on one of the skew lines to the planes passing through the second skew line and parallel to the first.

10.3 The shortest line segment between two skew lines

Def. Two non-coplanar lines which do not intersect are called skew lines.

We shall prove two results. (i) There is a line which meets both the skew lines at right angles and (ii) There is only one such line.

Let AB, CD be two given skew lines (fig. 1). Let a line perpendicular to both of them meet AB, CD in the points G, H respectively. Since the equations of the line AB is given, the co-ordinates of any point on it and in particular G can be written (see. 9.3) in terms of a paramater r. Similarly the co-ordinates of H can also be wirtten in terms of r'. Using the fact that GH is perpendicular to both AB and CD the two parameters r and r' can be determined. With the knowledge of r and r', the equation to the line GH can be written down. Thus there is a line GH which meets both the skew lines at right angles.

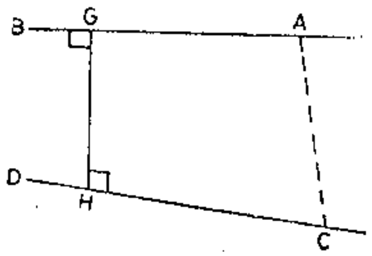


Fig. 1

In order to prove that there is only one such line, take any two points A, C one on each of the two given lines. Then GH is the projection of AC on GH itself and, therefore, $GH = AC \cos\theta$, where θ is the angle between GH and AC. Hence $GH \leq AC$. Thus GH is the only shortest line segment between two lines AB and CD.

10.3.1 Length and equations of the shortest line segment

Let the given lines AB and CD (see. fig. 1) have the equations

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

where $A \equiv (x_1, y_1, z_1)$ and $C \equiv (x_2, y_2, z_2)$.

Let l, m, n be the direction cosines of the shortest line segment GH. Since GH is perpendicular to both AB and CD, we get $ll_1 + mm_1 + nn_1 = 0$

and $ll_2 + mm_2 + nn_2 = 0$.

Solving,

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \frac{\sqrt{\Sigma l^2}}{\sqrt{\Sigma (m_1n_2 - m_2n_1)^2}} = \frac{1}{\sin\theta} \qquad (1)$$

where θ is the angle between the lines AB and CD and $\sqrt{\Sigma (m_1n_2 - m_2n_1)^2} \neq 0$ since the lines are skew. Therefore length of the shortest line segment GH = projection of AC on GH =

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \qquad \text{[Ref. Cor. of 7.8.1]}$$

where l, m, n have the values as given by (1).

To find the equations of shortest line segment, we observe that it is coplanar with both AB and CD. From 9.6, the equation of the plane containing the lines AB and GH is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad (2)$$

Similarly the equation of the plane containing the lines CD and GH is of the form

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \qquad (3)$$

Hence (2) and (3) are the equations of the shortest line segment.

The reader should observe that the condition for the given lines to intersect is also obtained by equating the length of the shortest line segment GH to zero.

Examples

Ex. 1: Find the length and the equations of the shortest line segment between the lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \qquad \dots (1)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \qquad \dots (2)$$

Sol : We shall work out this problem in three different methods.

First method

Let l, m, n be the direction cosines of the line of shortest segments. As it is perpendicular to the two lines, we have

$$3l - 16m + 7n = 0,$$

$$\text{and } 3l + 8m - 5n = 0.$$

Solving, $\frac{l}{24} = \frac{m}{36} = \frac{n}{12}$ or $\frac{l}{2} = \frac{m}{3} = \frac{n}{6} = \frac{1}{7}$.

Therefore, $l = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$.

The length of the shortest line segment is the projection of the join of the points (8, –9, 10), (15, 29, 5) on the shortest line segment and hence

$$= 7 \cdot \frac{2}{7} + 38 \cdot \frac{3}{7} - 5 \cdot \frac{6}{7} = 14.$$

Again from 9.6, the equation of the plane containing the line (1) and the shortest line segment is

$$\begin{vmatrix} x - 8 & y + 19 & z - 10 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or $117x + 4y - 41z - 490 = 0$.

Also the equation of the plane containing the line (2) and the shortest line segment is

$$\begin{vmatrix} x - 15 & y - 29 & z - 5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or $9x - 4y - z = 14$.

Hence the equtions of the shortest line segment are

$$117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 14.$$

Second method

$P\,(3r + 8, -16r - 9, 7r + 10)$, $P'\,(3r' + 15, 8r' + 29, -5r' + 5)$ are the general co-ordinates of the points on the two lines (1) and (2) respectively. The direction ratios of PP' are $3r - 3r' - 7, -16r - 8r' - 38, 7r + 5r' + 5$.

Now PP' will be the required shortest line segment, if it is perpendicular to both the given lines, i.e.,

$$3\,(3r - 3r' - 7) - 16\,(-16r - 8r' - 38) + 7\,(7r + 5r' + 5) = 0,$$

$$\text{and } 3\,(3r - 3r' - 7) + 8\,(-16r - 8r' - 38) - 5\,(7r + 5r' + 5) = 0.$$

or $\quad 157r + 77r' + 311 = 0,$

and $\quad 11r + 7r' + 25 = 0.$

Solving, $r = -1$, $r' = -2$.

Therefore, the co-ordinates of P and P' are (5, 7, 3) and (9, 13, 15) and the direction ratios of PP' are $-4, -6, -12$.

The length of the shortest line segment $PP' = 14$ and its equations are

$$\frac{x-5}{-4} = \frac{y-7}{-6} = \frac{z-3}{-12} \text{ or } \frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}.$$

Note: This method is sometimes very convenient and is specially useful when we require also the points of intersection of the shortest line segment and the two skew lines..

Third method

Let AB, CD be the two given skew lines. Let Π_1 be the plane passing through AB and parallel to CD (fig. 2). Then the perpendicular distance from any point on CD to the plane Π_1 gives the length of the shortest line segment. Let Π_2 be the plane containing AB, drawn perpendicular to the plane Π_1. Let Π_3 be the plane containing CD drawn perpendicular to the plane Π_1. The two planes Π_2 and Π_3 have in common PP' the shortest line segment.

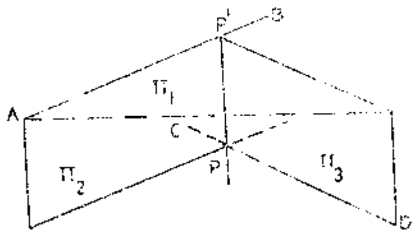


Fig. 2

We now solve the problem by the above method.

Since the plane Π_1 contains AB i.e., line (1) and is further parallel to CD i.e., line (2), its equation is given by

$$\begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 0$$

Perpendicular distance of the point (15, 29, 5) lying on CD from this plane

$$= \frac{30 + 87 + 30 - 49}{7} = 14,$$

which is the required length of the shortest line segment.

Since the plane Π_2 contains line (1) and is further perpendicular to Π_1, its equation is given by

$$\begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or $117x + 4y - 41z - 490 = 0$.

Since the plane Π_3 contains the line (2) and is further perpendicular to Π_1, its equation is given by

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0.$$

or $9x - 4y - z = 14$.

Hence $117x + 4y - 41z - 490 = 0$, $9x - 4y - z - 14 = 0$ are the required equations of the shortest line segment.

Ex. 2: Find the length of the shortest line segment between the axis of Z and the line·

$$ax + by + cz + d = 0, a'x + b'y + c'z + d' = 0.$$

Sol: The third method of Example 1 will prove convenient in this case.

Now the equation of any plane through the second given line is

$$ax + by + cz + d + k\ (a'x + b'y + c'z + d') = 0$$

or $(a + ka')\,x + (b + kb')\,y + (c + kc')\,z + (d + kd') = 0.$ ... (1)

This will be parallel to Z-axis whose direction cosines are 0, 0, 1 if the normal to (1) is perpendicular to Z-axis i.e., if

$$0 \cdot (a + ka') + 0 \cdot (b + kb') + 1 \cdot (c + kc') = 0,$$

$$\text{or } k = -c/c'.$$

Substituting this value of k in (1), we obtain

$$(ac' - a'c)\,x + (bc' - b'c)\,y + (dc' - d'c) = 0 \quad \text{... (2)}$$

as the required equation of the plane which passes through the second line and as one parallel to the first line.

The required length of the shortest line segment is the distance of any point on Z–axis (in particular the origin) from plane (2). The distance from (0, 0, 0) to the plane (2) is

$$\pm \frac{dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}$$

Ex. 3: Show that the length of the shortest segment between any two opposite edges of the tetrahedron formed by the planes $y + z = 0, z + x = 0, x + y = 0, x + y + z = a$ is $2a/\sqrt{6}$ and that three lines of shortest segment intersect at the point $x = y = z = -a$.

Sol: Let the planes OAC, OBC, OAB and ABC be respectively given in Fig. 3 by $y + z = 0, z + x = 0, x + y = 0, x + y + z = a$. We are required to find the lengths of shortest segments between OC and AB i.e., NE, OB and AC i.e., MD, OA and BC i.e., LF and show that the coordinates of the point R where these lines of shortest segment meet is $(-a, -a, -a)$.

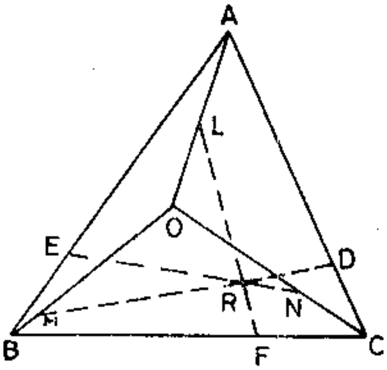


Fig. 3

Solving the equations $y + z = 0, z + x = 0$, we get the equation of OC as

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} = r_1 \text{ (say)} \quad \text{... (1)}$$

Let a_1, b_1, c_1, be the direction ratios of AB. Since AB is perpendicular to both the normals of the planes OAB and ABC, we have

$$a_1 + b_1 = 0,$$

$$a_1 + b_1 + c_1 = 0.$$

Solving, we get $\frac{a_1}{1} = \frac{b_1}{-1} = \frac{c_1}{0}$.

A point on AB can easily seen to be as $(0, 0, a)$. Thus the equation of AB becomes,

$$\frac{x}{1} = \frac{y}{-1} = \frac{z-a}{0} = r_2 \text{ (say)} \qquad \dots (2)$$

For some value of r_1, the co-ordinates of N are $\left(r_1, r_1, -r_1\right)$. Similarly the co-ordinates of E are $\left(r_2, -r_2, a\right)$.

Since EN is perpendicular to both (1) and (2), we get

$$1 \cdot \left(r_1 - r_2\right) + 1 \cdot \left(r_1 + r_2\right) + (-1) \cdot \left(-r_1 - a\right) = 0,$$

$$1 \cdot \left(r_1 - r_2\right) + (-1) \cdot \left(r_1 + r_2\right) + 0 \cdot \left(-r_1 - a\right) = 0.$$

Solving, $r_1 = -a/3$, $r_2 = 0$. Therefore the co-ordinates of N and E are respectively $(-a/3, -a/3, a/3)$, $(0, 0, a)$. Therefore

$$NE = \sqrt{\left(\frac{a}{3}\right)^2 + \left(\frac{a}{3}\right)^2 + \left(\frac{2a}{\sqrt{3}}\right)^2} = \frac{2a}{\sqrt{6}}$$

Similarly, $MD = LF = 2a/\sqrt{6}$. This proves the first part of the question.

The equation of NE is $\frac{x}{a/3} = \frac{y}{a/3} = \frac{z-a}{2a/3}$

$$\text{or } \frac{x}{1} = \frac{y}{1} = \frac{z-a}{2} = r_3 \text{ (say)} \qquad \dots (3)$$

By symmetry the equations of MD and LF are

$$\frac{x-a}{2} = \frac{y}{1} = \frac{z}{1} = r_4 \text{ (say)} \qquad \dots (4)$$

$$\frac{x}{1} = \frac{y-a}{2} = \frac{z}{1} = r_5 \text{ (say)} \qquad \dots (5)$$

The general co-ordinates of any point on (4) are $\left(2r_4 + a, r_4, r_4\right)$, while those of any point on (3) are $\left(r_3, r_3, 2r_3 + a\right)$. In order that (3) and (4) should intersect, we require

$$r_3 = 2r_4 + a,\ r_3 = r_4,\ 2r_3 + a = r_4 \qquad \dots (6)$$

The first two of (6) give $r_3 = r_4 = -a$, while the last of it ensures that the lines (3) and (4) should intersect. Thus the co-ordinates of the point R of intersection of (3) and (4) are $a, -a, -a$. It can be easily verified that this point lies on (5). Hence the three lines of shortest segment intersect at the point $x = y = z = -a$.

SAQ's : State whether the following statements are true or false. Support your answers with reasoning.

i. Two lines either intersect at a unique point or they will be parallel to each other.

ii. Line of shortest distance is perpendicular to both the skew lines between which the shortest distance is to be determined.

10.4 Summary

The equations of the line of shortest distance between the two skew lines.

$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

where l, m, n are direction consines of the line of shortest distance.

Length of the line of shortest distance $= \left|\left(x_2 - x_1\right) l + \left(y_2 - y_1\right) m + \left(z_2 - z_1\right) n\right|$

10.5 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Obtain the length and equations of the shortest line segment between two skew lines.

b) Find the length and the equations of the line of shortest segment between the two lines·

$$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}; \quad \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}.$$

(ii) a) Find the length and the equations of the shortest line segment between

$5x - y - z = 0, \quad x - 2y + z + 3 = 0.$

$7x - 4y - 2z = 0, \quad x - y + z - 3 = 0.$

b) Find the length of the shortest segment between the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}.$ Hence show that the lines are coplanar.

II. *Briefly answer the following questions.*

i) Find the length and the position of the shortest line segment between the lines

$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$; $5x - 2y - 3z + 6 = 0, x - 3y + 2z - 3 = 0.$

ii) Show that the length of the shortest segment between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$ is $2a$.

iii) Obtain the co-ordinates of the points where the shortest segment between the lines

$$\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}, \quad \frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}.$$

iv) Find the length and the equations of the shortest line segment between the two planes

$$\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}; \quad \frac{x-1}{1} = \frac{y-7}{3} = \frac{z+2}{2}.$$

v) Show that the equation to the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1, y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$ and if $2d$ is the length of the shortest segment, prove that $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$

vi) Show that the lengths of the shortest segments between a diagonal of a rectangular parallelopiped whose edges are a, b, c and the edges not meeting are $bc/\sqrt{b^2+c^2}$, $ca/\sqrt{c^2+a^2}$, $ab/\sqrt{a^2+b^2}$.

Answers

I. (ib) $4\sqrt{3}, x = y = z.$

(iia) $13/\sqrt{75}$, $17x + 20y - 19z - 39 = 0 = 8x + 5y - 31z + 67.$

(iib) 0.

II. (i) $17\sqrt{6}/39, 7x - 2y - 11z + 20 = 0 = 13x - 13z + 24.$

(iii) (11, 11, 31) and (3, 5, 7). (iv) $\sqrt{35}, (x-4) = (y-2)/3 = -(z+3)/5.$

10.6 Answers to Self Assessment Questions

i. False. Only when the lines are coplanar, they either intersect at a point or they are parallel to each other. In case, they lie in skew planes, they do not either intersect or parallel.

ii. True.

Unit–11 : TWO OR MORE PLANES

11.0 Contents

11.1 Aims and Objectives

After going through this unit, you will be able to :

i) investigate the nature of intersection of three planes,

ii) obtain the formula for volume of a tetrahedrón in terms of its vertices.

11.2 Introduction

In this unit we study two or more planes. The necessary conditions for a homogeneous second degree equation in x, y, z to represent a pair of planes have been obtained. Conversely, under these conditions it has been shown that a homogeneous second degree equation in x, y, z represents a pair of planes. The volume of a tetrahedron in terms of the co-ordinates of its four vertices has been obtained.

Finally, conditions have been obtained for three given planes as to when they form a single common line, a prism and a single common point.

11.3 Joint equation of two planes

We find necessary and sufficient conditions that a homogeneous second degree equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad ...(1)$$

may represent two planes.

Let us assume that (1) represents a pair of planes

$$lx + my + nz = 0 \qquad ...(2)$$

$$l'x + m'y + n'z = 0 \qquad ...(3)$$

There cannot appear constant terms in eqs. (2) and (3) for otherwise their joint equation will not be homogeneous. We may write without loss of generality

$$(lx + my + nz)(l'x + m'y + n'z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

Comparing coefficients, we obtain

$a = ll', b = mm', c = nn'$

and $2f = m'n + mn', 2g = l'n + ln', 2h = lm' + l'm$.

To obtain the necessary condition, we have to eliminate $l, m, n; l', m', n'$ from the above six relations. The eliminant can be obtained as follows.

We have always

$$0 = \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}$$

$$= \begin{vmatrix} ll'+l'l & lm'+l'm & ln'+l'n \\ lm'+l'm & mm'+m'm & m'n+mn' \\ ln'+l'n & m'n+mn' & nn'+n'n \end{vmatrix}$$

$$= 8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 8\,(abc + 2fgh - af^2 - bg^2 - ch^2).$$

Hence $$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

is a necessary condition.

Since $$(ll' - mm')^2 = (ll' + mm')^2 - 4\,ll'mm'$$
$$= 4h^2 - 4ab \geq 0,$$

we have also the condition $h^2 \geq ab$. Similarly the other conditions $g^2 \geq ac, f^2 \geq bc$ follow.

Summing up, the necessary conditions are

$D = 0,\ h^2 \geq ab,\ g^2 \geq ac$ and $f^2 \geq bc$.

Conversely, given the above conditions, we prove that (1) represents a pair of planes passing through the origin.

We consider the possibility $D = 0,\ h^2 \geq ab,\ g^2 \geq ac$ and $f^2 \geq bc$.

Then we can write

$$ax^2 + 2hxy + by^2 = (lx + my)\,(l'm + m'y).$$

Comparing coefficients, we have $a = ll', b = mm'$ and $2h = lm' + ml'$

Since $$(ll' - mm')^2 = 4\,(h^2 - ab) > 0.$$

We have $$\frac{l}{l'} \neq \frac{m}{m'}$$

If we write $$ln' + nl' = 2g,$$
$$mn' + nm' = 2f$$

n and n' can be determined in view of $lm' - ml' \neq 0$. If we set $nn' = c'$ then we can verify

$(lx + my + nz)\,(l'x + m'y + n'z) \equiv ax^2 + by^2 + c'z^2 + 2\,fyz + 2gzx + 2hxy.$

Then $ax^2 + by^2 + cz^2 + 2\,fyz + 2gzx + 2hxy = 0$ represents a pair of planes, the condition for which as already obtained is

$$D' = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c' \end{vmatrix} = 0$$

But we are given $$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\text{Therefore} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c' \end{vmatrix}$$

or $c\,(h^2 - ab) = c'\,(h^2 - ab)$.

Since by hypothesis $h^2 \neq ab$, we have $c = c'$.

Thus $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes

$$lx + my + nz = 0,$$

$$l'x + m'y + n'z = 0$$

passing through the origin.

Note : $D = 0$ is not a sufficient condition for the eq. (1) to represent a pair of planes, for example $x^2 + 2y^2 + z^2 + 2yz + 2zx + 2xy = 0$ satisfies the condition $D = 0$ but does not represent a pair of planes since $h^2 < ab$. In otherwords the quadratic expression in the equation cannot be split into real factors.

Cor : Let θ be the angle between the pair of planes represented by (1). Since the angle between two planes is same as the angle between their normals whose direction ratios are $l, m, n;\ l', m', n'$, we have from Cor. of 7.9.

$$\tan\theta = \frac{\sqrt{(mn' - m'n)^2 + (nl' - ln')^2 + (lm' - ml')^2}}{ll' + mm' + nn'}$$

$$= \frac{2\sqrt{f^2 + g^2 + h^2 - ab - ac - bc}}{a + b + c}$$

which is always real since $h^2 \geq ab$, $g^2 \geq ac$ and $f^2 \geq bc$

The planes will be at right angles if $a + b + c = 0$ i.e. then θ is 90°.

11.4 Orthogonal projection on a plane

Def : The foot of the perpendicular drawn from any point P to a given plane Π is called the orthogonal projection of the point P on the plane Π.

It is easily seen from the above that the projection of the area enclosed by a plane curve is the area enclosed by the projection of the curve on the plane of projection. Let A be the area enclosed by a plane curve and θ be the angle between the plane of area and the plane of projection. Then the area projected is $A \cos\theta$. This is well known result of pure solid geometry and therefore, assumed.

Theorem : If A_x, A_y, A_z be the areas of the projections of an area A and the three co-ordinate planes, then

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

Proof : Let l, m, n be direction cosines of the normal to the plane of the area A.

Since l is the cosine of the angle between the YZ plane and the plane of the area A, we have

$$A_x = l \,.\, A$$

Similarly

$$A_y = m \,.\, A$$

$$A_z = n \,.\, A$$

Hence $A_x^2 + A_y^2 + A_z^2 = (l^2 + m^2 + n^2)\, A^2 = A^2$

11.5 Volume of a Tetrahedron

We find the volume of a tetrahedron whose vertices are (x_r, y_r, z_r); $r = 1, 2, 3, 4$.

In a note of 8.4.6 using vectors it has been shown that the volume of a tetrahedron is the absolute value of $\frac{1}{6}\Delta_1$ where

$$\Delta_1 = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

We shall derive it without using vectors.

Let A, B, C, D form the vertices of a tetrahedron whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) repectively. Let L be the foot of the perpendicular from A to the plane BCD and let AL be p (fig. 1).

If V be the volume of the tetrahedron, then it is known from Block 1,

$$V = \frac{1}{3}\left|p\,\Delta\right| \qquad ...(1)$$

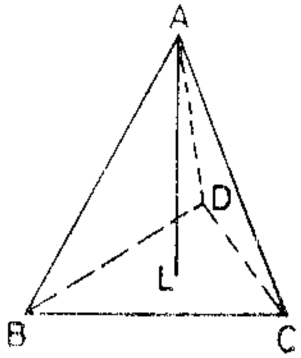


Fig. 1

where Δ is the area of the triangle BCD and $|p\,\Delta|$ stands for the absolute value of $p\,\Delta$.

The equation of the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

or
$$x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0$$

The length of the perpendicular from $A\left(x_1, y_1, z_1\right)$ to this plane is given by

$$p = \pm\frac{x_1\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y_1\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}}{\left\{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}}$$

the denominator of which is not zero. ... (2)

The numerator of $p = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

Let $\Delta_x, \Delta_y, \Delta_z$, be the areas of the projection of Δ on YZ, ZX, XY planes. Then from two-dimensional co-ordinate geometry, we know

$$\Delta_x = \pm\frac{1}{2}\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}; \Delta_y = \pm\frac{1}{2}\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}; \Delta_z = \pm\frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

Since $\Delta_x^2 + \Delta_y^2 + \Delta_z^2 = \Delta^2$, we see that the denominator of p in (2) is equal to $2\,\Delta$. Therefore from (2),

$$2\,p\,\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \qquad \ldots (3)$$

Therefore from (1) and (3), we get

$$V = \frac{1}{3}p \cdot \frac{1}{2p}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

whose absolute value is to be taken into account.

11.6 Intersection of Three Planes

Any three planes (no two of which are parallel) intersect in one of the following ways :

(i) The planes may have a common line of intersection.

(ii) The planes may form a triangular prism, if the line of interesection of two of them is parallel to the third but does not lie on it.

(iii) The planes may meet in a unique point, if the line of intersection of two of them is not parallel to the third.

The above are the only possibilities since of the three planes any two intersect in a line and in turn this line intersects the third or not. If the line interesects the plane (i) it may lie wholly in the plane i.e., intersect the third plane in a single point i.e., the planes may meet in a unique point. If the line of itersection of the two planes does not meet the third, then it shall be parallel to the third. In this case the planes will form a prism.

Let us assume the three planes are given by

$$a_1x + b_1y + c_1z + d_1 = 0, \qquad \ldots (1)$$

$$a_2x + b_2y + c_2z + d_2 = 0, \qquad \ldots (2)$$

and $\qquad a_3x + b_3y + c_3z + d_3 = 0, \qquad$ ... (3)

of which (1) and (2) are not parallel.

We shall investigate the conditions as to when the above planes satisfy (i), (ii) and (iii).

Since by our assumption (1) and (2) intersect, the equation to their common line of interesection from 9.3.4 is given by

$$\frac{x - \left(b_1d_2 - b_2d_1\right) \Big| \left(a_1b_2 - a_2b_1\right)}{b_1c_2 - b_2c_1} = \frac{y - \left(a_2d_1 - b_1d_2\right) \Big| \left(a_1b_2 - a_2b_1\right)}{a_2c_1 - a_1c_2}$$

$$= \frac{z}{a_1b_2 - a_2b_1} \qquad \text{... (4)}$$

This line will be parallel to the plane (3), if

$$a_3 \left(b_1 c_2 - b_2 c_1\right) + b_3 \left(a_2 c_1 - a_1 c_2\right) + c_3 \left(a_1 b_2 - a_2 b_1\right) = 0$$

i. e., $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Now two cases arise :

(a) The line (4) lies wholly in the plane (3). In this case the planes (1 to 3) have a common line of intersection (fig. 2a).

(b) The line (4) does not intersect the plane (3). In this case the three planes form a prism (fig. 2b).

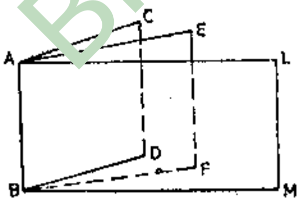


Fig. 2a

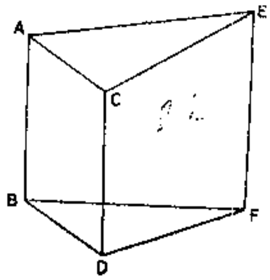


Fig. 2b

The line (4) lies wholly in the plane (3) if and only if a point of (4), say,

$\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1} ; \dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1} ; 0\right)$ lies in the plane i. e.,

$$a_3 \left(b_1 d_2 - b_2 d_1\right) + b_3 \left(a_2 d_1 - a_1 d_2\right) + d_3 \left(a_1 b_2 - a_2 b_1\right) = 0$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$$

If the common line of (1) and (2) intersects either YZ or ZX plane, we get similarly the condition for a common line of intersecton of the three planes to be $\Delta_1 = 0$ or $\Delta_2 = 0$ together with $\Delta = 0$.

If the common line of (1) and (2) does not intersect (3), none of $\Delta_1, \Delta_2, \Delta_3$ vanishes.

Finally if $\Delta \neq 0$, then the common line of intersection of (1) and (2) intersects (3) in a single point. In this case the three planes have only a single common point.

Concluding we say that three distinct non-parallel planes behave in relation to each other in any of the following three ways.

(i) They may intersect in a line which requires that of the four determinants, Δ and any one of $\Delta_1, \Delta_2, \Delta_3$ should vanish.

(ii) They may form a prism which requires Δ should vanish and none of $\Delta_1, \Delta_2, \Delta_3$ should vanish.

(iii) They may intersect in a unique point which requires $\Delta \neq 0$.

Examples :

Ex. 1 : Show that $12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$

represents a pair of planes and find the angle between them.

Sol. : Comparing the given equation with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$a = 12,\ b = -2,\ c = -6,\ f = 7/2,\ g = 3$ and $h = -1$.

In order that the given equation may represent a pair of planes, we require

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e., $12 \times (-2) \times (-6) + 7 \times 3 \times (-1) - 12 \times \frac{7}{2} \times \frac{7}{2} + 2 \times 3 \times 3 + 6 \times (-1) \times (-1) = 0.$

Also one can easily verify

$h^2 \geq ab,\ g^2 \geq ac$ and $f^2 \geq bc$.

If θ is the angle between the pair of planes then

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c} = \frac{2\sqrt{\frac{49}{4} + 9 + 1 + 24 - 12 + 72}}{4}$$

$$= \frac{\sqrt{425}}{4}.$$

Therefore $\cos\theta = \frac{4}{21}$ or $\theta = \cos^{-1}\left(\frac{4}{21}\right)$

Ex. 2 : Find the area of the triangle whose vertices are the points (1, 2, 3), (−2, 1, −4), (3, 4, −2).

Sol. : Let A be the required area and A_x, A_y, A_z be the areas of projection of the area A on XY, YZ, ZX planes. The vertices of the projection of the triangle on XY plane are (1, 2, 0), (−2, 1, 0), (3, 4, 0) so that

$$A_x = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ 3 & 4 & 1 \end{vmatrix} = -2.$$

Similarly $A_y = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ -2 & -4 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \frac{29}{2}$, $A_z = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 1 & -4 & 1 \\ 4 & -2 & 1 \end{vmatrix} = \frac{19}{2}$

Therefore the area of the triabgle A

$$= \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{4 + \left(\frac{29}{2}\right)^2 + \left(\frac{19}{2}\right)^2} = \frac{\sqrt{1218}}{2}.$$

Ex.3 : The vertices of a tetrahedron are (0, 1, 2), (3, 0, 1), (4, 3, 6), (2, 3, 2). Show that its volume is 6.

Sol. : The volume of the tetrahedron is

$$\frac{1}{6}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 3 & 6 & 1 \\ 2 & 3 & 2 & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 3 & -1 & -1 & 0 \\ 4 & 2 & 4 & 0 \\ 2 & 2 & 0 & 0 \end{vmatrix}$$

$$= -\frac{1}{6}\begin{vmatrix} 3 & -1 & -1 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} 3 & -1 & -1 \\ 16 & -2 & 0 \\ 2 & 2 & 0 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 16 & -2 \\ 2 & 2 \end{vmatrix}$$

$$= \frac{1}{6} \times 36 = 6.$$ Hence the result.

Ex. 4 : Find the volume of the tetrahedron in terms of three edges which meet in a point and of the angles which they make with each other.

Sol. : Let $OABC$ be a tetrahedron. Let $OA = a$, $OB = b$, $OC = C$, Angle $BOC = \lambda$, angle $COA = \mu$ and angle $AOB = \nu$ (fig. 3).

We take O as origin and any system of three mutually perpendicular lines through O as co-ordinate axes. Let the direction cosines of OA, OB, OC be l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 respectively.

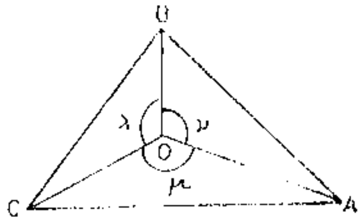


Fig. 3

From 7.7 (i), the co-ordinates of A, B, C are then given by $(l_1 a, m_1 a, n_1 a)$, $(l_2 b, m_2 b, n_2 b)$ $(l_3 c, m_3 c, n_3 c)$. The volume of the tetrahedron $OABC$ is the absolute value of

$$\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ l_1 a & m_1 a & n_1 a & 1 \\ l_2 b & m_2 b & n_2 b & 1 \\ l_3 c & m_3 c & n_3 c & 1 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} l_1 a & m_1 a & n_1 a \\ l_2 b & m_2 b & n_2 b \\ l_3 c & m_3 c & n_3 c \end{vmatrix} = -\frac{abc}{6}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Now $$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$ and

$$\begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_1 l_3 & \Sigma l_2 l_3 & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

$$\text{Therefore,} \quad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

We now get the volume of the tetrahedron $OABC$ as the absolute value of

$$\frac{abc}{6} \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

Ex. 5 : Show that the volume of the tetrahedron, the equations of whose faces are

$$a_r x + b_r y + c_r z + d_r = 0, \quad r = (1, 2, 3, 4)$$

is the absolute value of $\dfrac{\Delta^3}{6 D_1 D_2 D_3 D_4}$ where Δ is the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and D_1, D_2, D_3, D_4 are the cofactors of d_1, d_2, d_3, d_4 respectively in the determinant Δ and none of them is zero.

Sol. : Let (x_1, y_1, z_1) be the point of intersection of the three planes

$$a_r x + b_r y + c_r z + d_r = 0, \quad r = (2, 3, 4).$$

Similarly, let (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the other vertices obtained when $r = (3, 4, 1), (4, 1, 2), (1, 2, 3)$.

We write

$$a_1 x_1 + b_1 y_1 + c_1 z_1 + d_1 = k_1$$

$$\text{i.e.,} \; a_1 x_1 + b_1 y_1 + c_1 z_1 + (d_1 - k_1) = 0. \qquad \text{... (1)}$$

Also we have

$$a_2 x_1 + b_2 y_1 + c_2 z_1 + d_2 = 0, \qquad \text{... (2)}$$

$$a_3 x_1 + b_3 y_1 + c_3 z_1 + d_3 = 0, \qquad \text{... (3)}$$

$$a_4 x_1 + b_4 y_1 + c_4 z_1 + d_4 = 0, \qquad \text{... (4)}$$

Eliminating x_1, y_1, z_1 from (1), (2), (3), (4), we get

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 - k_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

or $$\Delta + k_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0$$

or $\Delta - k_1 D_1 = 0$, Hence $k_1 = \dfrac{\Delta}{D_1}$.

Similarly, $a_2 x_2 + b_2 y_2 + c_2 z_2 + d_2 = k_2 = \dfrac{\Delta}{D_2}$.

$a_3 x_3 + b_3 y_3 + c_3 z_3 + d_3 = k_3 = \dfrac{\Delta}{D_3}$.

$a_4 x_4 + b_4 y_4 + c_4 z_4 + d_4 = k_4 = \dfrac{\Delta}{D_4}$.

Remebering $a_r x_s + b_r y_s + c_r z_s + d_r = 0, r \neq s$, $(r = 1, 2, 3, 4;\ s = 1, 2, 3)$, we now have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \begin{vmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{vmatrix}$$

or $\Delta \times 6$ (volume of the tetrahedron) $= k_1 k_2 k_3 k_4 = \dfrac{\Delta^4}{D_1 D_2 D_3 D_4}$

or the volume of the tetrahedron is the absolute value of $\dfrac{\Delta^3}{6 D_1 D_2 D_3 D_4}$

Ex. 6 : Examine the nature of the intersection of the following sets of planes.

(i) $4x - 5y - 2z - 2 = 0,\ 5x - 4y + 2z + 2 = 0,\ 2x + 2y + 8z - 1 = 0.$

(ii) $2x + 3y - z - 2 = 0,\ 3x + 3y + z - 4 = 0,\ x - y + 2z - 5 = 0.$

(iii) $5x + 3y + 7z - 4 = 0,\ 3x + 26y + 2z - 9 = 0,\ 7x + 2y + 10z - 5 = 0.$

(i) $$\Delta = \begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{vmatrix} = 0,\quad \Delta_1 = \begin{vmatrix} -5 & -2 & -2 \\ -4 & 2 & 2 \\ 2 & 8 & -1 \end{vmatrix} \neq 0;$$

$$\Delta_2 = \begin{vmatrix} 4 & -2 & -2 \\ 5 & 2 & 2 \\ 2 & 8 & -1 \end{vmatrix} \neq 0, \quad \Delta_3 = \begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & -1 \end{vmatrix} \neq 0.$$

Since $\Delta = 0$ and none of $\Delta_1, \Delta_2, \Delta_3$ is zero, the set of given planes form a prism.

(ii) $$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} \neq 0.$$

Therefore the set of planes intersect in a point.

(iii) $$\Delta = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix} = 0; \quad \Delta_3 = \begin{vmatrix} 5 & 3 & -4 \\ 3 & 26 & -9 \\ 7 & 2 & -5 \end{vmatrix} = 0.$$

Since $\Delta = 0$ and one of $\Delta_1, \Delta_2, \Delta_3$ (here Δ_3) vanishes, the set of given planes intersect in a line.

Ex. 7 : Prove that the three planes

$2x + y + z = 3$; $x - y + 2z = 4$, $x + z = 2$

form a triangular prism and find the area of the normal section of the prism.

Sol. : Let Π_1, Π_2, Π_3 be the planes (fig. 4) respectively given by the equations

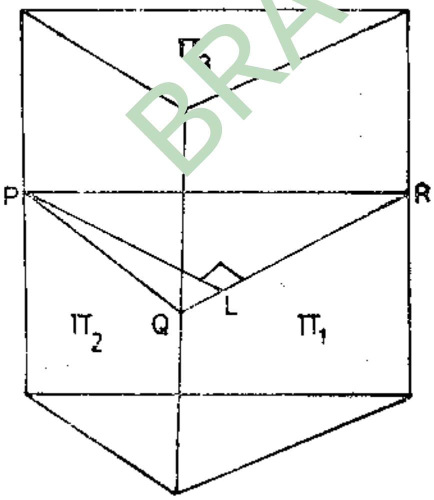


Fig. 4

$$2x + y + z = 3 \qquad \text{... (1)}$$

$$x - y + 2z = 4 \qquad \text{... (2)}$$

$$x + z = 2 \qquad \text{... (3)}$$

Let a, b, c be the direction ratios of the common line of intersection of the planes Π_2 and Π_3. Since this common line lies in both the planes, we have

$$a - b + 2c = 0,$$

$$a + c = 0.$$

Solving, $\dfrac{a}{-1} = \dfrac{b}{1} = \dfrac{c}{1}$ or $\dfrac{a}{1} = \dfrac{b}{-1} = \dfrac{c}{-1}$.

To find a point P on this common line, we may assume without loss of generality that the line intersects the YOZ plane i.e., the x co-ordinate of the point P is zero. To find its y, z co-ordinates we substitute $x = 0$ in (2) and (3) and obtain.

$$-y + 2z = 4$$

$$\text{and} \quad z = 2.$$

Solving, we get $y = 0$, $z = 2$. Thus P has the co-ordinates (0, 0, 2).

The common line of (2) and (3) is parallel to the plane Π_1 for

$$2.1 + 1.(-1) + (-1) = 0$$

and P is not a point on Π_1. Hence the given planes form a triangular prism.

Let the triangle PQR be its normal section. The equation of the plane through P perpendicular to the common line of (2) and (3) i.e., the equation of the normal section PQR is

$$1.(x-0) - 1.(y-0) - 1.(z-2) = 0$$

or $$x - y - z + 2 = 0 \qquad \text{... (4)}$$

The common point of the planes (1), (2) and (4) i.e., Q has the co-ordinates $\left(\frac{1}{3}, \frac{1}{2}, 2\right)$. This is obtained by solving eqs. (1), (2) and (4). Similarly the common point of the planes (1), (3) and (4) i.e., R has the co-ordinates $\left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)$. The length

$$QR = \sqrt{\left(\frac{1}{3} - \frac{1}{3}\right)^2 + \left(\frac{1}{2} - \frac{2}{3}\right)^2 + \left(2 - \frac{5}{3}\right)^2} = \sqrt{\frac{2}{9}}$$

Also perpendicular from P on Π_1 i.e., PL is $\frac{3-2}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}$.

Hence the area of the triangle PQR or the normal section of the prism

$$= \frac{1}{2} \times QR \times PL = \frac{1}{2}\sqrt{\frac{2}{9}} \cdot \frac{1}{\sqrt{6}} = \frac{1}{6\sqrt{3}}$$

11.7 Summary

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

represents a pair of planes through origin if and only if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

If $a + b + c = 0$ and $\Delta = 0$ then the planes are perpendicular to each other.

If A_x, A_y, A_z denote projections of the area A on the coordinate planes, then $A^2 = A_x^2 + A_y^2 + A_z^2$. Volume of a tetrahedron with vertices (x_r, y_r, z_r) $r = 1, 2, 3, 4$, is

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Any three planes may intersect in a common line or may form a triangular prism or may meet in a unique point.

11.8 Sample Examination questions

I. Answer the following questions in detail.

(i) (a) Obtain the necessary and sufficient condition that a homogeneous second degree equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ to represent two planes.

(b) Show that the equation $2x^2 - 2y^2 + 4z^2 + 6xz + 2yz + 3xy = 0$ represents a pair of planes and also find the angle between them.

(ii) (a) Obtain the volume of a tetrahedron whose vertices are (x_r, y_r, z_r); $r = 1, 2, 3, 4$.

(b) Find the volume of the tetrahedron formed by the planes, whose equations are $z + y = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$.

(iii) (a) Discuss various cases of the intersection of three planes (no two of which are parallel).

(b) Show that the planes $4x + 3y + 2z + 7 = 0$, $2x + y - 4z + 1 = 0$, $x - 7z - 2 = 0$ interesect in a line.

II. Briefly answer the following questions.

(i) From a point $P\ (x', y', z')$, a plane is drawn at right angles to OP to meet the co-ordinate axes at A, B, C. Prove that the area of the triangle ABC is $r^5/2x'y'z'$, where r is the measure of OP.

(ii) A variable plane makes with the co-ordinate planes a tetrahedron of constant volume $64k^3$. Find (a) the locus of the centroid of the tetrahedron (b) the locus of the foot of the perpendicular from the origin to the plane.

(iii) Show that the planes $x + y + z + 3 = 0$, $3x + y - 2z + 2 = 0$, $2x + 4y + 7z - 7 = 0$ form a triangular prism.

(iv) Examine the nature of the intersection of the planes $2x - y + z = 4$, $5x + 7y + 2z = 0$, $3x + 4y - 2z + 3 = 0$.

(v) Examine the nature of the intersection of the planes $3x - y + z = 5$, $2x + 4y + z + 10 = 0$, $6x - 2y + 2z + 9 = 0$.

(vi) Prove that the three planes $2x + y + z = 3$, $x - y + 2z = 4$, $x + z = 2$ form a triangular prism and find the area of a normal section of the prism.

(vii) Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ pass through one line if $a^2 + b^2 + c^2 + 2abc = 1$ and show that the line of intersection, then, is

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}.$$

Answers :

I. (ib) $\cos^{-1}(4/9)$ (iib) 2/3.

II. (ii) (a) $xyz = 64k^3$ (b) $(x^2 + y^2 + z^2)^3 - 384k^3\,xyz$.

(iv) planes intersect at a point $(1, -1, 1)$.

(v) two planes parallel and the third intersects them.

(vi) $\sqrt{3}\,/\,18$.

Unit–12 : CHANGE OF AXES

12.0 Contents

12.1 Aims and objectives

After going through this unit you will be able to :

(i) transform the coordinates of a rectangular coordinate system into another system of rectangular coordinates, both passing through the same origin,

(ii) convert a homogenous second degree expression from one system to the other and establish the invariance of certain expressions involving the coefficients of the coordinates of a second degree expression in each system.

12.2 Introduction

The co-ordinates of a point in space are always determined relatively to any assigned system of axes generally called the frame of reference and they change with the change in the frame of reference. In this unit we assume both the frames of reference have the same origin and obtain the formulae connecting the co-ordinates of a point relative to the two different frames of reference. We also studied the relations between the direction cosines of three mutually perpendicular lines of one system relative to the axes of the second rectangular system. We have shown by change of axes without change of origin a second degree homogenous expression remains a second degree homogenous expression. In doing so the invariance of certain expressions involving coefficients of the expression has been established. An elementary knowledge of matrices is assumed.

3 Change of the Directions of Axes

We change the directions of axes without changing the origin (fig. 1).

Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the respective direction cosines of the new axes OX', OY', OZ', referred to the original axes OX, OY, OZ.

Let (x, y, z) and (x', y', z') be the co-ordinates of any point P referred to the two systems of axes.

Draw PN' perpendicular to $X'OY'$ plane meeting it in N' and also $N'L'$ perpendicular to OX' meeting it in L' so that

$$OL' = x', \; L'N' = y', \; N'P = z'.$$

Now, the projection of OP on OX being equal to the sum of the projections of OL', $L'N'$, $N'P$ on OX, we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z'. \\ \text{Similarly,} \quad y &= m_1 x' + m_2 y' + m_3 z' \\ \text{and} \quad z &= n_1 x' + n_2 y' + n_3 z'. \end{aligned}\right\} \text{(A)}$$

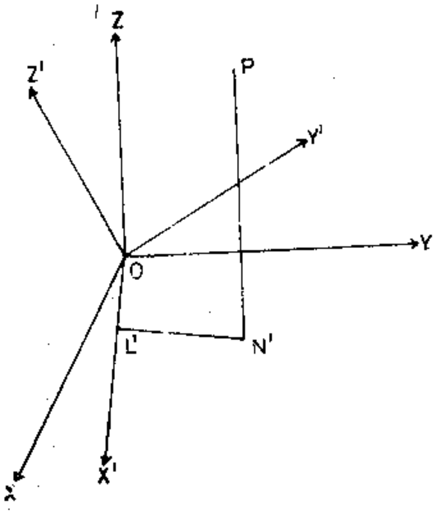


Fig. 1

In terms of matrices, we can express (A) as

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = AX'. \qquad \text{(A')}$$

Similarly, we can show that

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z, \\ y' &= l_2 x + m_2 y + n_2 z, \\ z' &= l_3 x + m_3 y + n_3 z, \end{aligned}\right\} \qquad \text{(B)}$$

and in matrix notation

$$X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A'X \qquad \text{(B')}$$

Thus we have $X = AA'X$; so that $AA' = I$, the identity matrix. Since A is the inverse of A, we have also $A'A = I$.

12.4 Relations between the direction cosines of the three mutually perpendicular axes of the two systems

l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 being the direction cosines of the three mutually perpendicular axes OX', OY', OZ' we have the relations

$$\left.\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1, \\ l_2^2 + m_2^2 + n_2^2 &= 1, \\ l_3^2 + m_3^2 + n_3^2 &= 1 \end{aligned}\right\} \qquad \text{(C)}$$

and

$$\left.\begin{aligned} l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0, \\ l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0, \\ l_3 l_1 + m_3 m_1 + n_3 n_1 &= 0 \end{aligned}\right\} \qquad \text{(C')}$$

The relations (C), (C') also follow from $A'A = I$.

Since $l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$ are clearly the direction cosines of the original axes OX, OY, OZ referred to the new, we have the relations

$$\left.\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1; \\ m_1^2 + m_2^2 + m_3^2 &= 1; \\ n_1^2 + n_2^2 + n_3^2 &= 1; \end{aligned}\right\} \quad \text{(D)}$$

and

$$\left.\begin{aligned} l_1 m_1 + l_2 m_2 + l_3 m_3 &= 0; \\ m_1 n_1 + m_2 n_2 + m_3 n_3 &= 0; \\ n_1 l_1 + n_2 l_2 + m_3 l_3 &= 0; \end{aligned}\right\} \quad \text{(D')}$$

which also follow from $AA' = I$.

We have written down the following table.

0	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Table 12.1

which is to be interpreted as $x = l_1x' + l_2y' + l_3z'$ etc. of (A) and $x' = l_1x + m_1y + n_1z$. etc. of (B). Also the table helps us to read the direction cosines of OX', OY', OZ' referred to OX, OY, OZ and vice-versa, e.g., D.C's of OX' (relative to OX, OY, OZ) are l_1, m_1, n_1 and D.C's of OX (relative to OX', OY', OZ') are l_1, l_2, l_3 etc.

Cor: If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the direction cosines of three mutually perpendicular lines, then

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$$

From 12.3, we have

$$\begin{aligned} AA' &= I \\ \Rightarrow |AA'| &= 1, \\ \Rightarrow |A|\ |A'| &= 1, \\ \Rightarrow |A'|^2 &= 1, \\ \Rightarrow |A'| &= \pm 1. \end{aligned}$$

12.5 Invariants

Def: Let OX, OY, OZ, and $O'X', O'Y', O'Z'$ be two orthogonal co-ordinate systems. Let the transform relations from one to the other be

$$\begin{aligned} x &= f_1(x', y', z'); \\ y &= f_2(x', y', z'); \\ z &= f_3(x', y', z'); \end{aligned}$$

Then the function $\phi(x, y, z)$ or the equation $\phi(x, y, z) = 0$ is said to be transformed into $\psi(x', y', z')$ or $\psi = 0$ where

$$\psi(x', y', z') = \phi\left\{f_1(x', y', z'), f_2(x', y', z'), f_3(x', y', z')\right\}.$$

Theorem: By any change of rectangular axes without change of origin, the expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

becomes

$$a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y'$$

then

(i) $$a + b + c = a' + b' + c'.$$

(ii) $$ab + bc + ca - f^2 - g^2 - h^2 = a'b' + b'c' + c'a' - f'^2 - g'^2 - h'^2,$$

(iii) $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$$

Proof: Consider two sets of rectangular axes OX, OY, OZ and OX', OY', OZ'. Let OX', OY', OZ' have the direction cosines l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 referred to OX, OY, OZ respectively. Let P be any point having co-ordinates (x, y, z), (x', y', z') relative to two systems of axes. We have

$$x^2 + y^2 + z^2 = OP^2 = x'^2 + y'^2 + z'^2.$$

Thus we see that $\quad x^2 + y^2 + z^2$

changes to $\quad x'^2 + y'^2 + z'^2.$

Since

$$x = l_1x' + l_2y' + l_3z';$$
$$y = m_1x' + m_2y' + m_3z';$$
$$z = n_1x' + n_2y' + n_3z';$$

as given in (A), the expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

changes to $$a\left(l_1x' + l_2y' + l_3z'\right)^2 + b\left(m_1x' + m_2y' + m_3z'\right)^2 + \ldots$$

which is evidently of the form

$$a'x'^2 + b'y'^2 \quad c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y'.$$

Thus the degree of an expression does not change as a result of the change of axes.

If λ be any constant, the expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + \lambda\left(x^2 + y^2 + z^2\right)$$

i.e., $$(a + \lambda)x^2 + (b + \lambda)y^2 + (c + \lambda)z^2 + 2fyz + 2gzx + 2hxy \qquad \ldots(1)$$

changes in the new coordinate system to

$$a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y' + \lambda\left(x'^2 + y'^2 + z'^2\right)$$

i.e., $$(a' + \lambda)x'^2 + (b' + \lambda)y'^2 + (c' + \lambda)z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y'. \qquad \ldots(2)$$

If now, for any value of λ, the expression (1) becomes a product of two linear factors, then for the same value of λ the expression (2) must also become a product of two linear factors, for the degree of each of the expressions (1) and (2) is two.

The values of λ for which the expression (1) and (2) are the products of linear factors are respectively the roots of the cubic equations.

$$\begin{vmatrix} a+\lambda & h & g \\ h & b+\lambda & f \\ g & f & c+\lambda \end{vmatrix} = 0, \quad \begin{vmatrix} a'+\lambda & h' & g' \\ h' & b'+\lambda & f' \\ g' & f' & c'+\lambda \end{vmatrix} = 0;$$

i. e., $\lambda^3 + \lambda^2 (a + b + c) + \lambda \left(bc + ca + ab - f^2 - g^2 - h^2\right) + D = 0,$ ... (3)

$\lambda^3 + \lambda^2 (a' + b' + c') + \lambda \left(b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2\right) + D' = 0,$ ... (4)

where $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad D' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$

As the equations (3) and (4) have the same roots, we get

$$\frac{1}{1} = \frac{a + b + c}{a' + b' + c'} = \frac{bc + ca + ab - f^2 - g^2 - h^2}{b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2} = \frac{D}{D'},$$

so that $a + b + c = a' + b' + c'$

$$bc + ca + ab - f^2 - g^2 - h^2 = b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2$$

$$D = D'.$$

Note 1 : The result obtained above shows that if a second degree homogeneous expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

is subjected to a change of rectangular axes without change of origin, then

$a + b + c$, $bc + ca + ab - f^2 - g^2 - h^2$, D are invariants.

Note 2 : It may be seen that

$$bc + ca + ab - f^2 - g^2 - h^2 = A + B + C$$

where A, B, C are the cofactors of a, b, c in the determinant D.

Examples :

Ex. 1 : OA, OB, OC are three mutually perpendicular lines through the origin and their direction cosines are

$$l_1, m_1, n_1 \; ; \; l_2, m_2, n_2 \; ; \; l_3, m_3, n_3 .$$

If $OA = OB = OC = a$, prove that the equation to the plane ABC is

$$\left(l_1 + l_2 + l_3\right) x + \left(m_1 + m_2 + m_3\right) y + \left(n_1 + n_2 + n_3\right) z = a.$$

Sol. : Let the required equation be

$$lx + my + nz + p = 0. \quad \text{... (1)}$$

($p \neq 0$, since the plane does not pass through origin).

The co-ordinates of A are (al_1, am_1, an_1). The plane (1) passes through A, Therefore we have

$$a(ll_1 + mm_1 + nn_1) + p = 0 \quad \ldots (2)$$

Similarly, we have

$$a(ll_2 + mm_2 + nn_2) + p = 0 \quad \ldots (3)$$

$$a(ll_3 + mm_3 + nn_3) + p = 0 \quad \ldots (4)$$

Multiplying (2), (3), (4) by l_1, l_2, l_3 respectively and additing we get from the eqs. (D) and (D') of 12.4

$$al + p(l_1 + l_2 + l_3) = 0 \quad \ldots (5)$$

Similarly,

$$am + p(m_1 + m_2 + m_3) = 0 \quad \ldots (6)$$

$$an + p(n_1 + n_2 + n_3) = 0 \quad \ldots (7)$$

Multiplying (5), (6), (7) by x, y, z respectively and additing we get the required result.

Ex. 2: Verify that the lines with direction cosines $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}; \frac{-14}{15}, \frac{2}{15}, \frac{1}{3}; \frac{2}{15}, \frac{-11}{15}, \frac{2}{5}$ from a mutually perpendicular system.

Sol: Here $l_1 = \frac{1}{3}, m_1 = \frac{2}{3}, n_1 = \frac{2}{3}$;

$l_2 = \frac{-14}{15}, m_2 = \frac{2}{15}, n_2 = \frac{1}{3}$,

$l_3 = \frac{2}{15}, m_3 = -\frac{11}{15}, n_3 = \frac{2}{3}$.

We can easily verify that from (C) and (C')

$$l_1^2 + m_1^2 + n_1^2 = \frac{1+4+4}{9} = 1;$$

$$l_2^2 + m_2^2 + n_2^2 = \frac{196+4+25}{225} = 1;$$

$$l_3^2 + m_3^2 + n_3^2 = \frac{4+121+100}{225} = 1;$$

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{-14+4+10}{45} = 0;$$

$$l_1l_3 + m_1m_3 + n_1n_3 = \frac{2-22+20}{45} = 0;$$

$$l_2l_3 + m_2m_3 + n_2n_3 = \frac{-28-22+50}{225} = 0, \text{ Hence the result.}$$

Ex. 3 : Let OX', OY', OZ' be a set of three mutually perpendicular lines having the direction cosines $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}; \frac{2}{3}, \frac{2}{3}, \frac{1}{3}; \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$ referred to the three co-ordinate axes OX, OY, OZ. Show that the expression $6x^2 + 11y^2 + 10z^2 - 12yz + 8zx - 4xy$ transforms to $3x'^2 + 6y'^2 + 18z'^2$ when referred to OX', OY', OZ'.

Sol: We know from (*A'*)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Taking the transpose of the above, we get

$$(x \;\; y \;\; z) = (x' \;\; y' \;\; z') \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

We can write $6x^2 + 11y^2 + 10z^2 - 12yz + 8zx - 4xy$ in the form

$$[\, x \;\; y \;\; z \,] \begin{bmatrix} 6 & -2 & 4 \\ -2 & 11 & -6 \\ 4 & -6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which transforms to

$$[\, x' \;\; y' \;\; z' \,] \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ -2 & 11 & -6 \\ 4 & -6 & 10 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$= 3x'^2 + 6y'^2 + 18z'^2$$

which can be verified by direct matrix multiplication.

Ex. 4: Let *AX', AY', AZ'* be a set of three mutually perpendicular axes having the direction cosines $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$; $\frac{-14}{15}$, $\frac{2}{15}$, $\frac{1}{3}$; $\frac{2}{15}$, $\frac{-11}{15}$, $\frac{2}{3}$ referred to the three co-ordinate axes *OX, OY, OZ*. Given *A* (1, 2, 3), show that the equation $3x + 4y - 5z = 11$ transforms to $5x' - 59y' - 88z' = 225$ when referred *AX', AY', AZ'*.

Sol : Shifting the origin *O* to *A* (1, 2, 3), the transform relations given by equation (*A*) become

$$x = 1 + \frac{x'}{3} - \frac{14y'}{15} + \frac{2z'}{15},$$

$$y = 2 + \frac{2x'}{3} + \frac{2y'}{15} - \frac{11z'}{15},$$

$$z = 3 + \frac{2x'}{3} + \frac{y'}{3} + \frac{2z'}{3}.$$

Making use of the above substitutions, the equation $3x + 4y - 5z = 11$ is transformed to

$$3\left(1 + \frac{x'}{3} - \frac{14y'}{15} + \frac{2z'}{15}\right) + 4\left(2 + \frac{2x'}{3} + \frac{2y'}{15} - \frac{11z'}{15}\right) - 5\left(3 + \frac{2x'}{3} + \frac{y'}{3} + \frac{2z'}{3}\right) = 11$$

or $5x' - 59y' - 88z' = 225$.

Note: It is well known in matrix theory that any symmertric matrix can be reduced to the diagonal form by means of an orthogonal matrix. Let A be a symmetric matrix and R be an orthogonal matrix such that RAR^T (R^T is transpose of R) is diagonal. This means that the quadratic form $X^T AX$ where $X^T = (x, y, z)$ is transformed to the diagonal form by means of the transformation.

$$X = R^T X'$$

where $$(X')^T = (x', y', z')$$

The rows (columns) of R are the direction cosines of one axis relative to the other as indicated in the theory (Ref. Table 12.1) The columns of R are the eigen vectors of A.

We use the above result in the following example.

Ex. 5 : Transform the expression $x^2 + 3y^2 + 5z^2 - 8yz - 8xy$ to the form $-3x'^2 + 3y'^2 + 9z'^2$. Verify the result obtained by finding the direction cosines of the new axes.

Sol: It is known if an expression of the form

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ transforms into

$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$

by a rotation of a rectangular axes, then $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \qquad \ldots (1)$$

and to each λ, the corresponding direction ratios of one of the new axes are given by

$$\left.\begin{aligned} (a-\lambda)\, l + hm + gn &= 0 \\ hl + (b-\lambda)\, m + fn &= 0 \\ gl + fm + (c-\lambda)\, n &= 0 \end{aligned}\right\} \qquad \ldots (2)$$

For this problem the roots of the equation (1) are given by $\lambda = -3, 3, 9$. Hence the expression $x^2 + 3y^2 + 5z^2 - 8yz - 8xy$ transforms to $-3x'^2 + 3y'^2 + 9z'^2$.

Corresponding to $\lambda_1 = -3$, we have from (2) $l_1 = m_1 = 2n_1$, or $\frac{l_1}{2} = \frac{m_1}{2} = \frac{n_1}{1}$.

Corresponding to $\lambda_2 = 3$, we have from (2) $-l_2 = 2m_2 = n_2$ or $\frac{l_2}{-2} = \frac{m_2}{1} = \frac{n_2}{2}$.

Corresponding to $\lambda_3 = 9$, we have from (2) $2l_3 = -m_3 = n_3$ or $\frac{l_3}{1} = \frac{m_3}{-2} = \frac{n_3}{2}$.

Taking the lines $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$; $\frac{x}{-2} = \frac{y}{1} = \frac{z}{2}$; $\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$ as the new axes OX', OY', OZ', respectively, the transform relations of equation (A) are given by

$$x = \frac{2}{3}x' - \frac{2}{3}y' + \frac{z'}{3},$$

$$y = \frac{2}{3}x' + \frac{y'}{3} - \frac{2}{3}z'.$$

$$z = \frac{x'}{3} + \frac{2}{3}y' + \frac{2}{3}z'.$$

Making use of the above substitutions, the expression $x^2 + 3y^2 + 5z^2 - 8yz - 8xy$ transforms to the form $-3x^2 + 3y^2 + 9z^2$.

Ex. 6 : Transform the equation $7x^2 - 8y^2 - 8z^2 - 2yz - 8zx + 8xy - 16x + 14y - 14z - 5 = 0$ into $x^2 + y^2 - z^2 = 1$. Verify this result by finding the direction cosines of the new axes.

Sol: [Consider $S(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$ (1)

Change the origin in (1) to (ξ,η,ζ) and let x', y', z' now denote the new co-ordinates. Then (1) becomes

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2(a\xi + h\eta + g\zeta + u)x'$$
$$+ 2(h\xi + b\eta + f\zeta + v)y' + 2(g\xi + f\eta + c\zeta + w)z' + S(\xi,\eta,\zeta) = 0. \quad \ldots (2)$$

Now let ξ, η, ζ be chosen, if possible, to satisfy the equations

$$\left.\begin{aligned} a\xi + h\eta + g\zeta + u &= 0, \\ h\xi + b\eta + f\zeta + v &= 0, \\ g\xi + f\eta + c\zeta + w &= 0. \end{aligned}\right\} \quad \ldots (3)$$

We then have $S(\xi,\eta,\zeta) = \xi(a\xi + h\eta + g\zeta + u) + \eta(h\xi + b\eta + f\zeta + v)$

$$+ \zeta(g\xi + f\eta + c\zeta + w) + u\xi + v\eta + w\zeta + d$$
$$= u\xi + v\eta + w\zeta + d \equiv d' \text{ say.} \quad \ldots (4)$$

So (2) becomes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0, \quad \ldots (5)$$

in which the dashes have been dropped.

Then if (x, y, z) satisfies (5), so does $(-x, -y, -z)$ i.e., S is symmetrical about the new origin, which is therefore a centre of S].

For the problem under consideration, the equations (3) for the centre are

$$\begin{aligned} 7\xi + 4\eta - 4\zeta - 8 &= 0, \\ 4\xi - 8\eta - \zeta + 7 &= 0, \\ -4\xi - \eta - 8\zeta - 7 &= 0, \end{aligned}$$

giving the unique solution $\xi = 0, \eta = 1, \zeta = -1$. Hence from (4),

$$d' = -8\xi + 7\eta - 7\zeta - 5 = 9.$$

The discriminating cubic is

$$\begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

having roots $-9, -9, 9$. The Quadratic referred to its axes is

$$-9x^2 - 9y^2 + 9z^2 + 9 = 0, \text{ i.e.,}$$
$$x^2 + y^2 - z^2 = 1.$$

It will be found that the three equations (2) of ex. 5 are each equivalent, when $\lambda = -9$, to

$$4l + m - n = 0,$$

thus verifying the Lemma for the case of a double root of the discriminating cubic, given at the end of this problem. According to it, X', Y' directions are any two orthogonal directions each orthogonal to the direction with direction ratios 4, 1, −1, which is therefore Z' direction. For instance the direction ratios of X', Y' directions could be taken as 0, 1, 1; 1, −2, 2 respectively. The transform-relations of equation (*A*) are now given by

$$x = \frac{y'}{3} + \frac{4z'}{\sqrt{18}},$$

$$y = \frac{x'}{\sqrt{2}} - \frac{2y'}{3} + \frac{z'}{\sqrt{18}}$$

$$z = \frac{x'}{\sqrt{2}} + \frac{2y'}{3} - \frac{z'}{\sqrt{18}}.$$

Making use of the above substitutions, one can verify that the equation $7x^2 - 8y^2 - 8z^2 - 2yz - 8zx + 8xy + 9 = 0$ transforms to $x'^2 + y'^2 + z'^2 = 1$.

Lemma

(a) $\lambda_1 \neq \lambda_2 \neq \lambda_3$; l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 form a unique triad of orthogonal directions.

(b) $\lambda_1 = \lambda_2 \neq \lambda_3$; l_3, m_3, n_3 is unique; l_1, m_1, n_1; l_2, m_2, n_2 can be any directions orthogonal to the direction l_3, m_3, n_3 and we shall take them so as to be orthogonal also to each other.

(c) $\lambda_1 = \lambda_2 = \lambda_3$; l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 can be any directions and we shall take them so as to be orthogonal to each other.

Ex. 7: Show that the equation $x + y + z = 0$ transforms to $x' = 0$, when referred to new axes through the same origin with direction cosines $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$; $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$. Hence show that the curve $x + y + z = 0$; $xy + yz + zx + a^2 = 0$ is a circle of radius $\sqrt{2}a$.

Sol : The transform relations given by equation (*A*) become

$$x = \frac{x'}{\sqrt{3}} + \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{6}}$$

$$y = \frac{x'}{\sqrt{3}} + \frac{2z'}{\sqrt{6}}$$

$$z = \frac{x'}{\sqrt{3}} - \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{6}}.$$

Obviously these substitutions transform the equation $x + y + z = 0$ into $\sqrt{3} \cdot x' = 0$ or $x' = 0$.

Also by above transformation the curve $x + y + z = 0$; $xy + yz + zx + a^2 = 0$ transforms to $x' = 0$ and

$$\left(\frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{6}}\right)\left(\frac{-2z'}{\sqrt{6}}\right) + \left(\frac{-2z'}{\sqrt{6}}\right)\left(\frac{-y'}{\sqrt{2}} + \frac{z'}{\sqrt{6}}\right) + \left(\frac{-y'}{\sqrt{2}} + \frac{z'}{\sqrt{6}}\right)\left(\frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{6}}\right) + a^2 = 0,$$

which on simplification turns out to be

$$y'^2 + z'^2 = 2a^2,$$

which is circle of radius $\sqrt{2}a$ in the $Y'Z'$ plane (i.e., $x' = 0$).

Ex. 8: If $F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$ is transformed into $F_1(x', y', z') = a_1x'^2 + b_1y'^2 + c_1z'^2 + 2f_1y'z' + 2g_1z'x' + 2h_1x'y' + 2u_1x' + 2v_1y' + 2w_1z' + d'$ by means of the transformations $x = x' + \xi,\ y = y' + \eta,\ z = z' + \zeta$. Show that a, b, c, f, g, h and hence $a + b + c, A + B + C, D$ are all invariant. Show also

$$S = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \text{ is an invariant.}$$

Sol: By means of the transformations $x = x' + \xi, y = y' + \eta, z = z' + \zeta$, $F(x, y, z)$ transforms into

$$F(x', y', z') + 2(a\xi + h\eta + g\zeta + u)x' + 2(h\xi + b\eta + f\zeta + v)y'$$
$$+ 2(g\xi + f\eta + c\zeta + w)z' + F(\xi, \eta, \zeta) \qquad \ldots (1)$$

which by hypothesis should be identical to $F_1(x', y', z')$.

Comparing coefficients of like tems of (1) and $F_1(x', y', z')$, we get

$$a = a_1, b = b_1, c = c_1, f = f_1, g = g_1, h = h_1 \qquad \ldots (2)$$

and
$$\left.\begin{aligned} a\xi + h\eta + g\zeta + u &= u_1, \\ h\xi + b\eta + f\zeta + v &= v_1, \\ g\xi + f\eta + c\zeta + w &= w_1, \\ F(\xi, \eta, \zeta) &= d_1. \end{aligned}\right\} \qquad \ldots (3)$$

From eq (2) we conclude a, b, c, f, g, h are invariant, Since by (2)

$$a + b + c = a_1 + b_1 + c_1$$
$$A + B + C = A_1 + B_1 + C_1$$
$$D = D_1$$

$a + b + c, A + B + C, D$ are also invariant.

Here A, B, C, are the cofactors of a, b, c in the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

We now prove $S = S_1$ so that S is an invariant. By definition

$$S_1 = \begin{vmatrix} a_1 & h_1 & g_1 & u_1 \\ h_1 & b_1 & f_1 & v_1 \\ g_1 & f_1 & c_1 & w_1 \\ u_1 & v_1 & w_1 & d_1 \end{vmatrix} = \begin{vmatrix} a & h & g & u_1 \\ h & b & f & v_1 \\ g & f & c & w_1 \\ u_1 & v_1 & w_1 & d_1 \end{vmatrix}$$

Multiplying the numbers in the first three columns of the above by ξ, η, ζ respectively and subtract the sum from the numbers in the fourth column.

Then
$$S_1 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u_1 & v_1 & w_1 & u\xi + v\eta + w\zeta + d \end{vmatrix}$$

Multiply the numbers in the first three rows of the above by ξ, η, ζ respectively and subtract the sum from the numbers in the fourth row.

Then $$S_1 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = S.$$

Here S is an invariant.

12.6 Summary

If $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ denote the the D.C's of new coordinate axes with respect to the original axes, then $(l_1, l_2, l_3), (m_1, m_2, m_3)$ and (n_1, n_2, n_3) are D.C's of original axes with respect to new axes.

If the coordinates of a point P in space with reference to original axes are (x, y, z) and (x', y', z') are the coordinates with respect to the new system, then

$$x' = l_1x + m_1y + n_1z \; ; \; y' = l_2x + m_2y + n_2z \; ; \; z' = l_3x + m_3y + n_3z$$

and $x = l_1x' + l_2y' + l_3z' \; ; \; y = m_1x' + m_2y' + m_3z' \; ; \; z = n_1x' + n_2y' + n_3z'$

$$\sum_{i=1}^{3} l_i^2 = 1 = \sum_{i=1}^{3} m_i^2 = \sum_{i=1}^{3} n_i^2$$

$$\Sigma l_1 l_2 = \Sigma m_1 m_2 = \Sigma n_1 n_2 = 0 \quad \Sigma l_1 m_1 = \Sigma m_1 n_1 = \Sigma n_1 l_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1, l_2^2 + m_2^2 + n_2^2 = 1, l_3^2 + m_3^2 + n_3^2 = 1$$

12.7 Sample Examination Questions

I. Answer the following in detail.

(i) (a) Obtain the relation between the co-ordinates of a point when we change the direction of the axes without changing the origin.

(b) If l_r, m_r, n_r ; $(r = 1, 2, 3)$ are the direction cosines of three mutually perpendicular straight lines and

$$\frac{a}{l_1} + \frac{b}{m_1} + \frac{c}{n_1} + 0 \; ; \; \frac{a}{l_2} + \frac{b}{m_2} + \frac{c}{n_2} + 0,$$

prove that $\frac{a}{l_3} + \frac{b}{m_3} + \frac{c}{n_3} + 0$, and

$$a : b : c = l_1 l_2 l_3 : m_1 m_2 m_3 : n_1 n_2 n_3.$$

(ii) (a) Obtain the relations between the direction cosines of the three mutually perpendicular axes of two systems.

(b) If l_r, m_r, n_r ; $(r = 1, 2, 3)$ are the direction cosines of three mutually perpendicular straight lines prove that

$$m_2 n_2 \left(m_3^2 - n_3^2\right) - m_3 n_3 \left(m_2^2 - n_2^2\right) = l_1 l_2 l_3 \; ;$$
$$\Sigma \; l_1 m_1 n_1 \left(m_2 n_3 + m_3 n_2\right) = l_1 l_2 l_3 .$$

II. Briefly answer the following.

(i) Verify that the lines OX', OY', OZ' with direction cosines $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}; \frac{2}{3}, \frac{2}{3}, \frac{1}{3}; -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ referred to OX, OY, OZ form a mutually perpendicular system. Show that the expression $4x^2 + 2y^2 + 3z^2 + 4yz - 4zx$ of $OXYZ$ system transforms to $3y'^2 + 6z'^2$ or $OX'Y'Z'$ system.

(ii) Verify that the lines OX', OY', OZ' given by

$$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1},$$

$$\frac{x}{1} = \frac{y}{-4} = \frac{z}{1},$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{2}.$$

form a mutually perpendicular system. Show that the expression $5x^2 - 16y^2 + 5z^2 + 8yz - 14zx + 8xy$ transforms to $12x'^2 - 18z'^2$ in the $OX'Y'Z'$ system. Verify the truth of the invariants in the two expressions.

(iii) Find the equation of the surface $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 1$ with reference to axes through the same origin and with direction cosines proportional to $-1, 0, 1$; $1, -1, 1$; $1, 2, 1$.

(iv) Transform the expression $6x^2 + 11y^2 + 10z^2 - 12yz + 8zx - 4xy$ to the form $3x'^2 + 6y'^2 + 18z'^2$. Verify the result obtained by finding the direction cosines of the new axes.

(v) Show that the equation $lx + my + nz = 0$ becomes $z = 0$, when referred to new axes through the same origin with direction cosines

$$\frac{-m}{\sqrt{l^2 + m^2}}, \frac{l}{\sqrt{l^2 + m^2}}, 0 ; \frac{-ln}{\sqrt{l^2 + m^2}}, \frac{-mn}{\sqrt{l^2 + m^2}},$$

Hence show that the curve $ax^2 + by^2 = 2z, lx + my + nz = 0$ is a rectangular hyperbola if $(a + b)n^2 + am^2 + bl^2 = 0$.

(vi) Find the conditions that the section of $ax^2 + by^2 + cz^2 = 1$ by $lx + my + nz = p$ should be a circle.

(vii) If $lx + my = 0$ is a circular section of $Ax^2 + By^2 + Cz^2 + 2Dxy = 1$, prove that $(B - C)\ l^2 - 2Dlm + (A - C)\ m^2 = 0$.

(viii) If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, $a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy$ are simultaneously transformed, show that

$a_1A + b_1B + c_1C + 2f_1F + 2g_1G + 2h_1H$ remains unaltered.

(A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Answers :

(ii) $2x'^2 + 3y'^2 + 6z'^2 = 1$.

(iv) $D.C$ s of new axes $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}; \frac{2}{3}, \frac{2}{3}, \frac{1}{3}; \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$.

(vi) $l = 0,\ m^2 (c - a) = n^2 (a - b)$;

or $m = 0,\ n^2 (a - b) = l^2\ (b - c)$;

or $n = 0,\ l^2 (b - c) = m^2\ (c - a)$.

BLOCK–3 : THE SPHERE

Unit–13 : EQUATION OF A SPHERE

13.0 Contents

13.1 Aims and Objectives

After going through this unit, you will be able to :

i) derive the equation of a sphere whose centre and radius are given

ii) obtain a necessary and sufficient condition for a general second degree equation in x, y, z to represent a sphere

iii) obtain the equation of a sphere which passes through four points, not all of which are coplanar.

13.2 Introduction

In this unit we shall define a sphere as the locus of a point whose distance from a fixed point, called the centre, is a constant and this constant distance is known as the radius of the sphere. We shall not only derive the equation of the sphere but also investigate the necessary and sufficient conditions that a general second degree equation in x, y, z may represent a sphere. Since the general equation of a sphere contains four effective constants, a sphere can be uniquely determined so as to satisfy four conditions. In particular we can find sphere through four non-coplanar points.

13.3 Equation of a sphere

Def : A sphere is the locus of a point which remains at a constant distance from a fixed point. The constant distance is called the radius and the fixed point, the centre of the sphere.

Let (a, b, c) be the centre and r, the radius of the given sphere. Equating the radius r to the distance of any point (x, y, z) on the sphere from its centre (a, b, c), we have

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$$

$$\text{or } x^2+y^2+z^2-2ax-2by-2cz+\left(a^2+b^2+c^2\right)=0 \qquad \text{... (A)}$$

which is the required equation of the given sphere.

We note the following characteristics of the equation (A) of the sphere :

1. It is of second degree in x, y, z ;
2. The coefficients of x^2, y^2, z^2 are all equal.
3. The product terms xy, yz, zx are absent.

Cor : The equation of a sphere of radius r and centre at origin is given by $x^2+y^2+z^2=r^2$.

13.3.1 Necessary and sufficient conditions that a general second degree equation to represent a sphere

The general second degree equation in (x, y, z)

$S = S(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ to represent a sphere are

$a = b = c \neq 0, f = g = h = 0, u^2 + v^2 + w^2 \geq ad.$

The conditions are necessary

Let $S = 0$ be a sphere of radius $r \neq 0$ and centre (ξ, η, ζ).

Substituting $x = x' + \xi,\ y = y' + \eta,\ z = z' + \zeta,\ S = 0$ becomes $ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2x'(a\xi + h\eta + g\zeta + u) + 2y'(h\xi + b\eta + f\zeta + v) + 2z'(g\xi + f\eta + c\zeta + w) + S(\xi, \eta, \zeta) = 0.$ ... (1)

We know the distance between the points $(r\cos\theta, r\sin\theta, 0)$ and $(0, 0, 0)$ is always r for all values of θ. Then $(r\cos\theta, r\sin\theta, 0)$ should be a point on the sphere $S = 0$ for all values of θ.

Writing $\xi_1 = a\xi + h\eta + g\zeta + u,\quad \eta_1 = h\xi + b\eta + f\zeta + v,$

$\zeta_1 = g\xi + f\eta + c\zeta + w,\quad d_1 = S(\xi, \eta, \zeta)$

and substituting $x' = r\cos\theta, y' = r\sin\theta, z' = 0$ in equation (1), we get

$$r^2(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta) + 2r(\xi_1\cos\theta + \eta_1\sin\theta) + d_1 = 0 \quad \text{... (2)}$$

Similarly, by substituting $x' = r\cos(\theta + \pi), y' = r\sin(\theta + \pi), z' = 0$ in equation (1), we also get

$$r^2(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta) - 2r(\xi_1\cos\theta + \eta_1\sin\theta) + d_1 = 0 \quad \text{... (3)}$$

From eqs. (2) and (3), we get

$$\xi_1\cos\theta + \eta_1\sin\theta = 0 \quad \text{... (4)}$$

$$r^2(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta) + d_1 = 0 \quad \text{... (5)}$$

Putting $\theta = 0$ and $\frac{\pi}{2}$, we get from (5)

$$ar^2 + d_1 = 0,\quad br^2 + d_1 = 0.$$

This shows $a = b$.

We shall show $a \neq 0$. For then $d_1 = 0 = S(\xi, \eta, \zeta)$ implying that the centre (ξ, η, ζ) is a point on the sphere $S = 0$ i.e., the radius of the sphere r is zero. Since by hypothesis, the radius of the sphere $S = 0$ is not zero; we have a contradiction. Therefore $a \neq 0$.

Putting $\theta = \pi/4$ and $-\pi/4$ in (5), we get

$$r^2\left(\frac{a+b}{2} + h\right) + d_1 = 0 = r^2\left(\frac{a+b}{2} - h\right) + d_1$$

Therefore $h = 0$

Thus by taking $(r\cos\theta, r\sin\theta, 0)$ as a point on the sphere of radius r and centre origin, we have established $a = b \neq 0, h = 0$.

Similarly by considering the points $(0, r\cos\theta, r\sin\theta)$ and $(r\cos\theta, 0, r\sin\theta)$ as points on the sphere, it can be shown

$b = c \neq 0, f = 0$ and $c = a \neq 0, g = 0$

Thus $S \equiv a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d$

$$= a\left[\left(x + \frac{u}{a}\right)^2 + \left(y + \frac{v}{a}\right)^2 + \left(z + \frac{w}{a}\right)^2 - \frac{(u^2 + v^2 + w^2 - ad)}{a^2}\right]$$

Since $\dfrac{u^2+v^2+w^2-ad}{a^2} = r^2 > 0$, we get $u^2+v^2+w^2 > ad$.

Therefore necessary conditions that $S = 0$ should represent a sphere are

$a = b = c \neq 0, f = g = h = 0,\ u^2+v^2+w^2 > ad.$

The conditions are sufficient

When $a = b = c \neq 0, f = g = h = 0$, we have

$$x^2+y^2+z^2+\frac{2ux}{a}+\frac{2vy}{a}+\frac{2wz}{a}+\frac{d}{a} = 0$$

and also $\dfrac{u^2+v^2+w^2-ad}{a^2} > 0$ since $u^2+v^2+w^2 > ad$.

Therefore we have a sphere of centre $\left(\dfrac{-u}{a}, \dfrac{-v}{a}, \dfrac{-w}{a}\right)$ and radius equal to

$$\frac{1}{a}\sqrt{u^2+v^2+w^2-ad} \neq 0.$$

Cor. 1 : $x^2+y^2+z^2+2ux+2vy+2wz+d = 0$ can be taken as the general equation of a sphere whose centre is $(-u, -v, -w)$ and radius equal to $\sqrt{u^2+v^2+w^2-d}$

Cor. 2 : The radius and therefore, the sphere is imaginary when $u^2+v^2+w^2-ad < 0$ and in this case we call it a virtual sphere. No real point lies on it.

13.4 Sphere through four given points

We shall find the equation of a sphere through four non-coplanar points

(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)

Let us see if we can find u, v, w, d such that

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0 \qquad \dots (1)$$

is the equation of a sphere through the four given points.

We have then the equations

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = 0 \qquad \dots (2)$$

$$x_2^2+y_2^2+z_2^2+2ux_2+2vy_2+2wz_2+d = 0 \qquad \dots (3)$$

$$x_3^2+y_3^2+z_3^2+2ux_3+2vy_3+2wz_3+d = 0 \qquad \dots (4)$$

$$x_4^2+y_4^2+z_4^2+2ux_4+2vy_4+2wz_4+d = 0 \qquad \dots (5)$$

Using Cramer's solution of determinants, we have

$$2u = \frac{1}{\Delta}\begin{vmatrix} -x_1^2-y_1^2-z_1^2 & y_1 & z_1 & 1 \\ -x_2^2-y_2^2-z_2^2 & y_2 & z_2 & 1 \\ -x_3^2-y_3^2-z_3^2 & y_3 & z_3 & 1 \\ -x_4^2-y_4^2-z_4^2 & y_4 & z_4 & 1 \end{vmatrix} = \frac{\Delta_1}{\Delta}.$$

$$2v = \frac{1}{\Delta}\begin{vmatrix} x_1 & -x_1^2 - y_1^2 - z_1^2 & z_1 & 1 \\ x_2 & -x_2^2 - y_2^2 - z_2^2 & z_2 & 1 \\ x_3 & -x_3^2 - y_3^2 - z_3^2 & z_3 & 1 \\ x_4 & -x_4^2 - y_4^2 - z_4^2 & z_4 & 1 \end{vmatrix} = \frac{\Delta_2}{\Delta}.$$

$$2w = \frac{1}{\Delta}\begin{vmatrix} x_1 & y_1 & -x_1^2 - y_1^2 - z_1^2 & 1 \\ x_2 & y_2 & -x_2^2 - y_2^2 - z_2^2 & 1 \\ x_3 & y_3 & -x_3^2 - y_3^2 - z_3^2 & 1 \\ x_4 & y_4 & -x_4^2 - y_4^2 - z_4^2 & 1 \end{vmatrix} = \frac{\Delta_3}{\Delta}.$$

$$d = \frac{1}{\Delta}\begin{vmatrix} x_1 & y_1 & z_1 & -x_1^2 - y_1^2 - z_1^2 \\ x_2 & y_2 & z_2 & -x_2^2 - y_2^2 - z_2^2 \\ x_3 & y_3 & z_3 & -x_3^2 - y_3^2 - z_3^2 \\ x_4 & y_4 & z_4 & -x_4^2 - y_4^2 - z_4^2 \end{vmatrix} = \frac{\Delta_4}{\Delta}.$$

where $$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

which is not zero (Ref. Cor 1 of 8.4.6) since by hypothesis all the four points are not coplanar. Substituting the values of $2u$, $2v$, $2w$ and d in (1), we get

$$\Delta\left(x^2 + y^2 + z^2\right) + \Delta_1 x + \Delta_2 y + \Delta_3 z + \Delta_4 = 0 \qquad \text{... (6)}$$

or in a determinant form

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \qquad \text{... (7)}$$

Since in (6), $\Delta \neq 0$ either (6) or (7) represents the equation of a sphere. This is not a virtual sphere since the four points lie on it as seen from (7).

Examples

Ex. 1 : Obtain the equation of the sphere described on the join of $A(2, -3, 4)$, $B(-5, 6, -7)$ as diameter.

Let P be an arbitrary point on the sphere with co-ordinates (x, y, z).

Then $CP = \sqrt{\left(x+\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2+\left(z+\frac{3}{2}\right)^2}$.

Since $CA^2 = CP^2$, we get

$$x^2+y^2+z^2+3x-3y+3z-56=0$$

as the required equation of the sphere.

Ex. 2 : A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere.

Sol. : Take the centre of the cube as origin and the planes through the centre parallel to its faces as the co-ordinate planes.

Let each edge of the cube be equal to $2a$. Then the equations of the faces of the cube are

$x = a,\ x = -a\ ;\ y = a,\ y = -a\ ;\ z = a,\ z = -a.$

If (f, g, h) be any point on the locus, we have

$(f-a)^2+(f+a)^2+(g-a)^2+(g+a)^2+(h-a)^2+(h+a)^2 = k^2$ (k, a constant)

or $2\left(f^2+g^2+h^2+3a^2\right) = k^2$

so that its locus is $2\left(x^2+y^2+z^2+3a^2\right) = k^2$

which is a real sphere if $\frac{k^2}{2}-3a^2 > 0$.

Ex. 3 : Find the equation of the sphere through the points (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3). Locate its centre and find its radius.

Sol. : Let the required equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0. \quad \dots (1)$$

Since it passes through the points (0, 0, 0), (0, 1, –1) ; (–1, 2, 0) and (1, 2, 3), we have

$$d = 0$$

$1+1+2v-2w+d=0$ or $v-w+1=0$ ... (2)

$1+4-2u+4v+d=0$ or $-2u+4v+5=0$ ... (3)

$1+4+9+2u+4v+6w+d=0$ or $u+2v+3w+7=0$ ... (4)

Multiplying (2) by (3) and adding it to (4), we get

$$u+5v+10=0. \quad (5)$$

Solving (3) and (5), we get $u = \frac{-15}{14},\ v = \frac{-25}{41}$.

From (2), $w = v+1 = \frac{-25}{14}+1 = \frac{-11}{14}$.

Sustituting the values of u, v, w, d in (1), we get

$$x^2+y^2+z^2-\frac{15x}{7}-\frac{25y}{7}-\frac{11z}{7}=0$$

which is the required equation of the sphere.

Its centre is $\left(\frac{15}{14}, \frac{25}{14}, \frac{11}{14}\right)$ being $(-u, -v, -w)$ and the radius

$= \sqrt{\left(\frac{-15}{14}\right)^2+\left(\frac{-25}{14}\right)^2+\left(\frac{-11}{14}\right)^2-0}$, being $\sqrt{u^2+v^2+w^2-d}$

$= \frac{\sqrt{971}}{14}$.

Ex. 4 : A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Sol. : Let the centre of the sphere $OABC$ be $P\ (f, g, h)$ so that its radius $OP = \sqrt{f^2 + g^2 + h^2}$.

The equation of the sphere is

$$(x-f)^2 + (y-g)^2 + (z-h)^2 = f^2 + g^2 + h^2$$

or $$x^2 + y^2 + z^2 + 2fx - 2gy - 2hz = 0.$$

To find OA, putting $y = 0,\ z = 0$ in (1), we have

$$x^2 - 2fx = 0 \quad \text{i.e.,} \quad OA = x = 2f$$

Similarly, $OB = 2g,\ OC = 2h.$

The equation of the plane ABC is

$$\frac{x}{2f} + \frac{y}{2g} + \frac{z}{2h} = 1.$$

Since this plane passes through the point (a, b, c)

$$\frac{a}{2f} + \frac{b}{2g} + \frac{c}{2h} = 1.$$

Hence the locus of the centre (f, g, h) of the sphere is

$$\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = 1 \quad \text{or} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Ex. 5 : Given $abc \neq 0$, find the equation of the sphere passing through the points $(0, 0, 0)$, $(-a, b, c)$, $(a, -b, c)$, $(a, -b, c)$; $(a, b, -c)$. Find also the centre and radius of the sphere.

Sol. : In order to show that the given points do not all lie in a plane (Ref. Cor. of 8.4.6)

$$\text{we require} \begin{vmatrix} 0 & 0 & 0 & 1 \\ -a & b & c & 1 \\ a & -b & c & 1 \\ a & b & -c & 1 \end{vmatrix} \neq 0.$$

In fact the value of the determinant gives $4\,abc$ which is not zero by hypothesis.

Hence the equation of the sphere passing through the four given points is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 0 & 0 & 0 & 0 & 1 \\ a^2+b^2+c^2 & -a & b & c & 1 \\ a^2+b^2+c^2 & a & -b & c & 1 \\ a^2+b^2+c^2 & a & b & -c & 1 \end{vmatrix} = 0 \qquad \ldots (1)$$

Expanding with the elements of the last column, we get

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z \\ a^2+b^2+c^2 & -a & b & c \\ a^2+b^2+c^2 & a & -b & c \\ a^2+b^2+c^2 & a & b & -c \end{vmatrix} = 0 \qquad \ldots (2)$$

$$\text{or } (x^2+y^2+z^2)\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} - x\begin{vmatrix} a^2+b^2+c^2 & b & c \\ a^2+b^2+c^2 & -b & c \\ a^2+b^2+c^2 & b & -c \end{vmatrix}$$

$$+ y\begin{vmatrix} a^2+b^2+c^2 & -a & c \\ a^2+b^2+c^2 & a & c \\ a^2+b^2+c^2 & a & -c \end{vmatrix} - z\begin{vmatrix} a^2+b^2+c^2 & -a & b \\ a^2+b^2+c^2 & a & -b \\ a^2+b^2+c^2 & a & b \end{vmatrix} = 0 \qquad \ldots (3)$$

$$\text{Since } \begin{vmatrix} 1 & b & c \\ 1 & -b & c \\ 1 & b & -c \end{vmatrix} = 4bc, \quad \begin{vmatrix} 1 & -a & c \\ 1 & a & c \\ 1 & a & -c \end{vmatrix} = -4ac,$$

$$\begin{vmatrix} 1 & -a & b \\ 1 & a & -b \\ 1 & a & b \end{vmatrix} = 4ab \text{ and } \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = 4abc$$

We get from (3)

$$4\,abc\,(x^2+y^2+z^2) - 4bc\,(a^2+b^2+c^2)\,x - 4ca\,(a^2+b^2+c^2)\,y - 4ab\,(a^2+b^2+c^2)\,z = 0$$

or $\frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$ as the required eqation of the sphere.

The centre of the sphere is

$$\left(\frac{a^2+b^2+c^2}{2a}, \frac{a^2+b^2+c^2}{2b}, \frac{a^2+b^2+c^2}{2c}\right)$$

and its radius is $\sqrt{\left(\frac{a^2+b^2+c^2}{2}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$

$$= \frac{(a^2+b^2+c^2)}{2} = \sqrt{(a^{-2}+b^{-2}+c^{-2})}.$$

SAQ'S : 1. What is the equation of a sphere with (a, b, c) as its centre and r as its radius?

2. What are the characters of the general second degree equation of a sphere?

3. What is a virtual sphere ?

13.5 Summary

The equation,

$$S = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

represents a sphere if and only if

$$a = b = c \neq 0, \quad f = g = h = 0; \; u^2 + v^2 + w^2 \geq d$$

Equation of a sphere through 4 given points

(x_i, y_i, z_i), $i = 1, 2, 3, 4$ is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

13.6 Sample Examination Questions

I. Answer the following in detail.

(i) (a) Obtain the necessary and sufficient conditions that a general second degree equation to represent a sphere.

(b) Find the equation of the sphere whose diameter is the line joining the origin to the point $(2, -2, 4)$. Also find its centre and radius.

(ii) (a) Obtain the equation of a sphere through four noncoplanar points.

(b) Find the equation of the sphere through the four points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Find its centre and radius.

II. Briefly answer the following.

(i) Show that the locus of all points which are twice as far from the point $(3, -1, 2)$ as from $(0, 2, -1)$ is a sphere. Find its centre and radius.

(ii) A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

(iii) A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC.

(iv) A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the

(i) triangle ABC lies on the sphere $9\left(x^2 + y^2 + z^2\right) = 4k^2$

(ii) tetrahedron $OABC$ lies on the sphere $x^2 + y^2 + z^2 = k^2$

(v) OA, OB, OC are mutually perpendicular lines through the origin and their direction cosines are $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$. If $OA = a$, $OB = b$, $OC = c$, prove that the equation to the sphere $OABC$ is

$$x^2 + y^2 + z^2 - x\left(al_1 + bl_2 + cl_3\right) - y\left(am_1 + bm_2 + cm_3\right) - z\left(an_1 + bn_2 + cn_3\right) = 0$$

(vi) Obtain the equation of the sphere which passes through the points (1, 0, 0) ; (0, 1, 0) ; (0, 0, 1) and has its radius as small as possible.

(vii) Find the equation to the sphere through the points (0, 0, 0), (0, 1, –1), (–1, 2, 0), (1, 2, 3).

(viii) Prove that the equation to the sphere circumscribing the tetrahedron whose sides are

$\frac{y}{b} + \frac{z}{c} = 0,\ \frac{z}{c} + \frac{x}{a} = 0,\ \frac{x}{a} + \frac{y}{b} = 0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is

$$\frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0 .$$

Answers :

I. (ib) $x^2 + y^2 + z^2 - 2x + 2y + 4z = 0$; centre $(1, -1, 1)$; radius $= \sqrt{6}$.

(iib) $x^2 + y^2 + z^2 - ax - by - cz = 0$; centre $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$, radius $\frac{\sqrt{a^2 + b^2 + c^2}}{2}$,

II. (i) $x^2 + y^2 + z^2 + 2x - 6y - \frac{4z}{3} + 2 = 0$; centre $(-1, 3, 2/3)$; radius $\frac{\sqrt{76}}{3}$.

(iii) $\left(x^2 + y^2 + z^2\right)^2 \left(x^{-2} + y^{-2} + z^{-2}\right) = 4r^2$

(iv) $3\left(x^2 + y^2 + z^2\right) - 2(x + y + z) - 1 = 0$;

(vii) $7\left(x^2 + y^2 + z^2\right) - 15x - 25y - 11z = 0$.

13.7 Answers to SAQ's

SAQ 1 : $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

SAQ 2 : The coefficients of x^2, y^2, z^2 are equal, the coefficients of xy, yz, zx are zeros and $u^2 + v^2 + w^2 \geq ad$ where, u, v, w are half of the coefficients of x, y, z respectively, a is the coefficient of x^2, y^2, z^2 and d is the constant term.

SAQ 3 : If the sum of the squares of the half of the coefficients of x, y, z terms is less than the product of the coefficients of x^2, y^2, z^2 and the constant term, then the sphere is called virtual sphere.

Unit-14 : PLANE SECTIONS OF A SPHERE

14.0 Contents

14.1 Aims and objectives

After going through this unit, you will be able to :

(i) prove that a plane section of a sphere or the intersection of two spheres is a circle and obtain the equation of a sphere passing through this circle,

(ii) prove that a straight line meets a sphere, in general, in two points.

14.2 Introduction

In this Unit we shall show that the set of points common to a sphere $S = 0$ and a plane $U = 0$ or to two spheres $S = 0, S' = 0$ is a circle. Not only the equation of the circle can be written down as $S = 0$, $U = 0$ or $S = 0, S' = 0$, but also the general equation of a sphere passing through this circle can be obtained as $S + kU = 0$ or $S + kS' = 0$. An additional condition has to be specified to determine k. It has been shown that a line from a given point and with given direction ratios intersects a given sphere in general in two points which may be coincident. The power of a given point with respect to the given sphere is the square of the length of a tangent from the point to the sphere.

14.3 Plane section of a Sphere

A plane section of a sphere, i.e., the points common to a sphere and a plane is a circle.

Let O be the centre of the sphere and P, any point on the plane section (fig. 1). Let ON be the perpendicular to the given plane; N being the foot of the perpendicular.

As ON is perpendicular to the plane which contains the line NP, we have $ON \perp NP$. Hence $NP^2 = OP^2 - ON^2$. Now O and N being fixed points and OP being fixed (radius of the sphere) this relation shows that NP is constant for all positions of P on this section.

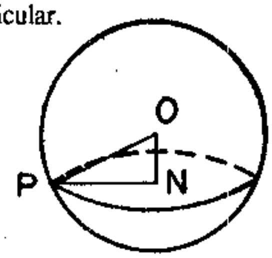


Fig. 1

Hence the locus of P is a circle whose centre is N, the foot of the perpendicular from the centre of the sphere to the plane.

The section of a sphere by a plane through its centre is known as a great circle. The centre and the radius of a great circle are the same as those of the sphere.

Cor.: The circle through three given non-collinear points lies entirely on any sphere through the same three points. Thus the condition for a sphere to contain a given circle is equivalent to that of its passing through any three of its points.

14.4 Intersection of two Spheres

The intersection of two spheres is a circle. The co-ordinates of points common to any two spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

satisfy both the equations and therefore they also satisfy the equation

$$S_1 - S_2 \equiv 2x\left(u_1-u_2\right) + 2y\left(v_1-v_2\right) + 2z\left(w_1-w_2\right)+\left(d_1-d_2\right) = 0$$

which being of the first degree represents a plane.

Also $S_1 - S_2 = 0, S_1 = 0$ (or $S_2 = 0$) $\Rightarrow S_2 = 0$ (or $S_1 = 0$)

Thus the points of intersection of the two spheres are the same as those of any one of them and the plane and therefore, they lie on a circle.

14.5 Sphere with a given Diameter

We shall find the equation of the sphere described on the line joining the points $A\left(x_1, y_1, z_1\right)$, $B\left(x_2, y_2, z_2\right)$ as diameter.

Let P be any point (x, y, z) on the sphere described on AB as diameter.

Since the section of the sphere by the plane through the three points P, A, B is a great circle having AB as diameter, P lies on a semicircle and therefore

$$PA \perp PB$$

The direction cosines of PA, PB are proportional to $x - x_1, y - y_1, z - z_1$ and $x - x_2, y - y_2, z - z_2$ respectively. Therefore they are perpendicular, if and only if

$$\left(x - x_1\right)\left(x - x_2\right) + \left(y - y_1\right)\left(y - y_2\right) + \left(z - z_1\right)\left(z - z_2\right) = 0$$

which is the required equation of the sphere.

14.6 Equation of a Circle

Given a circle there is a sphere through it. Any circle is the intersection of its plane with some sphere through it. Therefore a circle can be represented by two equations, one being of a sphere and the other of the plane.

Thus the two equations

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ lx + my + nz = p$$

taken together represent a circle.

A circle can also be represented by the equations of any two spheres through it.

Note: The student may note that the equations

$$x^2 + y^2 + 2gx + 2fy + c = 0,\ z = 0$$

also represent a circle which is the intersection of the cylinder $x^2 + y^2 + 2gx + 2fy + c = 0$ with the plane $z = 0$.

14.7 Spheres through a given circle

The equation $\quad S + \lambda U = 0 \quad$ ... (1)

obviously represents a sphere passing through the circle C with equations

$$S = 0,\ U = 0.$$

Also any sphere through C can be put in the form (1) [for some λ] where

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$$
$$U \equiv lx + my + nz - p$$

Also the equation $\quad S + \lambda S' = 0 \qquad \ldots (2)$

represents a sphere through the circle with equations $S = 0, S' = 0$.

Also any sphere through the circle $S = 0, S' = 0$ can be put in the form (2) [for some λ] , where

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$$
$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d'$$

Here λ is an arbitrary constant which can be chosen to fulfil one more condition.

Note 1: We notice that the equation of the plane of the circle through the two given spheres is

$$S - S' \equiv 2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0$$

From this we see that the equation of any sphere through the circle $S = 0, S' = 0$ can also be taken of the form $S + K(S - S') = 0$, K being an arbitrary constant.

This form proves sometimes comparitively more convenient than the condition (2).

Note 2: It is important to remember that the general equation of a sphere through the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad z = 0$$

is $\quad x^2 + y^2 + z^2 + 2gx + 2fy + kz + c = 0$

with k as a parameter.

14.8 Intersection of a Sphere and a Line

We will show that in general every straight line meets a sphere in two points which may be coincident.

Let $\quad x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \ldots (1)$

and $\quad \dfrac{x - \alpha}{l} = \dfrac{y - \beta}{m} = \dfrac{z - \gamma}{n} \qquad \ldots (2)$

be the equations of a sphere and a line respectively. If each ratio in (2) is r, then r represents the distance of any point from (α, β, γ) provided l, m, n are direction cosines. The point $(lr + \alpha, mr + \beta, nr + \gamma)$ will also lie on the given sphere (1), if r satisfies the equation

$$r^2 + 2r\,[l(\alpha+u) + m(\beta+v) + n(\gamma+w)] + \left(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right) = 0 \qquad \ldots (A)$$

and this latter being a quadratic equation in r gives two values say r_1, r_2, of r. Then

$$\left(lr_1+\alpha,\ mr_1+\beta,\ nr_1+\gamma\right);\ \left(lr_2+\alpha,\ mr_2+\beta,\ nr_2+\gamma\right)$$

are points of intersection. If (A) has no real roots then the sphere and the line do not intersect.

14.9 Power of a Point

If l, m, n are the direction cosines of the line (2) above so that $l^2 + m^2 + n^2 = 1$, then r_1, r_2 are the distances of the point $A(\alpha, \beta, \gamma)$ from the points of intersection of P and Q.

$\therefore\ AP.AQ = r_1 r_2 = \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d$ which is independent of the direction cosines l, m, n and called the power of the given point A with respect to the given sphere.

In the limit $P \to Q$ i.e., APQ becomes the tangent at the point, say T, to the sphere, then $AT^2 = AP.\,AQ$ i.e., the power of a given point with respect to the given sphere is the square of the length of the tangent from the point to the sphere.

Cor.: If A is the origin, then $r_1 r_2 = d$, i.e., the power of the origin with respect to the given sphere is d.

Def.: If from a fixed point A, chords be drawn in any direction to intersect a given sphere in P and Q, then $AP.\ AQ$ is constant. This constant is called the power of the point A with respect to the sphere.

Examples

Ex. 1: Find the equation of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$, $x + y + z = 3$ as a great circle.

Sol : The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 + 10y - 4z - 8 + k(x + y + z - 3) = 0 \qquad \ldots (1)$$

i.e., $x^2 + y^2 + z^2 + kx + (10 + k)y - (4 - k)z - (8 + 3k) = 0$

In order that (1) may have the given circle as its great circle, its centre $\left(\frac{-k}{2}, -\frac{10+k}{2}, \frac{4-k}{2}\right)$ must lie on the plane $x + y + z - 3 = 0$.

$$\therefore \frac{-k}{2} - \frac{10+k}{2} + \frac{4-k}{2} = 3 \text{ or } -3k = 12 \text{ i.e., } k = -4; \text{ hence (1) becomes}$$

$$x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$$

which is the required equation.

Ex. 2 : Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point (1, 2, 3).

Sol : The sphere $x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0$ passes through the given circle for all values of λ. It will pass through (1, 2, 3) if

$$1^2 + 2^2 + 3^2 - 9 + \lambda(2.1 + 3.2 + 4.3 - 5) = 0$$

i.e., $5 + 15\lambda = 0$ or $\lambda = -\frac{1}{3}$.

The required equation of the sphere, therefore is $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$

Ex. 3: Find the equation of the sphere described on the line joining the points (2, –1, 4) and (–2, 2, –2) as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x + y - z = 3$.

Sol: We know that the equation of the sphere with (x_1, y_1, z_1) and (x_2, y_2, z_2) as the end points of one of its diameters is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0 \qquad \ldots (1)$$

Since the end points are now given as (2, –1, 4) and (–2, 2, –2), the equation (1) of the sphere is

$$(x - 2)(x + 2) + (y + 1)(y - 2) + (z - 4)(z + 2) = 0$$

$$\text{or } x^2 + y^2 + z^2 - y - 2z - 14 = 0 \qquad \ldots (2)$$

which is the required equation. Its centre is $C\left(0, \frac{1}{2}, 1\right)$ and radius

$$r = \sqrt{0 + \frac{1}{4} + 1 + 14} = \sqrt{\frac{61}{4}}.$$

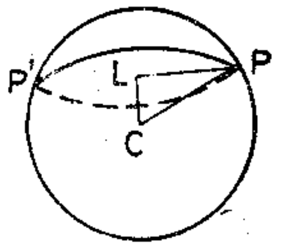


Fig. 2

Let the given plane $2x + y - z - 3 = 0$ ... (3)

cut the sphere (2) in the circle PP' having centre L (fig. 2)

$\therefore\ p$ = perperndicular CL from C on the plane (3)

$$= \left| \frac{\frac{1}{2} - 1 - 3}{\sqrt{4 + 1 + 1}} \right| = \frac{7}{2\sqrt{6}}.$$

If a be the radius of the circle PP',

then $a^2 = r^2 - p^2 = \frac{61}{4} - \frac{49}{24} = \frac{317}{24}$.

Hence the area of circle $PP' = \pi a^2 = \frac{317}{24}\pi$.

Ex. 4: Show that the two circles,

$$x^2 + y^2 + z^2 - y + 2z = 0,\ x - y + z - 2 = 0;$$

$$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0,\ 2x - y + 4z - 1 = 0$$

lie on the same sphere and find its equation.

Sol : The equation of any sphere through the first circle is

$$x^2 + y^2 + z^2 - y + 2z + \lambda(x - y + z - 2) = 0 \qquad \text{... (1)}$$

and that of any sphere through the second circle is

$$x^2 + y^2 + z^2 + x - 3y + z - 5 + \mu(2x - y + 4z - 1) = 0 \qquad \text{... (2)}$$

The equations (1) and (2) will represent the same sphere if λ, μ can be chosen so as to satisfy the four equations $\lambda = 2\mu + 1$, $-1-\lambda = -\mu-3$, $2+\lambda = 4\mu + 1$, $-2\lambda = -\mu-5$.

The first two of these equations give $\lambda = 3, \mu = 1$. These values clearly satisfy the remaining two equations also. These four equations in λ, μ being consistent, the two circles lie on the same sphere, viz;

$$x^2 + y^2 + z^2 + 2z - y + 3(x - y + z - 2) = 0$$

i.e., $$x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$$

Ex. 5: Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular straight lines through a fixed point is constant.

Sol: Take the fixed point O as the origin and any three mutually perpendicular lines through it as the co-ordinate axes. With this choice of axes, let the equation of the given sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

The x–axis ($y = 0 = z$) meets the sphere in points given by

$$x^2 + 2ux + d = 0$$

so that if x_1, x_2 be its roots, the two points of intersection are $(x_1, 0, 0)$ and $(x_2, 0, 0)$.

Also we have $x_1 + x_2 = -2u,\ x_1x_2 = d.$

$$\therefore \quad (\text{intercept on } x\text{–axis})^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$

$$= 4(u^2 - d).$$

Similarly, $$(\text{intercept on } y\text{–axis})^2 = 4(v^2 - d)$$

$$(\text{intercept on } z\text{–axis})^2 = 4(w^2 - d)$$

The sum of the squares of the intercepts is

$$4\left(u^2 + v^2 + w^2 - 3d\right) = 4\left(u^2 + v^2 + w^2 - d\right) - 8d = 4r^2 - 8p$$ where r is the radius of the given sphere and p is the power of the given point i.e., origin with respect to the sphere. Since the sphere and the point are both given, r and p are both constants.

Hence the result.

14.10 Summary

Intersection of a plane and a sphere is a circle. Intersection of two spheres is also a circle. If $S = 0$, $U = 0$ is a circle, then any sphere through this circle is $S + \lambda U = 0$, where λ is a parameter. If a line through a given point A (α, β, γ) intersects a sphere $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, in points P, Q then power of A with respect to the sphere S is $AP.AQ = \alpha^2+\beta^2+\gamma^2+2u\alpha + 2v\beta + 2w\gamma + d$.

14.11 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Find the quation of the sphere described on the line joining the points $A\left(x_1, y_1, z_1\right)$, $B\left(x_2, y_2, z_2\right)$ as diameter.

b) Obtain the equation of a sphere through a given circle.

(ii) a) Show that in general every stright line meets a sphere in two points which may be coincident.

b) Find the co-ordinates of the points where the line

$$\frac{1}{4}(x + 3) = \frac{1}{3}(y + 4) = -\frac{1}{5}(z - 8)$$

intersects the sphere $x^2 + y^2 + z^2 + 2x - 10y = 23$

II. *Briefly answer the following questions.*

i) Show that the centre of all sections of the sphere $x^2 + y^2 + z^2 = r^2$ by planes through a point (x', y', z') lie on the sphere $x(x - x') + y(y - y') + z(z - z') = 0$.

ii) Find the equation of the circle circumscribing the triangle formed by the three points $(a, 0, 0)$; $(0, b, 0)$; $(0, 0, c)$. Obtain also the co-ordinates of the centre of this circle.

iii) Obtain the equation of the sphere which passes through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $z + 2y + 2z = 0$ in a circle of radius 3.

iv) Find the equation of the sphere through the circle $x^2+y^2+z^2+2x+3y+6 = 0$, $x-2y+4z-9 = 0$ and passing through the centre of the sphere

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0.$$

v) Show that the following points are concyeclic

(a) $(5, 0, 2)$, $(2, -6, 0)$, $(7, -3, 8)$, $(4, -9, 6)$

(b) $(-8, 5, 2)$, $(-5, 2, 2)$, $(-7, 6, 6)$, $(-4, 3, 6)$

vi) Find the equation of the diameter of the sphere $x^2 + y^2 + z^2 = 29$ such that a rotation about it will transfer the point $(4, -3, 2)$ to the point $(5, 0, -5)$ along a great circle of the sphere. Find also the angle through which the sphere must be so rotated.

vii) Show that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$, $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere and find its equation.

Answers

I. (iib) $(1, -1, 3); (5, 2, -2)$

II. (ii) $\left[\frac{a\left(b^{-2}+c^{-2}\right)}{2\Sigma a^{-2}}, \frac{b\left(c^{-2}+a^{-2}\right)}{2\Sigma a^{-2}}, \frac{c\left(a^{-2}+b^{-2}\right)}{2\Sigma a^{-2}}\right]$

(iii) $x^2 + y^2 + z^2 + 6z - 4 = 0$

(iv) $x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$

(vi) $\frac{x}{2} = \frac{y}{6} = \frac{z}{5}$, $\cos^{-1}\left(\frac{16}{99}\right)$.

Unit–15 : TANGENT PLANE AND NORMAL AT A POINT ON THE SPHERE

15.0 Contents

15.1 Aims and Objectives

After going through this unit, you will be able to :

(i) obtain the equation of a tangent plane at any point of a sphere and,

(ii) find the locus of the points of contact of the tangent planes which passes through a given point.

15.2 Introduction

In this Unit we derive the equation of the tangent plane at any point of a given sphere. We have shown that the locus of points of contact of the tangent planes through a given point is the circle in which the plane cuts the sphere. Conditions for two spheres to touch at a point internally (externally) have been also derived.

15.3 Equation of a tangent plane

We shall derive the equation of the tangent plane at any point (α, β, γ) of the sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

As (α, β, γ) lies on the sphere, we have

$$S_1 \equiv \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = 0 \qquad \dots (1)$$

The points of intersection of any line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \qquad \dots (2)$$

through (α, β, γ) with the given sphere are $(lr + \alpha, mr + \beta, nr + \gamma)$ where the values of r are the roots of the equation $r^2\left(l^2 + m^2 + n^2\right) + 2r\,[l\,(\alpha + u) + m\,(\beta + v) + n\,(\gamma + w)] + S_1 = 0$, got by substituting (2) in the equation of the sphere. By virtue of the condition (1), one root of this quadratic is zero so that one of the points of intersection coincides with (α, β, γ).

In order that the second point of intersection may also coincide with (α, β, γ), the second value of r must also vanish and this requires

$$l\,(\alpha + u) + m\,(\beta + v) + n\,(\gamma + w)\} = 0 \qquad \dots (3)$$

Thus the line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ meets the sphere in two coincident points at (α, β, γ) and so is a tangential line to it there at for any set of values of l, m, n which satisfy (3).

The locus of the tangent lines at (α, β, γ) is, thus, obtained by eliminating l, m, n between (3) and the equations (2) of the line and this gives

$$(x-\alpha)(\alpha + u) + (y-\beta)(\beta + v) + (z-\gamma)(\gamma + w) = 0$$

or $\quad \alpha x + \beta y + \gamma z + u\,(x + \alpha) + v\,(y + \beta) + w\,(z + \gamma) + d = S_1 = 0$

from (1); which is a plane known as the tangent plane at (α, β, γ). Hence

$$(\alpha + u)\, x + (\beta + v)\, y + (\gamma + w)\, z + (u\,\alpha + v\,\beta + w\,\gamma + d) = 0 \qquad \dots (4)$$

is the equation of the tangent plane to the sphere $S = 0$ at (α, β, γ).

Cor. 1: The line joining the centre of a sphere to any point on it is perpendicular to the tangent plane there at.

For the direction cosines of the line joining the centre $(-u, -v, -w)$ to the point (α, β, γ) on the sphere are proportional to $\alpha + u, \beta + v, \gamma + w$ which are also the coefficients of x, y, z in the equation of the tangent plane at (α, β, γ).

Cor. 2: If a plane or line touches a sphere, then the length of the perpendicular from its centre to the plane or the line is equal to its radius as can be easily verified from the equation (4) of the tangent plane.

15.4 Plane of contact

We shall find the locus of the points of contact of the tangent planes which pass through a given point (α, β, γ).

Let (x', y', z') be any point on the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. The tangent plane $x\,(x' + u) + y\,(y' + v) + z\,(z' + w) + (ux' + vy' + wz' + d) = 0$ at this point passes through (α, β, γ), if

$$\alpha\,(x' + u) + \beta\,(y' + v) + \gamma\,(z' + w) + ux' + vy' + wz' + d = 0$$

or $$x'(\alpha + u) + y'(\beta + v) + z'(\gamma + w) + (\alpha u + \beta v + \gamma w + d) = 0$$

which is the condition that the point (x', y', z') should lie on the plane

$$x\,(\alpha + u) + y\,(\beta + v) + z\,(\gamma + w) + (u\,\alpha + v\,\beta + w\,\gamma + d) = 0.$$

It is called the plane of contact for the point (α, β, γ). Thus the locus of points of contact is the circle in which the plane cuts the sphere.

15.5 Conditions for two spheres to touch at a point internally or externally

Def: If two spheres S_1, S_2 with centres B, C and radii r_1, r_2 (>0) have a common tangent plane Π at common point A, then

(i) S_1, S_2 touch internally when B, C lie on the same side of Π (fig. 1(a)).

(ii) S_1, S_2 touch externally when B, C lie on either side of Π (fig. 1(b)).

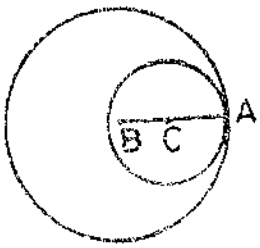


Fig. 1 (a)

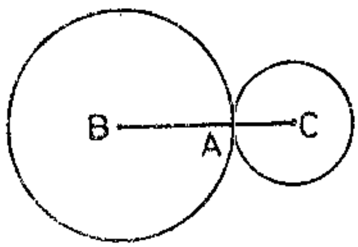


Fig. 1 (b)

Theorem : Two spheres S_1, S_2 having centres B, C and radii r_1, r_2 touch

(i) internally when $BC = |r_1 - r_2|$ and (ii) externally when $BC = r_1 + r_2$.

Proof: Let the two spheres S_1, S_2 touch at A internally. Then Π is common tangent plane at A to both S_1, S_2. B and C are on the same side of plane Π.

$\therefore$ B, C, A are on a straight line since CA and BA are both perpendicular to the common tangent plane Π. Also B and C are on the same side of A.

$\therefore \quad BC = |BA - CA| = |r_1 - r_2|$.

Conversely, if $BC = |r_1 - r_2|$, we can choose a point A outside and on BC such that $BA = r_1$, $CA = r_2$.

Since $BA = r_1$, $CA = r_2$, A is a common point to S_1, S_2. Let the plane through A drawn perpendicular to BC be Π.

$BA \perp \Pi \Rightarrow \Pi$ is tangent plane to the sphere S_1

$CA \perp \Pi \Rightarrow \Pi$ is tangent plane to the sphere S_2

$\therefore$ Π is the common tangent plane to the spheres S_1, S_2. To this plane Π, B and C are on the same side ($\because$ B and C are on the same side of A).

$\therefore$ S_1, S_2 touch internally.

Similarly, S_1, S_2 touch externally $\Leftrightarrow BC = r_1 + r_2$.

Note: Two spheres having centres B, C and radii r_1, r_2 touch at A

(i) internally, then A divides BC externally in the ratio $r_1 : r_2$.

(ii) externally, then A divides BC internally in the ratio $r_1 : r_2$.

Examples

Ex. 1: Find the equations of the spheres passing through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, y = 0$ and touching the plane $3y + 4z + 5 = 0$..

Sol : The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + ky = 0$$

or $$x^2 + y^2 + z^2 - 6x + ky - 2z + 5 = 0 \qquad \ldots (1)$$

Its centre is $(3, -k/2, 1)$ and radius $= \sqrt{9 + \frac{k^2}{4} + 1 - 5} = \sqrt{5 + \frac{k^2}{4}}$

The sphere (1) will touch the plane $3y + 4z + 5 = 0$, its perpendicular from the centre $(3, -k/2, 1)$ on the plane equals the radius of the sphere.

i.e., $$\pm \frac{3(-k/2) + 4 + 5}{\sqrt{9 + 16}} = \sqrt{5 + \frac{k^2}{4}}$$

or if $$4k^2 + 27k + 44 = 0$$

$$\therefore \quad k = \frac{-27 \pm \sqrt{27^2 - 704}}{8} = \frac{-11}{8} \text{ or } -4.$$

Substituting these values of k in (1), we get

$$x^2 + y^2 + z^2 - 6x - \frac{11}{4}y - 2z + 5 = 0$$

and $$x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0$$

as the two required spheres.

Ex. 2 : Show that the two spheres $x^2 + y^2 + z^2 = 25$, $x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch each other externally and find their point of contact.

Sol : The centre of the first sphere is $B\ (0, 0, 0)$ and its radius is $r_1 = 5$.

The centre of the first sphere is $C\ (12, 20, 9)$ and its radius is $r_2 = \sqrt{12^2 + 20^2 + 9^2 - 225} = 20$.

Now $BC = \sqrt{12^2 + 20^2 + 9^2} = 25 = 5 + 20 = r_1 + r_2$.

Therefore the two spheres touch each other externally. If A is the point of contact then A divides BC in the ratio $r_1 : r_2$ i.e., $1 : 4$. Therefore A has the co-ordinates

$$\left(\frac{1.12 + 4.0}{5}, \frac{1.20 + 4.0}{5}, \frac{1.9 + 4.0}{5}\right) = \left(\frac{12}{5}, 4, \frac{9}{5}\right)$$

Ex. 3: Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at $(1, 1, -1)$ passes through the origin.

Sol: The tangent plane to the given sphere at $(1, 1, -1)$ is $x + 5y - 6 = 0$

The equation of the required sphere is, therefore, of the form

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 + k\,(x + 5y - 6) = 0.$$

This will pass through the origin if $k = -\frac{1}{2}$. Thus the required equation is

$$2\left(x^2 + y^2 + z^2\right) - 3x + y + 4z = 0.$$

Aliter.

Let the required equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

As it passes through the origin $d = 0$.

The equation of the tangent plane at $(1, 1, -1)$ to this sphere is

$$x\,(u + 1) + y\,(v + 1) + z\,(w - 1) + u + v - w = 0. \qquad \ldots (1)$$

The tangent plane to the given sphere at $(1, 1, -1)$ is

$$x + 5y - 6 = 0 \qquad \ldots (2)$$

Since (1) and (2) should be identical, we get

$$\frac{u + 1}{1} = \frac{v + 1}{5} = \frac{w - 1}{0} = \frac{u + v - w}{-6}$$

$$\therefore\ w = 1 \qquad \ldots (3)$$

$$5u - v = -4 \qquad \ldots (4)$$

$$7u + v - w = -6 \qquad \ldots (5)$$

Putting $w = 1$ in (5), we get $\quad 7u + v = -5 \qquad \ldots (6)$

Solving (4) and (6), we get

$$u = -\frac{3}{4},\ v = -\frac{15}{4} + 4 = \frac{1}{4}.$$

Therefore the required equation of the sphere is

$$x^2 + y^2 + z^2 - \frac{3x}{2} + \frac{y}{2} + 2z = 0$$

or $\quad 2\left(x^2 + y^2 + z^2\right) - 3x + y + 4z = 0$

Ex. 4: Find the two tangent planes to

(i) the sphere $x^2 + y^2 + z^2 - 2y - 6z + 5 = 0$ which are parallel to the plane $2x + 2y = z$

(ii) the sphere $x^2 + y^2 + z^2 = 9$ which pass through the line $x + y = 6, x - 2z = 3$.

Sol: (i) The general equation of a plane parallel to the given plane $2x + 2y - z = 0$ is

$$2x + 2y - z + \lambda = 0$$

This will be a tangent plane, if its distance from the centre (2, –1, 3) of the spehre is equal to the radius 3 and this requires

$$\frac{-1 + \lambda}{\pm 3} = 3$$

Thus $\lambda = 10$ or -8.

Hence the required tangent planes are

$$2x + 2y - z + 10 = 0 \text{ and } 2x + 2y - z - 8 = 0$$

(ii) Any plane

$$x + y - 6 + \lambda (x - 2z - 3) = 0$$

through the given line will touch the given sphere if

$$\pm \frac{-6 - 3\lambda}{\sqrt{(1 + \lambda)^2 + 1 + \lambda^2}} = 3$$

or $2\lambda^2 - \lambda - 1 = 0$.

This gives $\lambda = 1, -\frac{1}{2}$.

The corresponding two planes, therefore are

$2x + y - 2z = 9, x + 2y + 2z = 9$.

Ex. 5: Find the equation of the sphere inscribed in the tetrahedron whose faces are

$x = 0, y = 0, z = 0, 2x + 6y + 3z - 14 = 0$.

Sol: Let the required equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

The centre of this sphere is $(-u, -v, -w)$ and its radius $r = \sqrt{u^2 + v^2 + w^2 - d}$.

The equation of the plane $2x + 6y + 3z - 14 = 0$ can be written in the intercept form so that its intercepts on the x, y, z axes are 7, 7/3, 14/3 respectively.

Since the required sphere is inscribed in the tetrahedron, we have

$0 < -u < 7, 0 < -v < 7/3, 0 < -w < 14/3$. ... (1)

Since $x = 0$ is a tangent plane to the required sphere $|u| = r$. Therefore $r = -u$.

Since $y = 0, z = 0$ are also tangent planes to the sphere, we have $r = -v = -w$.

Since $u^2 + v^2 + w^2 - d = r^2$ we get $d = 2r^2$.

$\therefore u = v = w = -r, d = 2r^2 = 2u^2$.

Since $2x + 6y + 3z - 14 = 0$ is a tangent plane to the sphere, we get

$$\left|\frac{-2u - 6v - 3w - 14}{7}\right| = r$$

$$\therefore (2u + 6v + 3w + 14)^2 = 49u^2$$

$$\text{or } (11u + 14)^2 = 49u^2$$

$$\text{or } 11u + 14 = \pm 7u \therefore u = -7/9 \text{ or } -7/2.$$

Therefore from (1), we get $u = v = w = -\frac{7}{9}$ and $d = 2u^2 = \frac{98}{81}$.

Therefore the required equation of the sphere is

$$x^2 + y^2 + z^2 - \frac{14}{9}(x + y + z) + \frac{98}{81} = 0.$$

or $81(x^2 + y^2 + z^2) - 126(x + y + z) + 98 = 0.$

15.6 Summary

Equation of a tangent plane at A (α, β, γ) of a sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $(\alpha + u)x + (\beta + v)y + (\gamma + w)z + (u\alpha + v\beta + w\gamma + d) = 0$. Two spheres $S_1 = 0$, $S_2 = 0$ touch each other if and only if the distance between the centres of the spheres is equal to the sum of their radii or equal to the absolute value of the difference of the radii.

15.7 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Derive the equation of a tangent plane at a point (α, β, γ) of a sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

b) Obtain the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which pass through the line $3(16 - x) = 3z = 2y + 30$.

(ii) a) Obtain the conditions for two spheres to touch at a point internally (or externally).

b) Show that the equation of the sphere which touches the sphere

$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ at $(1, 2, -2)$ and passes through the point $(-1, 0, 0)$ is $x^2 + y^2 + z^2 + 2x - 6z + 1 = 0$.

II. *Briefly answer the following questions.*

i) Find the co-ordinates of the points on the sphere $x^2 + y^2 + z^2 - 4x + 2y = 4$ the tangent planes at which are parallel to the plane $2x - y + 2z = 1$.

ii) Obtain the equations of the sphere which pass through the circle $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$; $2x + y + z = 4$ and touch the plane $2x + 4y = 14$.

iii) Find the equation of the sphere inscribed in the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$, $2x - 6y + 3z + 6 = 0$.

iv) Show that the plane $lx + my + nz = p$ will touch the shpere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

v) Obtain the equaions of the spheres that pass through the points (4, 1, 0), (2, −3, 4), (1, 0, 0) and touch the plane $2x + 2y - z = 11$.

vi) Find the equation of the sphere which has its centre at the origin and which touches the line $2(x + 1) = 2 - y = z + 3$.

vii) Find the centres of the two spheres which touch the plane $4x + 3y = 47$ at the point (8, 5, 4) and which touch the sphere $x^2 + y^2 + z^2 = 1$.

viii) Show that the planes of contact of all points on the line $\frac{x}{2} = \frac{(y - a)}{2} = \frac{(z + 3a)}{4}$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ pass through the line $-(2x + 3z)/13 = (y - a)/3 = z = 1$.

Answers

I. (ib) $2x + 2y - z - 2 = 0$, $x + 2y - 2z + 14 = 0$.

II. (i) $(4, -2, 2)$, $(0, 0, -2)$

(ii) $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$;

$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$.

(iii) $9(x^2 + y^2 + z^2) + 6(x - y + z) + 2 = 0$.

(v) $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$;

$16(x^2 + y^2 + z^2) - 102x + 50y - 49z + 86 = 0$.

(vi) $9(x^2 + y^2 + z^2) = 5$

vii) $(4, 2, 4)$, $(64/21, 27/21, 4)$.

Unit–16 : POLAR PLANE AND SYSTEM OF SPHERES

16.0 Contents

16.1 Aims and Objectives

After going through this unit, you will be able to:

i) derive the equation of a polar plane of a point

ii) define and obtain poles, polars, conjugate points with respect to a sphere

iii) define the radical planes of coaxal system of spheres.

16.2 Introduction

In this unit we have derived the equation of a polar plane of a given point with respect to a given sphere. Various results of poles, polars, conjugate points, conjugate planes have been discussed. We define the angle of intersection of two spheres and obtain the condition for their orthogonality. Radical plane of two spheres, radical line of three spheres and radical centre of four spheres have been defined. Finally we have shown that $x^2 + y^2 + z^2 + 2kx + d = 0$ for different k and constant d represents a coaxal system of spheres, having $x = 0$ (YZ plane) as the radical plane and the x – axis as the line of their centres. The point spheres of this system called the limiting points have been obtained.

16.3 The polar plane

Def : If a variable line is drawn through a fixed point A meeting a given sphere in P, Q and the point R is taken on this line such that the points A, R divide PQ internally and externally in the same ratio, then the locus of R is a plane called the polar plane of A with respect to the sphere.

Theorem 1 : The equation of the polar plane of the point (α, β, γ) with respect to the sphere

$$x^2 + y^2 + z^2 = a^2$$

is

$$\alpha x + \beta y + \gamma z = a^2.$$

Proof : We use well known result that if the points A, R divide PQ internally and externally in the same ratio, then the points P, Q also divide AR internally and externally in the same ratio.

Consider the sphere

$$x^2 + y^2 + z^2 = a^2 \qquad \text{... (1)}$$

and let A be the point (α, β, γ).

Let (x, y, z) be the co-ordinates of the point R on any line through A. The co-ordinates of the point dividing AR in the ratio $\lambda : 1$ are

$$\left[\frac{\lambda x + \alpha}{\lambda + 1}, \frac{\lambda y + \beta}{\lambda + 1}, \frac{\lambda z + \gamma}{\lambda + 1}\right].$$

This point will be on the sphere (1) for all values of λ which are the roots of the quadratic equation

$$\left(\frac{\lambda x + \alpha}{\lambda + 1}\right)^2 + \left(\frac{\lambda y + \beta}{\lambda + 1}\right)^2 + \left(\frac{\lambda z + \gamma}{\lambda + 1}\right)^2 = a^2$$

i. e., $\lambda^2\left(x^2 + y^2 + z^2 - c^2\right) + 2\lambda\left(\alpha x + \beta y + \gamma z - a^2\right) + \left(\alpha^2 + \beta^2 + \gamma^2 - a^2\right) = 0.$ ... (2)

Its roots λ_1 and λ_2 are the ratios in which the points P, Q divide AR.

Since P, Q divide AR internally and externally in the same ratio, we have

$$\lambda_1 + \lambda_2 = 0.$$

Thus from (2) we have

$$\alpha x + \beta y + \gamma z - a^2 = 0 \qquad \text{... (3)}$$

which is the relation satisfied by the co-ordinates (x, y, z) of R.

Hence (3) is the locus of R. Clearly it is a plane. Thus we have proved that the equation of the polar plane of the point (α, β, γ) with respect to the sphere

$$x^2 + y^2 + z^2 = a^2$$

is $$\alpha x + \beta y + \gamma z = a^2.$$

It may similarly be shown that the polar plane of (α, β, γ) with respect to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $$(\alpha + u)x + (\beta + v)y + (\gamma + w)z + (u\alpha + v\beta + w\gamma + d) = 0.$$

Def : If Π be the polar plane of a point P, then P is called the pole of the plane Π.

Theorem 2 : The pole of the plane

$$lx + my + nz = P \qquad \text{... (1)}$$

with respect to the sphere

$$x^2 + y^2 + z^2 = a^2$$

is $$\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$$

Proof : Let (α, β, γ) be the required pole. Then its polar plane with respect to the given sphere is

$$\alpha x + \beta y + \gamma z = a^2 \qquad \text{... (2)}$$

so that, on comparing (1) and (2), we get

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{a^2}{p},$$

or $$\alpha = \frac{a^2 l}{p}, \ \beta = \frac{a^2 m}{p}, \ \gamma = \frac{a^2 n}{p}$$

Thus $\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$ is the required pole.

16.4 Some results concerning poles and polars

In the following discussion we take the equation of the sphere in the form

$$x^2 + y^2 + z^2 = a^2.$$

Result 1: The line joining the centre O of a sphere to any point P is perpendicular to the polar plane of P.

For the direction ratios of the line joining the centre O $(0, 0, 0)$ to the point P (α, β, γ) are α, β, γ and these are also the direction ratios of the normal to the polar plane $\alpha x + \beta y + \gamma z = a^2$ of P (α, β, γ).

Result 2 : If the line joining the centre O of a sphere to any point P meets the polar plane of P in Q then

$$OP \,.\, OQ = a^2,$$

where a is the radius of the sphere.

$$\text{We have } OP = \sqrt{\alpha^2 + \beta^2 + \gamma^2}.$$

Also OQ which is the length of the perpendicular from the centre O $(0, 0, 0)$ to the polar plane $\alpha x + \beta y + \gamma z = a^2$ of P, is given by

$$|OQ| = \frac{a^2}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}.$$

Hence the result.

Result 3 : If the polar plane of a point P passes through another point Q, then the polar plane of Q passes through P.

The condition that the polar plane

$$\alpha_1 x + \beta_1 y + \gamma_1 z = a^2,$$

of P $(\alpha_1, \beta_1, \gamma_1)$ passes through Q $(\alpha_2, \beta_2, \gamma_2)$ is

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = a^2,$$

which is also, by symmetry, or directly the condition that the polar plane of Q passes through P.

16.5 Conjugate points, conjugate planes and polar lines

Def. : Two points such that the polar plane of either passes through the other are called conjugate points.

Result 1 : If the pole of a plane Π_1 lies on another plane Π_2 then the pole of Π_2 lies on Π_1.

The condition that the pole

$$\left(\frac{a^2 l_1}{p_1}, \frac{a^2 m_1}{p_1}, \frac{a^2 n_1}{p_1}\right)$$

of the plane Π_1

$$l_1 x + m_1 y + n_1 z = p_1$$

lies on the palne Π_2

$$l_2 x + m_2 y + n_2 z = p_2$$

is

$$a^2\left(l_1 l_2 + m_1 m_2 + n_1 n_2\right) = p_1 p_2$$

which is also, clearly, the condition that the pole

$$\left(\frac{a^2 l_2}{p_2}, \frac{a^2 m_2}{p_2}, \frac{a^2 n_2}{p_2}\right)$$

of Π_2 lies on Π_1.

Def. : Two planes such that the pole of either lies on the other are called conjugate planes.

Result 2 : The polar planes of all the points on a line L pass through another line L'.

The polar plane of any point

$$(lr + \alpha, mr + \beta, nr + \gamma)$$

on the line L

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

is $\quad (lr+\alpha)\, x + (mr+\beta)\, y + (nr+\gamma)\, z = a^2$

or $\quad \left(\alpha x + \beta y + \gamma z - a^2\right) + r\,(lx + my + nz) = 0$

which clearly passes through the line

$$\alpha x + \beta y + \gamma z - a^2 = 0, \; lx + my + nz = 0$$

whatever value r may have,

Let this line be L.

Now as the polar plane of any arbitrary point P on L passes through every point of L' the pola plane of every point of L' passes through the point P on L and as P is arbitrary, it passes throug every point of L i.e., it passes through L.

Thus we see that if L' is the line such that the polar planes of all points on the line L, pass through it then the polar planes of all the points on L' pass through L.

Def. : Two lines such that the polar plane of every point on either line passes through the other are called polar lines.

16.6 Angle of intersection of two spheres

Def. : The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point. This is, therefore, also equal to the angle between the radii of the spheres to the common point, the radii being perpendicular to the respective tangent planes at the point.

The angle of intersection at every common point of the spheres is the same, for if P, P' be any two common points and C, C' be the centres of the spheres the triangles $CC'P$ and $CC'P'$ are congruent and accordingly

$$\angle CPC' = \angle CP'C'.$$

If the angle of intersection of two spheres is a right angle, the spheres are said to be orthogonal, In this case

$$CC'^2 = CP^2 = C'P^2.$$

16.6.1 Condition for the orthogonality of two spheres

We shall obtain the condition for the two spheres

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

to be orthognal.

The spheres will be orthogonal if the square of the distance between their centres is equal to the sum of the squares of their radii and this requires

$$\left(u_1 - u_2\right)^2 + \left(v_1 - v_2\right)^2 + \left(w_1 - w_2\right)^2$$

$$= \left(u_1^2 + v_1^2 + w_1^2 - d_1\right) + \left(u_2^2 + v_2^2 + w_2^2 - d_2\right)$$

$$2\,u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$$

16.7 Radical plane, line and centre

Radical plane : We will show that the locus of points whose powers with respect to two spheres are equal is a plane perpendicular to the line joining their centres.

The powers of the point $P \ (x, y, z)$ with respect to the spheres

$$S_1 = x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1 = 0$$

$$S_2 = x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0$$

are S_1 and S_2 respectively.

Equating them we obtain

$$2x\left(u_1 - u_2\right) + 2y\left(v_1 - v_2\right) + 2z\left(w_1 - w_2\right) + d_1 - d_2 = 0$$

which is the required locus and being of the first degree in x, y, z represents a plane which is obviously perpendicular to the line joining the centres of the two spheres and is called the radical plane of the two spheres.

Thus the radical plane of two spheres $S_1 = 0$, $S_2 = 0$ in which the coefficients of the second degree terms are each equal to unity is

$$S_1 - S_2 = 0.$$

In case the two spheres intersect, the plane of their common circle is their radical plane.

Radical Line :

We will show that the three radical planes of three spheres taken two by two intersect in a line.

For let the three spheres be given by

$$S_1 = 0, \ S_2 = 0, S_3 = 0.$$

Then their radical planes are

$$S_1 - S_2 = 0; S_2 - S_3 = 0; S_3 - S_1 = 0.$$

They clearly meet in the line

$$S_1 = S_2 = S_3$$

This line is called the radical line of the three spheres.

Radical Centre :

The four radical lines of four spheres taken three by three intersect at a point.

The point common to the three planes

$$S_1 = S_2 = S_3 = S_4$$

is clearly common to the radical lines of the four spheres

$$S_1 = 0, \ S_2 = 0, S_3 = 0, S_4 = 0.$$

This point is called the radical centre of the four spheres.

16.8 A simplified form of the equation of the two given spheres, Limiting points

Therorem : If $S_1 = 0, S_2 = 0$ be two spheres, then the equation

$$S_1 + \lambda S_2 = 0,$$

λ being the parameter, represents a system of spheres such that any two members of the system have the same radical plane.

Let $S_1 + \lambda_1 S_2 = 0$ and $S_1 + \lambda_2 S_2 = 0$ be any two members of system.

Making the coefficients of second degree term unity, we write them in the form

$$\frac{S_1 + \lambda_1 S_2}{1 + \lambda_1} = 0, \quad \frac{S_1 + \lambda_2 S_2}{1 + \lambda_2} = 0$$

The radical plane of these two spheres is

$$\frac{S_1 + \lambda_1 S_2}{1 + \lambda_1} - \frac{S_1 + \lambda_2 S_2}{1 + \lambda_2} = 0$$

or $$S_1 - S_2 = 0.$$

Since this equation is independent of λ_1 and λ_2, we see that every two members of the system have the same radical plane.

Coaxal System : *Def :* A system of spheres such that any two of them have the same radical plane is called a coaxal system of spheres.

Thus the system of spheres

$$S_1 + \lambda S_2 = 0$$

is coaxal and we say that it is determined by the spheres

$$S_1 = 0 \text{ and } S_2 = 0.$$

The coaxal system is also given by the equation

$$S_1 + K (S_1 - S_2) = 0 \qquad \dots (1)$$

Cor : The locus of the centres of spheres of a coaxal system is a line,

for if (x, y, z) be the centre of the sphere

$$S_1 + \lambda S_2 = 0$$

we have

$$x = -\frac{u_1 + \lambda u_2}{1 + \lambda}, y = -\frac{v_1 + \lambda v_2}{1 + \lambda}, z = -\frac{w_1 + \lambda w_2}{1 + \lambda}.$$

On eliminating λ, we find that it lies on the line

$$\frac{x + u_1}{u_1 - u_2} = \frac{y + v_1}{v_1 - v_2} = \frac{z + w_1}{w_1 - w_2}$$

This result is also otherwise obvious as the line joining the centres of any two spheres is perpendicular to their common radical plane.

Simplified form of the equation of two spheres

By taking the line joining the centres of two given spheres as x-axis say $(-u_1, 0, 0)$ and $(-u_2, 0, 0)$ and radii $\sqrt{u_1^2 - d_1}$, $\sqrt{u_2^2 - d_2}$ respectively, their equations take the form

$$x^2 + y^2 + z^2 + 2 u_1 x + d_1 = 0, \qquad x^2 + y^2 + z^2 + 2 u_2 x + d_2 = 0$$

so that their radical plane is

$$2x (u_1 - u_2) + (d_1 - d_2) = 0.$$

Further, if we take their radical plane as the *YZ* plane i.e., $x = 0$, we have $d_1 = d_2 = d$ (say).

Thus by taking the line joining the centres as x-axis and their radical plane as the *YZ* plane, the equation of any two spheres can be put in the simplified form

$$x^2 + y^2 + z^2 + 2 u_1 x + d = 0, \qquad x^2 + y^2 + z^2 + 2 u_2 x + d = 0$$

where u_1, u_2 are different.

Cor.1 : The equation $x^2 + y^2 + z^2 + 2kx + d = 0$ represents a coaxal system of spheres for different values of k, d being constant. The plane *YZ* is the radical plane and X-axis is the line of centres.

Cor. 2 : Limiting points: The equation $x^2 + y^2 + z^2 + 2kx + d = 0$ can be written as

$$(x + k)^2 + y^2 + z^2 = k^2 - d$$

For $k = \pm\sqrt{d}$, we get spheres of the system with radius zero and thus the system includes two point spheres.

$$\left(-\sqrt{d}, 0, 0\right), \qquad \left(\sqrt{d}, 0, 0\right).$$

These two points are called the limiting points and only if d is positive i.e., when the spheres do not meet the radical plane in a real circle.

Def : Limiting points of a coaxal system of spheres are the point spheres of the system.

Examples

Ex. 1: Find the angle of intersection of the two spheres namely $x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$ and the other described on the line joining the points (1, 2, –3), (5, 0, 1) as diameter.

Sol : Let S' be the sphere, described on the line joining the points (1, 2, –3), (5, 0, 1) as diameter. Let its centre be C' and radius r'.

Then C' has the co-ordinates (3, 1, –1).

Then $r = \frac{1}{2}\sqrt{4^2 + 2^2 + 4^2} = 3.$

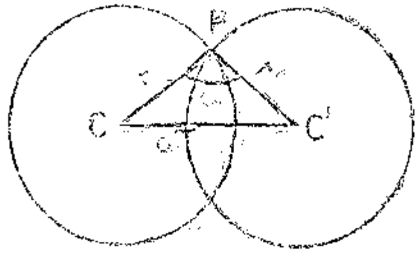


Fig. 1

Let S be the sphere whose equation is $x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$. The centre C of this sphere has the co-ordinates (1, 2, 3) and radius equal to $\sqrt{1^2 + 2^2 + 3^2 - 10} = 2$. The two spheres intersect each other if $r + r' > d$. Since $d = CC' = \sqrt{(3-1)^2 + (1-2)^2 + (-1-3)^2} = \sqrt{21}$, we have $r + r' = 2 + 3 = 5 > d = \sqrt{21}$ in this problem. Let the angle of intersection $\angle CPC'$ be α. We know from Trigonometry

$$d^2 = r^2 + r'^2 - 2rr' \cos\alpha$$

$$\text{or } 2rr' \cos\alpha = r^2 + r'^2 - d^2$$

$$\therefore \cos\alpha = \frac{r^2 + r'^2 - d^2}{2rr'}$$

$$= \frac{2^2 + 3^2 - 21}{2 \cdot 2 \cdot 3} = \frac{-2}{3}.$$

Hence the angle of intersection of the two spheres is $\cos^{-1}\left(-\frac{2}{3}\right)$.

Ex. 2 : Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is

$$r_1 r_2 \Big/ \sqrt{r_1^2 + r_2^2}$$

Sol : Let the common circle be $x^2 + y^2 = a^2, z = 0$.

The general equation of the sphere through this circle being

$$x^2 + y^2 + z^2 + 2kz - a^2 = 0,$$

let the two given spheres through this circle be

$$x^2 + y^2 + z^2 + 2k_1 z - a^2 = 0, \qquad x^2 + y^2 + z^2 + 2k_2 z - a^2 = 0.$$

We have

$$r_1^2 = k_1^2 + a^2, \qquad r_2^2 = k_2^2 + a^2.$$

Since the spheres cut orthogonally,

$$\left(k_2 - k_1\right)^2 = k_1^2 + a^2 + k_2^2 + a^2$$

$$\text{or } k_1 k_2 = -a^2$$

Squaring we get $\quad k_1^2 k_2^2 = a^4$

i.e. $\left(r_1^2 - a^2\right)\left(r_2^2 - a^2\right) = a^4$

$$\text{or } r_1^2 r_2^2 - a^2\left(r_1^2 + r_2^2\right) = 0$$

$$\text{or } a^2 = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}$$

$$\text{or } a = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

giving the required result.

Ex. 3 : Find the limiting points of the coaxal system defined by the spheres

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0, \quad x^2 + y^2 + z^2 - 6y - 6z + 6 = 0.$$

Sol : The equation of the plane of the circle through the two given spheres is

$$3x + 3y + 6z = 0, \qquad \text{i.e, } x + y + 2z = 0.$$

Then the equation of the co-axal system determined by the given spheres is

$$x^2 + y^2 + z^2 + 3x - 3y + 6 + \lambda(x + y + 2z) = 0$$

$$x^2 + y^2 + z^2 + (3 + \lambda)x + (\lambda - 3)y + 2\lambda z + 6 = 0.$$

Its centre is $\left[-\frac{3+\lambda}{2}, -\frac{\lambda-3}{2}, -\lambda\right]$

and radius is $\sqrt{\left[\left(\frac{3+\lambda}{2}\right)^2 + \left(\frac{\lambda-3}{2}\right)^2 + \lambda^2 - 6\right]}$

Equating this radius to zero, we obtain

$$6\lambda^2 - 6 = 0$$

i.e., $\quad \lambda = \pm 1.$

The spheres corresponding to these values of λ become point spheres coinciding with their centres and are the limiting points of the system.

The limiting points, therefore, are

$$(-1, 2, 1) = (-2, 1, -1)$$

Ex. 4 : Find the equations to the two spheres of the coaxal system

$$x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$$

which touch the plane $3x + 4y = 15$.

Sol. : Let the equation of the sphere S be given by $x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$

Then S has the centre $C\left(-\lambda, \frac{-\lambda}{2}, \frac{-3\lambda}{2}\right)$ and radius equal to $\sqrt{\lambda^2 + \frac{\lambda^2}{4} + \frac{9\lambda^2}{4} + 3\lambda + 5}.$

Since the plane $3x + 4y = 15$ touches the sphere S, we require, length of the perpendicular from C to the plane $3x + 4y - 15 = 0$ is equal to the radius of S.

$$\therefore \left|\frac{-3\lambda - 2\lambda - 15}{\sqrt{9 + 16}}\right| = \sqrt{\lambda^2 + \frac{\lambda^2}{4} + \frac{9\lambda^2}{4} + 3\lambda + 5}$$

This on simplification gives $\lambda = 2$ or $-4/5$.

Substituting these values of λ in the equation of the sphere, we get

$$x^2 + y^2 + z^2 + 4x + 2y + 6z - 11 = 0,$$

$$5(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0$$

as the required equations of the two spheres.

Ex. 5 : Three spheres of radii r_1, r_2, r_3 have their centres A, B, C at the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ and $r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2$. A fourth sphere passes through the origin and A, B, C. Show that the radical centre of the four spheres lies on the plane $ax + by + cz = 0$.

Sol : The equation of the sphere S_1 having $(a, 0, 0)$ as its centre r_1 radius is $(x-a)^2 + y^2 + z^2 = r_1^2$.

Let the sphere S_2 have its centre $(0, b, 0)$ and radius r_2. Then its equation is

$$x^2 + (y-b)^2 + z^2 = r_2^2$$

Let the sphere S_3 have its centre $(0, 0, c)$ and radius r_3. Then its equation is

$$x^2 + y^2 + (z-c)^2 = r_3^2.$$

Since the sphere S_4 passes through origin and A, B, C its equation can easily be derived as

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

The equation to the radical planes of S_1, S_2 is

$$2ax - 2by + r_1^2 - r_2^2 + b^2 - a^2 = 0. \quad \text{... (1)}$$

The equation to the radical plane S_1, S_3 is

$$2ax - 2cz + r_1^2 - r_3^2 + c^2 - a^2 = 0. \quad \text{... (2)}$$

The equation to the radical plane of S_1, S_4 is

$$ax - by - cz + r_1^2 - a^2 = 0 \quad \text{... (3)}$$

$$\text{Since} \begin{vmatrix} 2a & -2b & 0 \\ 2a & 0 & -2c \\ a & -b & -c \end{vmatrix} = 4\,abc \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4abc \neq 0,$$

We have a unique point of intersection $\left(\dfrac{a^2 - r_1^2}{2a}, \dfrac{b^2 - r_2^2}{2b}, \dfrac{c^2 - r_3^2}{2c}\right)$ of the three radical planes (1), (2) and (3). We also know this point as the radical point of the four spheres S_1, S_2, S_3, S_4.

Multiplying eq. (3) throughout by 3 and subtracting from the sum of the equations (1) and (2), we get

$$ax + by + cz - \left(r_1^2 + r_2^2 - r_3^2\right) + a^2 + b^2 + c^2 = 0.$$

or $ax + by + cz = 0$ since $r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2$.

It can easily be verified that the radical centre $\left(\dfrac{a^2 - r_1^2}{2a}, \dfrac{b^2 - r_2^2}{2b}, \dfrac{c^2 - r_3^2}{2c}\right)$ lies on the plan $ax + by + cz = 0$.

16.9 Summary

Equation of the polar plane of a point $A(\alpha, \beta, \gamma)$ with respect to a sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $(\alpha + u)x + (\beta + v)y + (\gamma + w)z + u\alpha + v\beta + \gamma w + d = 0$.

Pole of the plane $lx + my + nz = p$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is

$$\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$$

The points such that the polar plane of either passes through the other are called conjugate points. Corresponding polar planes are called the conjugate planes. Two spheres $S_1 = 0, S_2 = 0$ cut each other orthogonally if $2u_1 u_2 + 2v_1 v_2 + 2w_1 w_2 = d_1 + d_2$.

If $S_1 = 0, S_2 = 0$ are equations of two spheres in their standard form, $S_1 - S_2 = 0$ is a plane called the radical plane of S_1, S_2. If $S_1 = 0, S_2 = 0, S_3 = 0$ are equations of three spheres, then the line of intersection of the planes $S_1 - S_2 = 0$, $S_2 - S_3 = 0$ is called the radical line. Three planes $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$ intersect in a point, called the radical centre of S_1, S_2 and S_3.

16.10 Sample Examination Questions

I. *Answer the following in detail.*

(i) (a) Define polar plane. Prove that the equation of the polar plane of the point (α, β, γ) with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $\alpha x + \beta y + \gamma z = a^2$.

(b) Define the pole of a plane. Show that the pole of the plane $lx + my + nz = p$, with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$

(ii) (a) Define the angle of intersection of two spheres. Obtain the condition for the orthogonality of two spheres.

(b) Find the equation of the sphere that passes through the circle $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$, $3x - 4y + 5z - 15 = 0$ and cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

(iii) (a) Define radical plane, radical line and radical centre.

(b) Show that the radical planes of the sphere of a coaxal system and of any given sphere pass through a line.

II. *Briefly answer the following.*

(i) Find the equation of the sphere that passes through the two points (0, 3, 0), (–2, –1, –4) and cuts orthogonally the two spheres
$x^2 + y^2 + z^2 + x - 3z - 2 = 0$, $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$.

(ii) Show that the polar line of $\frac{x+1}{2} = \frac{y-2}{3} = z + 3$ with respect to the sphere $x^2 + y^2 + z^2 = 1$ is the line $\frac{7x+3}{11} = \frac{2-7y}{5} = \frac{z}{-1}$.

(iii) Two points P, Q are conjugate with respect to a sphere S; show that the sphere on PQ as diameter cuts S orthogonally.

(iv) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point (1, –2, 1) and cuts orthogonally the sphere
$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

(v) Show that the sphere which cuts two spheres orthogonally will cut every member of the coaxal system determined by them orthogonally.

(vi) Find the limiting points of the coaxal systems of spheres
$x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$

Answers

I. (iib) $5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0$

II. (i) $x^2 + y^2 + z^2 + 2x - 2y + 4z - 3 = 0.$

(iv) $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$

(vi) (2, –3, 4); (–2, 3, –4).

[Author of the units 7-16 : Dr. V.V. Ramana Rao].

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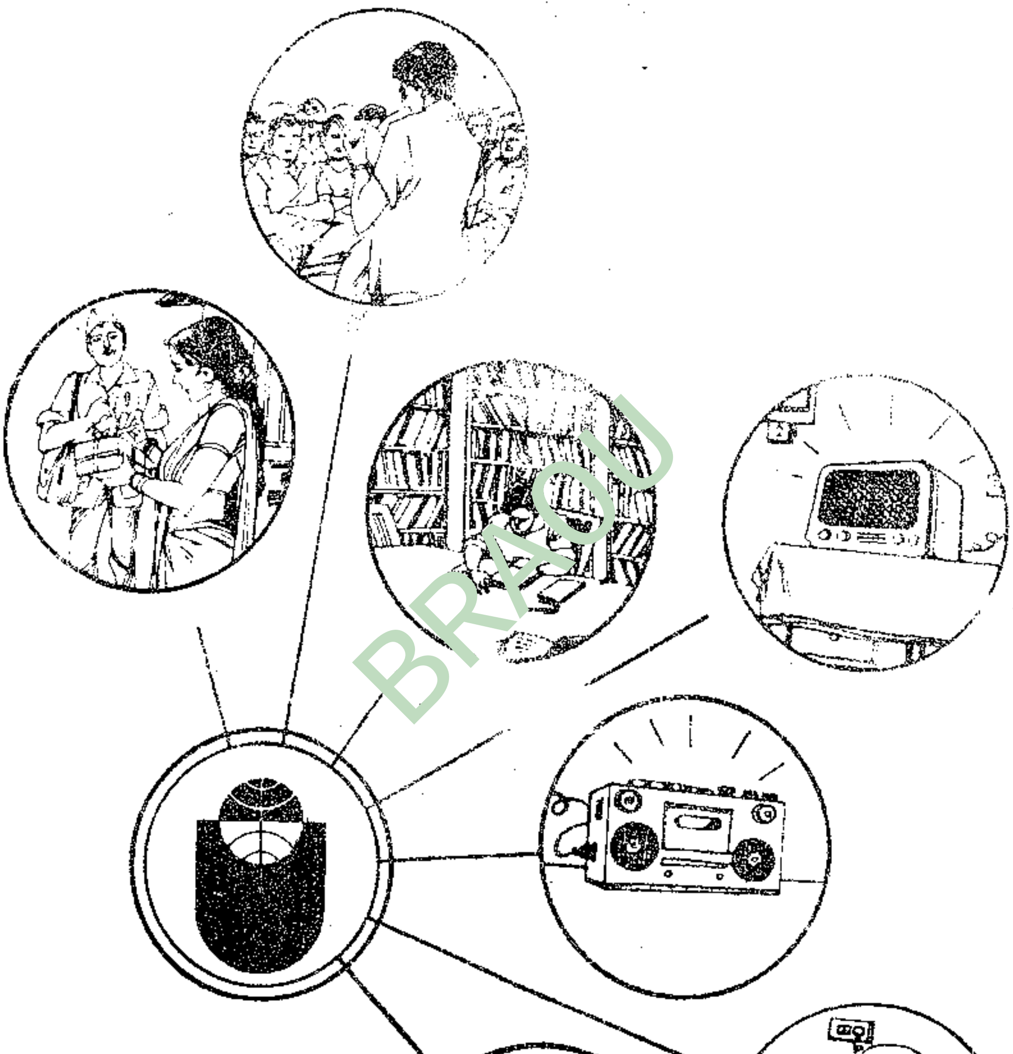

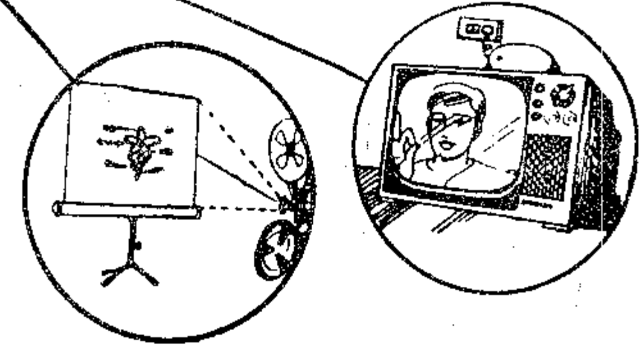

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ANDHRA PRADESH OPEN UNIVERSITY

M2

MATHEMATICS

Conics, Conicoids and Theory of Equations

BLOCKS : 4-6

$$\vec{r} = (1-t)\vec{a} + t\vec{b}$$

$$A^{-1} = \frac{\text{Adj } A}{\text{Det } A}$$

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$$

MATHEMATICS

COURSE – II

Vectors, Three Dimensional Geometry, Theory of Equations

Blocks : IV – VI

ANDHRA PRADESH OPEN UNIVERSITY

HYDERABAD

1991

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PREFACE

This book deals with the topics in Vectors, Three Dimensional Geometry, Theory of equations included in the syllabus for the second year of the B.Sc. Course offered by the Andhra Pradesh Open University. These topics cover the "core" area of the subject to be studied in the Second Year of the Three Year Degree Course in Science. The syllabus for the sake of convenience is divided into Blocks, each of which comprises a number of units. Each Block generally covers a specific area of the subject. The units are prepared by specialists in accordance with a format so designed as to enable the student to read and understand them without much difficulty. Each unit begins with a statement of its objectives and has at its end assignments intended to test the students comprehension of its subject matter.

In Block–1, vector is taken as a quantity which has magnitude and direction to make the student to apply the concepts to geometry and physics. Analytical Geometry provides a general method for solving geometric problems. This analytic geometry was explained in Blocks 2, 3, 4 and 5. Roots of an equation, relation between the roots and coeffcients, nature of roots, and solutions of cubic and biquadratic equations are introduced through the theory of equations in Block 6.

The university hopes that the course material will help the student to get acquainted with the concepts and principles of Mathematics.

Contents

BLOCK – 4 : CONICS

Introduction

The locus of a point which moves such that its distance from a fixed point is proportional to its distance from a fixed line is called a conic. The constant of proportionality e (> 0) is called eccentricity. The conic is an ellipse, parabola or a hyperbola according as e is less than, equal to or greater than unity. Geometrically, the section of a right circular cone by a plane gives a conic.

The theory of conics is useful in the study of conicoids in three dimensional geometry. Often we come across conics in mathematics and sciences. In astronomy, we know that every planet describes an ellipse with sun at one of the foci. In mechanics, it is established that if a particle moves in a path so that the acceleration is always directed towards a fixed point and inversely proportional to the square of its distance to the fixed point, its path is a conic. If a particle is projected under the action of gravity of the earth, the path is a parabola if the air resistance is negligible. If a source of rays is placed at the focus of a parabola, the rays are reflected in parallel lines. This principle is made use of in heat and light reflectors.

UNIT–17 : PARABOLA

17.0 Contents

17.1 Aims and objectives

After going through this unit, you will be able to :

(i) define a Conic and obtain the equation of a parabola

17.2 Conic

In a plane, if (i) l is a given line

(ii) S is a point on l

(iii) $e > 0$ is a real number and

(iv) PM is the distance of a point P from line l

then the locus of points P such that $SP = e.PM$ is called a conic. If C is the given conic, $C = \{P / SP = e \,.\, PM\}$. The point S is called a focus, the line l is called a directirx and e is called the eccentricity. If $e < 1$, the conic is called ellipse, if $e = 1$ it is called parabola and if $e > 1$ it is called hyperbola.

Equation of a Conic

The equation of a conic with $S(x_1, y_1)$ as the focus, the line $x \cos\alpha + y \sin\alpha - p = 0$ as the directrix and e as eccentricity :

If P has coordinates (x, y), $SP = \sqrt{(x - x_1)^2 + (y - y_1)^2}$.

The distance of P to the given line, $PM = |x \cos\alpha + y \sin\alpha - p|$.

The equation to the conic is $SP = e.PM$.

$$\therefore \sqrt{(x - x_1)^2 + (y - y_1)^2} = e\,|x \cos\alpha + y \sin\alpha - p| \qquad \dots (1)$$

Squaring and rearranging terms in (1) the equation of the conic is obtained as

$$x^2\left(1-e^2\cos^2\alpha\right)-2e^2\sin\alpha\,.\cos\alpha\,.\,xy+y^2\left(1-e^2\sin^2\alpha\right)$$
$$-2\left(x_1-pe^2\cos\alpha\right)x-2\left(y_1-pe^2\sin\alpha\right)y+\left(x_1^{\,2}+y_1^{\,2}-p^2e^2\right)=0 \qquad \ldots (2)$$

This equation is of degree two in x and y.

Notation : We use the following notation

$$\text{If } S \equiv ax^2+by^2+2hxy+2fy+c, \text{ then}$$
$$S_i = axx_i+byy_i+h\left(xy_i+yx_i\right)+g\left(x+x_i\right)+f\left(y+y_i\right)+c,$$
$$\left(S_i\right)_j = ax_jx_i+by_jy_i+h\left(x_jy_i+y_jx_i\right)+g\left(x_j+x_i\right)+f\left(y_j+y_i\right)+c,$$

It immediately follows that $\left(S_i\right)_j = S_{ij} = S_{ji} = \left(S_j\right)_i$

17.? Parabola

The standard or canonical form of the equation of the parabola is $y^2 = 4ax$ $(a > 0)$.

Suppose S is a focus and ZM is a directrix corresponding to focus S in the definition of a conic. From S draw SZ perpendicular to the directrix. Let length $SZ = 2a$. By definition S is not on the directrix and hence $a > 0$. If A is the mid-point of $\overleftrightarrow{SZ}$, choose $\overleftrightarrow{AS}$ as the x-axis and $\overleftrightarrow{AD}$ perpendicular to $\overleftrightarrow{AS}$, as the y-axis. Then the focus S is $(a, 0)$ and the directrix is $x + a = 0$. If PM is the perpendicular distance from P on the directrix, then

$$SP = e.PM$$
$$\Leftrightarrow SP^2 = e^2\,.\,PM^2$$
$$\Leftrightarrow (x-a)^2+y^2 = (x+a)^2 \quad (\because e = 1)$$
$$\Leftrightarrow y^2 = 4ax\ (a>0)$$

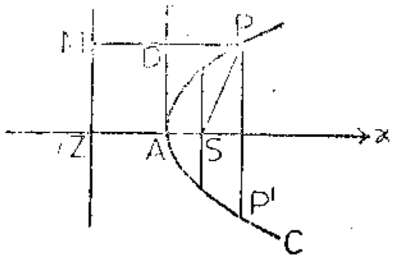


Fig. 1

Properties : If C is the parabola $S(x, y) = y^2 - 4ax = 0$, then

(i) $(x, y)\ \varepsilon\ C \Rightarrow (x, -y)\ \varepsilon\ C$.

$\therefore$ The parabola is symmetric about the x-axis.

(ii) The parabola $y^2 = 4ax$ passes through the origin.

(iii) The line $y = k$ (constant) cuts the parabola in only one point.

(iv) If $x < 0$, y cannot be determined to satisfy $y^2 = 4ax$ $(a > 0)$. Hence the curve is only to the right of the y-axis.

(iv) For any $x > 0$, two values of y, namely $y \pm \sqrt{4ax}$ are determined by the parabola. Hence the parabola branches in the first and the fourth quadrants.

Other forms of the equation

(i) The parabola $y^2 = -4ax$ $(a > 0)$ has $(-a, 0)$ as the focus and $x - a = 0$ as the directrix. It lies in the second and third quadrants. It is the mirror image of the parabola in figure1, about the y-axis. The curve is shown in Figure 2.

(ii) The parabola $x^2 = 4ay$ $(a > 0)$ has $(0, a)$ as the focus and $y + a = 0$ as the directrix. It lies in the first and second quadrants. It is symmetric about the y–axis.

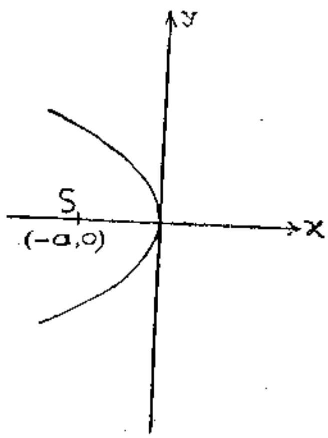


Fig. 2

(iii) The equation of the parabola with (x_1, y_1) as the focus and the directrix parallel to the x-axis has the form $y = ax^2 + bx + c$, where a, b, c are constants.

To prove this, let $y = k$ be the directrix. Since this does not pass through (x_1, y_1), $y_1 \neq k$. Using the definition of a parabola,

$$(x - x_1)^2 + (y - y_1)^2 = (y - k)^2.$$

$$\therefore y = \frac{1}{2(y_1 - k)}x^2 - \frac{x_1}{(y_1 - k)}x + \frac{x_1^2 + y_1^2 - k^2}{2(y_1 - k)}.$$

$\therefore$ The parabola has the form $y = ax^2 + bx + c$.

(iv) The equation of the parabola with (x_1, y_1) as the focus and the directrix parallel to the y-axis has the form $y = ay^2 + by + c$.

17.4 Parametric equations of the parabola

Given any point (x_1, y_1) on the parabola, it is possible to find a real number t such that $x_1 = at^2$, $y_1 = 2at$. Also, for any real t, the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. The coordinates $(at^2, 2at)$ are called parametric coordinates and t is called a parameter. The point $(at^2, 2at)$ is also referred to as the point t on the parabola. The equations $x = at^2$, $y = 2at$ are called parametric equations of the parabola.

Note: Classification of points in the plane of a parabola :

The point (x, y) in the xy –plane is inside the parabolic region if and only if $y^2 - 4ax < 0$. Such a point (x, y) is called an interior point with respect to the parabola. The point (x, y) is outside the parabolic region if and only if $y^2 - 4ax > 0$. Such a point is called an exterior plane with respect to the parabaola. The point (x, y) is on the parabola if and only if $y^2 - 4ax = 0$.

Theorem-1 : A parabola has only one focus and one directrix.

Proof: Without loss of generality, let the equation of the parabola be $y^2 = 4ax$. Suppose (x_1, y_1) is a focus and $x \cos \alpha + y \sin \alpha - p = 0$ is a directrix corresponding to this focus. Then the equation of the parabola is (17.2, eq. (2))

$$x^2 \sin^2 \alpha - 2xy \sin \alpha \cos \alpha + y^2 \cos^2 \alpha - 2\left(x_1 - p \cos^2 \alpha\right)x$$
$$- 2\left(y_1 - p \sin^2 \alpha\right)y + \left(x_1^2 + y_1^2 - p^2\right) = 0 \qquad \ldots (1)$$

This is the same as $y^2 = 4ax$. A general point on it is $(at^2, 2at)$.

$\therefore$ (1) gives $a^2t^4 \sin^2 \alpha - 4a^2t^3 \sin \alpha \,.\, \cos \alpha + 4a^2t^2 \cos^2 \alpha - 2\left(x_1 - p \cos^2 \alpha\right)at^2$

$$- 2\left(y_1 - p \sin^2 \alpha\right).2at + \left(x_1^2 + y_1^2 - p^2\right) = 0$$

Since this equation is true for all t, all the coefficients of the polynomial should vanish.

$$\therefore \sin^2 \alpha = 0 \qquad \text{(i)}$$

$$\sin \alpha \cos \alpha = 0 \qquad \text{(ii)}$$

$$\cos^2 \alpha = \frac{x_1 - p \cos^2 \alpha}{2\alpha} \qquad \text{(iii)}$$

$$y_1 = p \sin^2 \alpha \qquad \text{(iv)}$$

$$x_1^2 + y_1^2 = p \qquad \text{(v)}$$

From (i) $\quad \sin \alpha = 0$ and hence $\cos \alpha = \pm 1$.

From (iv) $\quad y_1 = 0$.

From (v) $\quad x_1^2 = p^2$ and hence $x_1 = \pm p$.

From (iii) $\quad x_1 - p = 2a \ (a > 0)$.

$$\therefore x_1 = -p$$

Then $x_1 - p = 2a$ gives $x_1 = a$ and $p = -a$.

Hence $(a, 0)$ is the only focus. The equation to the directrix is

$$x \cos \alpha + y \sin \alpha = p$$

$$\text{i.e., } \pm x + a = 0.$$

The focus $(a, 0)$ should not be on the directrix.

$\therefore$ The equation of the directrix is $x + a = 0$.

This is the only directrix.

Latus rectum: The length of the chord thorugh the focus parallel to the directrix is called the latus rectum of the parabola. Half of the latus rectum is called semi-latus rectum.

If $2l$ is the latus rectum, then the point (a, l) lies on the parabola. Hence $l^2 = 4a.a$ or $l = 2a$.

Theorem-2: The equation of the chord joining two distinct points $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ on the parabola $S \equiv y^2 - 4ax = 0$ is

$$S_1 + S_2 = S_{12}$$

proof : Since A and B are different points on the parabola, $y_1 \neq y_2$.

$$\therefore S_1 = yy_1 - 2a\left(x + x_1\right) = 0$$

$$S_2 = yy_2 - 2a\left(x + x_2\right) = 0$$

are intersecting lines. The equation of any line in the plane of the two intersecting lines $S_1 = 0$ and $S_2 = 0$ is

$$\lambda_1 S_1 + \lambda_2 S_2 = \lambda_3 \quad \left(\lambda_1, \lambda_2, \lambda_3 \text{ are constants}\right).$$

If this is the required chord it passes through $A\left(x_1, y_1\right)$.

$$\therefore \lambda_1 S_{11} + \lambda_2 S_{21} = \lambda_3$$

$$\lambda_2 S_{21} = \lambda_3 \quad \left(\because S_{11} = 0\right)$$

$$\therefore \lambda_2 : \lambda_3 = 1 : S_{21}$$

$$= 1 : S_{12}$$

Similarly, since this chrod passes through $B\left(x_2, y_2\right)$.

$$\lambda_1 : \lambda_3 = 1 : S_{12}$$

$$\therefore \lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : S_{12}.$$

$\therefore$ The equation of the chord AB is $S_1 + S_2 = S_{12}$.

$$\text{or, } \left[yy_1 - 2a\left(x + x_1\right)\right] + \left[yy_2 - 2a\left(x + x_2\right)\right] = y_1y_2 - 2a\left(x_1 + x_2\right)$$

Theorem-3: The equation of the tangent at $\left(x_1, y_1\right)$ on the parabola $S \equiv y^2 - 4ax = 0$ is $S_1 = 0$.

Proof : The equation of the chrod joining $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ is $S_1 + S_2 = S_{12}$. As the point P approaches A along the curve, $\overleftrightarrow{AB}$ becomes the tangent at A. Therefore, the equation of the tangent at A is obtained by taking the limit

$$\underset{\left(x_2, y_2\right) \to \left(x_1, y_1\right)}{\text{Lt}} \left\{S_1 + S_2 = S_{12}\right\}$$

$\therefore$ The required equation is $S_1 + S_1 = S_{11} = 0$. $\left(\because \left(x_1, y_1\right) \text{ lies on the parabola}\right)$.

$\therefore S_1 \equiv yy_1 - 2a\left(x + x_1\right) = 0$ is the equation of the tangent at $\left(x_1, y_1\right)$

Note : (i) The equation of the tangent at $\left(x_1, y_1\right)$ can also written as

$$\left(y - y_1\right) y_1 = 2a\left(x - x_1\right).$$

(ii) Since $a \neq 0$, the tangent at $\left(x_1, y_1\right)$, namely $yy_1 - 2a\left(x + x_1\right) = 0$ is not parallel to x - axis. If θ is the angle the tangent makes with the y - axis, then

$$\tan\theta = \frac{y_1}{2a}.$$

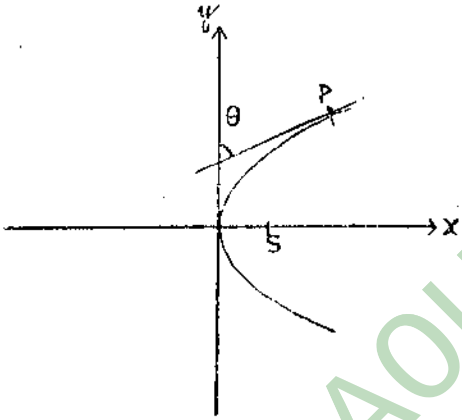


Fig. 3

Corresponding to $\left(x_1, y_1\right)$ on the curve, it is possible to choose a real number t such that $t = \tan\theta$. It is this parameter 't' that appeared in the parametric equations to the parabola. The equation to the tangent at $(at^2, 2at)$ is got by substituting $x_1 = at^2$, $y_1 = 2at$ in the equation of the tangent at $\left(x_1, y_1\right)$. This gives

$$y \cdot 2at - 2a\left(x + at^2\right) = 0$$

$$\text{or } yt - x - at^2 = 0$$

(ii) The normal $\left(x_1, y_1\right)$ to the parabola $y^2 - 4ax = 0$ is perpendicular to the tangent at $\left(x_1, y_1\right)$. It's slope is defined at every point is equal to $\frac{-y_1}{2a}$. The equation of the normal at $\left(x_1, y_1\right)$ is

$$2a\left(y - y_1\right) + y_1\left(x - x_1\right) = 0$$

At the point $(at^2, 2at)$, this simplifies to

$$y + tx = 2at + at^3.$$

If the normal at point 't' on the parabola passes through a fixed point $\left(x_1, y_1\right)$, then

$$y_1 + tx_1 = 2at + at^3$$

Since this is a cubic equation in t, it gives three values for t (one of them is real or all of them are real). In general, three normals to the parabola can be drawn from a point $\left(x_1, y_1\right)$.

(iv) The equation of the chord joining two distinct points 't_1' and 't_2' on the parabola is

$$S_1 + S_2 = S_{12}$$

i.e., $\left\{y\left(2at_1\right)-2a\left(x+at_1^2\right)\right\}+\left\{y\left(2at_2\right)-2a\left(x+at_2^2\right)\right\}$

$$=\left(2at_2\right)\left(2at_1\right)-2a\left(at_2^2+at_1^2\right)$$

$$\therefore 2ay\left(t_1+t_2\right)-4ax-2a^2\left(t_1^2+t_2^2\right)=4a^2t_1t_2-2a^2\left(t_1^2+t_2^2\right)$$

$$\therefore y\left(t_1+t_2\right)-2x-2at_1t_2=0.$$

Theorem-4: If $\left(x_1, y_1\right)$ is the mid-point of a chord of a parabola $S \equiv y^2-4ax=0$, the equation of the chord is $S_1 = S_{11}$.

Proof : If the chord whose mid-point is $\left(x_1, y_1\right)$ meets the parabola in $A\left(x_2, y_2\right)$ and $B\left(x_3, y_3\right)$, then

$$x_2+x_3 = 2x_1,\ y_2+y_3=2y_1.$$

The equation to $\overleftrightarrow{AB}$ is $S_2+S_3 = S_{23}$.

i.e., $yy_2-2a\left(x+x_2\right)+yy_3-2a\left(x+x_3\right) = y_2y_3-2a\left(x_2+x_3\right)$

$$\therefore y\left(y_2+y_3\right)-2a\left(2x+x_2+x_3\right) = 2k \qquad \text{... (constant)}$$

$$\therefore y\left(2y_1\right)-2a\left(2x+2x_1\right) = 2k.$$

$$\therefore yy_1-2a\left(x+x_1\right) = k.$$

Since this chord passes through $\left(x_1, y_1\right)$,

$$y_1^2-2a\left(x_1+x_1\right) = k$$

$$\text{or } y_1^2-4ax_1 = k.$$

The value of k is determined and the equation of the chord is

$$\therefore yy_1-2a\left(x+x_1\right) = y_1^2-4ax_1$$

$$\text{or } S_1 = S_{11}.$$

Theorem-5: The locus of the mid-points of all parallel chords of a parabola is a straight line.

Proof: Suppose the chords are parallel to the line $lx+my=0$, $l \neq 0$ through the origin, since any line parallel to $y=0$ (x-axis) cuts the parabola in only one point.

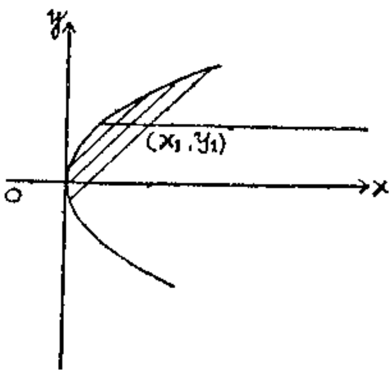


Fig. 4

The equation of the chord having (x_1, y_1) as the mid-point is

$$S_1 = S_{11}.$$

$$\text{or, } yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1.$$

$$\text{Since this is parallel to } lx + my = 0,$$

$$ly_1 + 2am = 0.$$

$\therefore$ The locus of the mid-points of chords is $ly + 2am = 0$, which is a straight line parallel to the x-axis.

Note : Suppose $lx + my = p$ is the tangent at a point (x_1, y_1).

Then comparing with $yy_1 - 2a(x + x_1) = 0$,

$$\frac{y_1}{m} = -\frac{2a}{l} = \frac{2ax_1}{p}$$

$$\therefore ly_1 + 2am = 0.$$

$\therefore (x_1, y_1)$ lies on the line $ly + 2am = 0$.

Therefore, the tangent to the parabola parallel to $lx + my = 0$ touches the parabola at a point on the line $ly + 2am = 0$.

Diameter: The lines on which the mid-points of a system of parallel chords of a parabola lie is called a diameter of the parabola.

Principal axis : If a diameter bisects every chord perpendicular to it, it is called a principal axis or a symmetrical axis of the parabola.

Theorem-6: The parabola has only one principal axis.

Proof: The equation to the diameter which cuts chords parallel to $lx + my = 0$ is $ly + 2am = 0$.

$ly + 2am = 0$ is a principal axis of the parabola

$$\Leftrightarrow ly + 2am = 0, lx + my = 0 \text{ cut orthogonally}$$

$$\Leftrightarrow l \cdot m = 0$$

$$\Leftrightarrow m = 0 \qquad (\because l \neq 0)$$

$\therefore ly + 2am = ly = 0$ or $y = 0$ is the only principal axis of the parabola.

Vertex : The point of intersection of the principal axis of the parabola with the parabola is called the vertex of the parabola.

The parabola $y^2 = 4ax$ has only one vertex, namely the origin.

Chord of contact : If the tangents from two points A and B on the parabola cut in a point P, then the chord AB is called the chord of contact with respect to the point P.

Theorem-7: The equation of the chord of contact of the parabola $S \equiv y^2 - 4ax = 0$, w.r.t. a point $P(x_1, y_1)$ is $S_1 = 0$.

Proof: Suppose the tangents at $A(x_2, y_2)$ and $B(x_3, y_3)$ meet at $P(x_1, y_1)$. Then P is an exterior point w.r.t. the parabola. The equation of the tangents at A and B are

$$yy_2 - 2a(x + x_2) = 0$$

$$yy_3 - 2a(x + x_3) = 0$$

Since these lines pass through $P(x_1, y_1)$,

$$y_1y_2 - 2a\left(x_1 + x_2\right) = 0$$
$$y_1y_3 - 2a\left(x_1 + x_3\right) = 0.$$

Hence $\left(x_2, y_2\right)$ and $\left(x_3, y_3\right)$ satisfy the equation

$$S_1 = yy_1 - 2a\left(x + x_1\right) = 0.$$

This is the required equation of the chord of contact w.r.t. $P\left(x_1, y_1\right)$.

Theorem-8: If the chord of contact $S \equiv y^2 - 4ax = 0$ w.r.t. the point Q pass through a fixed point $P\left(x_1, y_1\right)$, then the point Q lies on the line $S_1 = 0$.

Proof : The equation of the chord of contact w.r.t. $Q\ \left(x_2, y_2\right)$ is

$$S_2 \equiv yy_2 - 2a\left(x + x_2\right) = 0 \qquad \text{... (1)}$$

Since this passes through $\left(x_1, y_1\right)$,

$$S_{21} = S_{12} = y_1y_1 - 2a\left(x_1 + x_2\right) = 0 \qquad \text{... (2)}$$

Hence $Q\left(x_2, y_2\right)$ lies on the line

$$S_1 \equiv yy_1 - 2a\left(x + x_1\right) = 0 \qquad \text{... (3)}$$

17.5 Pole and Polar

If the chord of contact of tangents w.r.t. a point Q passes through a fixed point P, then the locus of Q is a straight line l called the polar line of P w.r.t. the parabola. The point P is called the pole of line l w.r.t. the parabola.

If the given point $P\left(x_1, y_1\right)$ is an interior point w.r.t. the parabola, then the locus of Q is $S_1 = 0$. (Theorem. 8). If the point P is on the parabola, the locus of Q is the tangent at P, namely $S_1 = 0$. If P is an exterior point w.r.t. the parabola, the locus of Q is that part of the line which lies in the exterior of the parabola.

If the polar line of $P\left(x_1, y_1\right)$ passes through $Q\left(x_2, y_2\right)$, then equation (2) of theorem 8, means that the polar of $Q\left(x_2, y_2\right)$ passes through $P\left(x_1, y_1\right)$.

Conjugate points : If two points P and Q are such that the polar of either point w.r.t. the parabola passes through the other, they are called *Conjugate points* w.r.t. the parabola.

Theorem-9: If the pole of a given line l is on a line m, then the pole of m is on l (w.r.t. the parabola).

Proof: Let P and Q be the poles of line l and m respectively, with respect to the parabola. Using 17.5, if the line m passes through P, then the line l passes through Q.

Conjugate lines : If two lines l and m are such that the pole of either is on the other w.r.t. the parabola, they are called *conjugate lines* w.r.t. the parabola.

Theorem. 10: The equation of the pair of tangents from an exterior point $P\left(x_1, y_1\right)$ is $S_1{}^2 - SS_{11} = 0$.

Proof: Take any point $Q\left(x_2, y_2\right)$ different from P, on the pair of tangents. Then the coordinates of a point R that divides PQ in the ratio $1 : \lambda\ (\lambda \neq -1)$ are

$$\left(\frac{\lambda x_1 + x_2}{\lambda + 1}, \frac{\lambda y_1 + y_2}{\lambda + 1}\right)$$

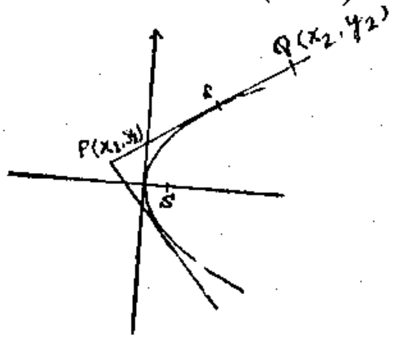


Fig. 5

R is on the curve $\Leftrightarrow \left(\dfrac{\lambda y_1 + y_2}{\lambda + 1}\right)^2 - 4a\left(\dfrac{\lambda x_1 + x_2}{\lambda + 1}\right) = 0$

$\Leftrightarrow \lambda^2 S_{11} + 2\lambda S_{12} + S_{22} = 0$ ... (1)

$\overleftrightarrow{PQ}$ is a tangent $\Leftrightarrow \overleftrightarrow{PQ}$ cuts the curve in only one point

$\Leftrightarrow$ Discriminant of (1) is zero

$\Leftrightarrow (S_1)_2{}^2 - S_{11}S_{22} = 0$

$\therefore$ The locus of $Q(x_2, y_2)$ is

$$S_1{}^2 - S_{11}S = 0.$$

Since $P(x_1, y_1)$ also satisfies this equation, the equation of the pair of tangents is

$$S_1{}^2 - SS_{11} = 0$$

or, $\left\{yy_1 - 2a(x + x_1)\right\}^2 - \left\{(y^2 - 4ax)(y_1{}^2 - 4ax_1)\right\} = 0.$

Examples

1. Find the equatuion of the parabola whose focus is the point (3, 4) and whose directrix is the straight line $2x - 3y + 5 = 0$.

Sol: Let $P(x, y)$ be a point on the parabola. If S is the focus and PM is drawn perpendicular to the directrix, then $SP = e.\ PM$.

Since $e = 1$, $SP = PM$ or $SP^2 = PM^2$

$$\text{i.e., } (x-3)^2 + (y-4)^2 = \left(\frac{2x - 3y + 5}{\sqrt{2^2 + 3^2}}\right)^2$$

i.e., $9x^2 + 12xy + 4y^2 - 98x - 7y + 300 = 0.$

2. Find the focus, vertex and directrix of the parabola

$$y^2 - 2x - 6y + 5 = 0$$

Sol: The equation can be written as

$$y^2 - 6y = 2x - 5$$

$$\text{or } (y-3)^2 = 2(x+2)$$

Making the transformations of coordinates,$X = x + 2$, $Y = y - 3$ (Shifting the orign to (–2, 3) and translating the axes), the equation of the parabola is $Y^2 = 2X$. The vertex is (–2, 3). Comparing the equation with $Y^2 = 4aX$, it follows that $a = \frac{1}{2}$. The length of the latus rectum is $4a = 2$. The focus in the new coordinates system is $(a, 0) = \left(\frac{1}{2}, 0\right)$. In the original system, the focus is

$$\left(\frac{1}{2} - 2, 0 + 3\right), \text{ i.e., } \left(-\frac{3}{2}, 3\right).$$

The directrix is $X + a = 0$

i.e., $x + 2 + a = 0$

i.e., $x + \frac{5}{2} = 0.$

3. If $y = mx + c$ is a tangent to the parabola, prove that $c = \frac{a}{m}$.

Sol: Suppose line $y = mx + c$ cuts the parabola $y^2 = 4ax$ in a point 't'. Then the equation of the tangent at 't' is

$$yt - x - at^2 = 0.$$

Comparing the two equations representing the same tangent,

$$\frac{1}{t} = \frac{-m}{-1} = \frac{c}{at^2}$$

$$\therefore c = at = \frac{a}{m}.$$

Therefore, the line whose equation is $y = mx + \frac{a}{m}$, touches the parabola $y^2 = 4ax$.

4. If $y = mx + c$ is a normal to the parabola, prove that $c + 2am + am^3 = 0$.

Sol : If the line $\quad y = mx + c \qquad \ldots (1)$

is normal to $y^2 = 4ax$ at a point 't', then the normal at 't' has the equation

$$y + tx = 2at + at^3 \qquad \ldots (2)$$

Comparing (1) and (2),

$$\frac{1}{1} = \frac{t}{-m} = \frac{2at + at^3}{c}$$

$$\therefore t = -m \text{ and hence}$$

$$c + 2am + am^3 = 0$$

5. Find the point in which the tangents at the points 't_1' and 't_2' intersect. Find also the point in which normals at 't_1' and 't_2' intersect.

Sol : The equation of the tangents at 't_1' and 't_2' are

$$yt_1 - x - at_1^2 = 0$$

$$yt_2 - x - at_2^2 = 0$$

Solving for x and y, the point of intersection of the two tangents is $\left(at_1t_2,\ a\left(t_1 + t_2\right)\right)$.

The equations of the normals at 't_1' and 't_2' are

$$y + t_1x = 2at_1 + at_1^3$$

$$y + t_2x = 2at_2 + at_2^3.$$

The point of intersection of these two normals is

$$\left[2a + a\left(t_1^2 + t_1t_2 + t_2^2\right),\ -at_1t_2\left(t_1 + t_2\right)\right]$$

6. Find the pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$.

Sol. : If $\left(x_1, y_1\right)$ is the pole, then the equation of the polar line is

$$yy_1 - 2a\left(x + x_1\right) = 0 \qquad \ldots (1)$$

This line (1) is same as the given line

$$lx + my + n = 0. \qquad \ldots (2)$$

From (1) and (2),

$$\frac{y_1}{m} = \frac{-2a}{l} = \frac{-2ax_1}{n}$$

$$\therefore \left(x_1, y_1\right) = \left(\frac{n}{l}, \frac{-2am}{l}\right), \text{ provided } l \neq 0.$$

The polar of $\left(x_1, y_1\right)$, namely $yy_1 - 2a\left(x + x_1\right) = 0$ will not be parallel to x–axis. Hence $\left(\frac{n}{l}, \frac{-2am}{l}\right)$, is the pole of $lx + my + n = 0$ only when $l \neq 0$.

If $l = 0$, the line $my + n = 0$ is parallel to x – axis and has no pole.

7. Prove that the locus of the point of intersection of perpendicular tangents to the parabola is the directix of the parabola.

Sol. : The equation of the tangents at 't_1' and 't_2' are

$$yt_1 - x - at_1^2 = 0$$

$$yt_2 - x - at_2^2 = 0$$

If these lines are perpendicular,

$$t_1 t_2 + 1 = 0$$

$$t_1 t_2 = -1.$$

These two lines intersect in $\left(at_1t_2, a\left(t_1 + t_2\right)\right)$.

$\therefore$ The locus of this point is given by

$$\left.\begin{aligned} x &= at_1t_2 \\ y &= a\left(t_1 + t_2\right) \\ t_1t_2 &= -1 \end{aligned}\right\}$$

i.e., the directix $x = -a$.

8. Prove that the locus of poles of chords of a parabola which subtend a right angle at the vertex is a straight line.

Sol. : The polar of $\left(x_1, y_1\right)$ w.r.t. $y^2 = 4ax$ is

$$yy_1 - 2a\left(x + x_1\right) = 0.$$

Let this cut the parabola in A and B. Then, the equation of the pair of lines OA and OB is (O is the vertex)

$$y^2 - 4ax\left(\frac{yy_1 - 2ax}{2x_1a}\right) = 0.$$

or, $\quad 4ax^2 - 2y_1 . xy + x_1 . y^2 = 0.$

Since these lines (OA and OB) are perpendicular, $4a + x_1 = 0$.

$\therefore$ The locus of $P\left(x_1, y_1\right)$ is the straight line $x + 4a = 0$.

9. If the normal at t_1 ($\neq 0$) on the parabola cuts the curve again in t_2, then show that $t_2 = -t_1 - \frac{2}{t_1}$.

Sol. : The normal at t_1 is $y + t_1x = 2at_1 + at_1^3$.

If this passes through 't_2', then

$$2at_2 + t_1 at_2^2 = 2at_1 + at_1^3.$$

$$\therefore \left(t_1 - t_2\right)\left(2 + t_1^2 + t_1t_2\right) = 0$$

$$\therefore t_1 + t_2 = -\frac{2}{t_1} \qquad \left(\because t_1 \neq t_2\right)$$

10. Find the direction of the parabola $y^2 = \frac{7x}{2}$ which bisects chords parallel to $3x - 2y = 0$.

Sol. : The equation of the chord whose mid-point is (x_1, y_1) is

$$yy_1 - \frac{7}{4}(x + x_1) = y_1^2 - \frac{7x_1}{2}.$$

or $$7x - 4y_1 y = 7x_1 - 4y_1^2.$$

If this is parallel to $3x - 4y = 0$, then

$$\frac{7}{3} = \frac{-4y_1}{-4}$$

i. e., $$3y - 7 = 0.$$

This is the required diameter.

11. Prove that the locus of the mid-points of chords of a parabola passing through the vertex is another parabola.

Sol. : Take any point P on the parabola $y^2 = 4ax$. If P is $(at^2, 2at)$, then the mid-point of OP is $Q\left(\frac{1}{2}at^2, at\right)$.

$\therefore$ The locus of Q is $x = \frac{at^2}{2}$, $y = at$

or $$y^2 = 2ax.$$

This is a parabola with (0, 0) as vertex and $y = 0$ as the axis. The latus rectum of this parabola is half the latus rectum of the given parabola.

12. Find the locus of poles w.r.t. the parabola $y^2 = 4bx$, of all tangents to the parabola $y^2 = 4ax$.

Sol. : Let $P(x_1, y_1)$ be the pole of a tangent to the parabola $y^2 = 4bx$. The polar of $P(x_1, y_1)$ $y^2 = 4bx$ is

$$yy_1 = 2b(x + x_1)$$

i.e., $$y = \frac{2b}{y_1}x + \frac{2bx_1}{y_1}$$

This is a tangent to $y^2 = 4ax$ if

$$\frac{2bx_1}{y_1} = \frac{a}{(2b/y_1)} \quad \text{(Example - 3).}$$

i.e., $$ay_1^2 = 4b^2 x_1$$

$\therefore$ The locus of (x_1, y_1) is $ay^2 = 4b^2 x$.

13. Prove that the tangents at the ends of a focal chord (chord which passes through the focus) of the parbola intersect at right angles on the directix.

Sol. : Let $P\,(at_1^2 ; 2at_1)$ and $Q\,(at_2^2 ; 2at_2)$ be the extremities of a focal chord. Evidently t_1, $t_2 \neq 0$. The equation of PQ is

$$y(t_1 + t_2) = 2x + 2a\,t_1 t_2.$$

Since it passes through the focus $(a, 0)$, $2a + 2a\,t_1 t_2 = 0$, i.e., $t_1 t_2 = -1$. The slopes of the tangent at 't_1' and 't_2' are respectively $\frac{1}{t_1}$ and $\frac{1}{t_2}$. Since $\left(\frac{1}{t_1}\right) \cdot \left(\frac{1}{t_2}\right) = \frac{1}{t_1 t_2} = \frac{1}{-1} = -1$, the two tangents drawn at the ends of a focal chord are perpendicular.

The coordinates of the point of intersection of the tangents at t_1 and t_2 is $\left[at_1 t_2, a\left(t_1 + t_2\right)\right]$.

Since $x = at_1 t_2 = -a$, the point of intersection lies on the directrix of the parabola.

14. The orthocentre of the triangle formed by three tangents to a parabola is on the directrix.

Sol. : Let the triangle formed by three tangents at t_1, t_2 and t_3 on the parabola be DEF. Then D is the point.

$$\left[at_1 t_2, a\left(t_1 + t_2\right)\right].$$

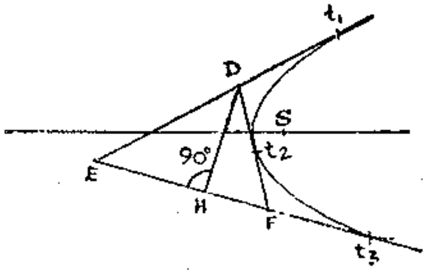


Fig. 6

The equation of the line EF is

$$yt_3 = x + at_3^2.$$

Slope of the altitude DH drawn from D perpendicular to EF is $-t_3$. The equation of the line DH is

$$y - a\left(t_1 + t_2\right) = -t_3\left(x - at_1 t_2\right).$$

$$\text{i.e., } y + t_3 x = a\left(t_1 + t_2\right) + a\,t_1 t_2 t_3. \quad \dots (1)$$

Similarly, the equation of the altitude through E is

$$y + t_2 x = a\left(t_1 + t_3\right) + a\,t_1 t_2 t_3. \quad \dots (2)$$

The orthocentre is the intersection of the altitudes of the triangle. Solving (1) and (2), the point of intersection is

$$x = -a, \; y = a\left(t_1 + t_2 + t_3\right) + at_1 t_2 t_3.$$

These are the coordinates of the orthocentre of $\Delta\, DEF$.

15. If normals at two points of a parabola intersect on the curve, show that the product of the ordinates at the point is $8a^2$.

Sol. : Let the normals at $t_1 \neq 0$ and $t_2 \neq 0$ intersect at the point t_3 on the curve. Then by example 9,

$$t_3 = -t_1 - \frac{2}{t_1}$$

$$t_3 = -t_2 - \frac{2}{t_2}$$

$$\therefore \; t_1 + \frac{2}{t_1} = t_2 + \frac{2}{t_2}$$

i.e., $t_1 - t_2 = 2\left(\frac{1}{t_2} - \frac{1}{t_1}\right) = \frac{2(t_1 - t_2)}{t_1 t_2}$.

Since $t_1 \neq t_2$, it follows that $t_1 t_2 = 2$.

$\therefore$ The product of the ordinates of the points 't_1' and 't_2'

$$= 2at_1 . 2at_2 = 4a^2 t_1 t_2 = 8a^2.$$

16. Show that the locus of the poles of normal chords of the parabola $y^2 = 4ax$ is $(x + 2a) y^2 + 4a^3 = 0$.

Sol. : Let PQ be a normal chord of the parabola, which is normal to the parabola at $(at^2, 2at)$. Let (x_1, y_1) be the pole. Then the equation of $\overleftrightarrow{PQ}$ is

$$y + xt = 2at + at^3. \qquad \text{... (1)}$$

Treated as the polar of (x_1, y_1) the equation of $\overleftrightarrow{PQ}$ is

$$yy_1 = 2a(x + x_1).$$

Comparing (1) and (2), $\frac{1}{y_1} = \frac{-t}{2a} = \frac{2at + at^3}{2ax_1}$

$\therefore \quad t = \frac{-2a}{y_1}$ and $\frac{2ax_1}{y_1} = 2at + at^3$

Eliminating t between these equations

$$\frac{2ax_1}{y_1} = 2a\left(\frac{-2a}{y_1}\right) + a\left(\frac{-2a}{y_1}\right)^3.$$

i.e., $x_1 y_1^2 + 2ay_1^2 + 4a^3 = 0$

$\therefore$ The locus of (x_1, y_1) is $y^2(x + 2a) + 4a^3 = 0$.

17.6 Summary

Locus of a point P which moves such that its distance from a fixed point bears a constant ratio with its distance from a fixed line is called a conic. The fixed point is called the focus, fixed line is called directrix and the constant ratio 'e' is called eccentricity. If $e < 1$, the conic is called an ellipse. If $e = 1$, the conic is called a parabola. If $e > 1$, the conic is called a hyperbola. The standard equation of a parabola is $y^2 = 4ax$ $(a > 0)$, and the parametric equations of a parabola are $x = at^2$, $y = 2at$. A parabola has one focus and one directrix. Eqatuion of the tangent at (x_1, y_1) of a parabola $y^2 = 4ax$ is $yb_1 = 2a(x + x_1)$.

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 . (S_1 = S_{11}).$$

The locus of the mid points of all parallel chords of a parabola is a straight line. The parabola has only one principal axis. Equation to the polar of a point $P(x_1, y_1)$ with respect to a parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1).$$

17.7 Sample Examination Questions

I. *Answer the following in detail.*

(i) a) Obtain the standard equation of a parabola symmetric about x-axis and state its properties.

b) Find the equation of the parabola whose focus and directrix are given by

(1) $(1, -1)$; $x + y + 7 = 0$

(2) $(3, -4)$; $x - y + 5 = 0$

(3) $(-1, -1)$; $2x - 3y + 6 = 0$

(ii) a) Prove that the equation of the tangent at (x_1, y_1) on the parabola $S \equiv y^2 - 4ax = 0$ is $S_1 = 0$. Also obtain the equation of the normal at (x_1, y_1)

b) Show that the line $y = x - 6$ is a normal to the parabola $y^2 = 8x$. Find the point at which this line is a normal.

II. *Briefly answer the following.*

i) For the parabolas $y^2 = 8x + 9$, $y^2 + 4x - 2y + 3 = 0$, $x^2 + 2y = 4x - 3$, obtain the vertex, focus and the directrix of the parabola.

ii) Show that the locus of the point of intersection of two normals to $y^2 = 4ax$ which are mutually perpendicular is $y^2 = a\ (x - 3a)$.

iii) If the line $lx + my + n = 0$ is normal to $y^2 = 4ax$ show that $al^2 + 2alm^2 = m^2n$.

iv) Show that the locus of the mid-points of chords of the parabola $y^2 = 6x$ passing through the point (9, 5) is

$$y^2 - 5y - 3x + 27 = 0.$$

v) Show that the locus of the intersection of tangents to $y^2 = 4ax$ which intercept a constant length d on the directrix is

$$(y^2 - 4ax)(x + a)^2 = d^2x^2.$$

vi) If three normals from a point to the parabola $y^2 = 4ax$ cut the axis in points whose distances from the vertex are in arithmetic progression, show that the point lies on the curve

$$27ay^2 = 2(x - 2a)^3$$

Answers

I. (ib) (1) $x^2 - 2xy + y^2 - 18x - 10y - 45 = 0$;

(2) $x^2 + 2xy + y^2 - 22x + 18y + 25 = 0$

(3) $9x^2 + 12xy + 4y^2 + 2x + 62y - 10 = 0$.

(iib) $(1, -3)$, $(1, -2)$, $y + 4 = 0$; $\left(-\frac{1}{2}, 1\right)$, $\left(\frac{-3}{2}, 1\right)$, $2x + 1 = 0$;

$\left(2, \frac{1}{2}\right)$, $(2, 0)$, $y = 1$.

UNIT–18 : ELLIPSE

18.0 Contents

18.1 Aims and Objectives

After going through this unit, you will be able to:

derive the standard equation of an ellipse and verify its properties.

18.2 Standard equation of the ellipse

Suppose the given conic has eccentricity $e < 1$. Let S be a focus and $\overleftrightarrow{ED}$ be a directrix corresponding to this focus, Draw SD perpendicular to the directrix. Choose $\overleftrightarrow{DS}$ as X axis and $\overleftrightarrow{DE}$ as the Y axis. Let $SD = c$. From any point $P\ (X, Y)$ on the ellipse, draw PM perpendicular to the directrix. Then

$$SP = e\,PM$$

$$SP^2 = e^2 . PM^2$$

$$(X - c)^2 + Y^2 = e^2 X^2$$

$$X^2 (1 - e)^2 - 2cX + Y^2 + c^2 = 0.$$

$$\left(X - \frac{c}{1 - e^2}\right)^2 + \frac{Y^2}{1 - e^2} = \frac{c^2}{\left(1 - e^2\right)^2} - \frac{c^2}{\left(1 - e^2\right)}$$

$$= \frac{c^2 e^2}{\left(1 - e^2\right)^2}$$

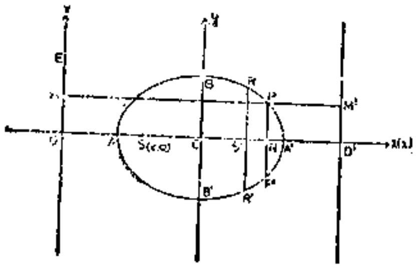


Fig. 1

Denote the point $\left(\frac{c}{1 - e^2}\ ,\ 0\right)$ by C. Shifting the origin to C with the axes remaining parallel, the equations of transformation of coordinates are

$$x = X - \frac{c}{1 - e^2}\ , y = Y.$$

The equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\left(1 - e^2\right)} = 1, \text{ where } a = \frac{ce}{1 - e^2}.$$

Let $b^2 = a^2\,(1 - e^2)$. Since $e < 1$, b is a real number. The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Properties : If L is the ellipse $\equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ then

(i) $(x, y) \in L \Rightarrow (x, -y), (-x, y), (-x, -y) \in L$. The ellipse is symmetric about the x and y axes. The point C bisects all chords of the ellipse passing through it. C is called the centre of the ellipse.

(ii) The ellipse L passes through $A(-a, 0), A'(a, 0), B(0, b), B'(0, -b)$.

(iii) The line AA' is called the major axis and BB' is called the minor axis.

(iv) If $x < -a$ or $x > a$, (i.e., $|x| > a$), there is no y satisfying the ellipse. The ellipse L therefore lies between the lines $x = \pm a$. Similarly, it follows that the curve L lies between $y = \pm b$. Thus, the ellipse is a closed curve described within the rectangle $x = \pm a, y = \pm b$.

(v) The coordinates of the focus S are $(-ae, 0)$ and directrix

$$\overleftrightarrow{DE} \text{ is } x + \frac{a}{e} = 0.$$

This follows if we note that

$$SA = e.AD$$
$$SA' = e.A'D$$
$$\therefore SA + SA' = e\,(AD + A'D) = e.DD'$$
$$AA' = 2e\,.\,CD$$
$$2 \cdot CA = 2e.CD$$
$$\therefore CA = e\,.\,CD$$
$$\text{or } CD = \frac{a}{e}$$

Also $SA' - SA = e\,(A'D - AD)$

$(SC + CA') - (CA - CS) = e.\,AA' = e\,.\,2\,.\,CA$

$$\therefore CS = e\,.\,CA = e.a$$

(vi) Since C is symmetric about the axes, there is another focus $(ae, 0)$ and correspondingly another directrix $x = \frac{a}{e}$, for the ellipse.

(vii) $SP + SP' = 2a$.

Since $SP = e.PM$ and $S'P = e.PM'$, it follows that

$$SP + S'P = e\,(PM + PM') = e.\,MM' = e\,.\,DD' = e\,.\,\frac{2a}{e} = 2a$$

18.3 Parametric equations of the ellipse

Let C be the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Draw the circles with centre C and radii b and a. The line through C making an angle θ with the x - axis cuts the circles of radius b in Q and cuts the circle of radius a in P. Draw the line parallel to the x-axis through Q and the line parallel to the y - axis through P meeting in a point M. Let P_1, Q_1 be the projections of P, Q on the x - axis. If the coordinates of M are (x, y), then

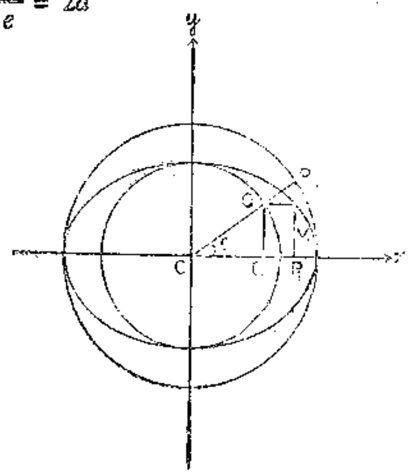


Fig. 2

$$x = CP_1 = CP \cos\theta = a \cos\theta$$

$$y = P_1M = Q_1Q = CQ \sin\theta = b \sin\theta$$

But the coordinates $(a \cos\theta, b \sin\theta)$ satisfy the equation of the ellipse. Hence the point M lies on the ellipse. The equations

$$x = a\cos\theta, \quad y = b\sin\theta$$

are called parametric equations of the ellipse. The coordinates of any point on the ellipse can be written in the form $(a\cos\theta, b\sin\theta)$. The coordinates $(a\cos\theta, b\sin\theta)$ are called parametric coordinates of a point on the ellips. The angle θ (parameter) is called the eccentric angle and varies from $\theta = 0$ to 2π. The point on the ellipse corresponding to the eccentric angle θ is also called the point 'θ' on the ellipse.

18.4 Classification of points in the plane of an ellipse

A point (x, y) in the xy plane is inside the elliptic region if and only if $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0$. Such a point is called an interior point w.r.t. the ellipse. The point (x, y) is outside the elliptic region if and only if $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 > 0$. Such a point is called an exterior point w.r.t. the ellipse. The point (x, y) is on the ellipse if and only if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

18.5 A geometric property of ellipse

If PN is the perpendicular to the x - axis from a point P on the ellipse, then

$$\frac{CN^2}{a^2} + \frac{PN^2}{b^2} = 1$$

$$\therefore \frac{PN^2}{b^2} = 1 - \frac{CN^2}{a^2} = \frac{a^2 - CN^2}{a^2} = \frac{CA^2 - CN^2}{a^2}$$

$$= \frac{(CA + CN)(CA - CN)}{a^2} = \frac{NA \,.\, NA'}{a^2}$$

$$\therefore \quad \frac{PN^2}{NA \,.\, NA'} = \frac{b^2}{a^2} = \frac{CB^2}{CA^2}.$$

Theorem 1 : The ellipse has only two foci and two directrices corresponding to these foci.

Proof : Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$ ... (1)

be the equation of the ellipse. Let (x_1, y_1) be a focus and $x\cos\alpha + y\sin\alpha = p$ be a directrix corresponding to this focus. Then the equation of the ellipse is

$$\left(1 - e^2\cos^2\alpha\right)x^2 - 2e^2xy\sin\alpha\cos\alpha + \left(1 - e^2\sin^2\alpha\right)y^2$$

$$-2\left(x_1 - pe^2\cos\alpha\right)x - 2\left(y_1 - pe^2\sin\alpha\right)y + \left(x_1^2 + y_1^2 - e^2p^2\right) = 0 \quad \text{... (2)}$$

Comparing the two equations (1) and (2) representing the ellipse,

$$\sin\alpha\cos\alpha = 0 \quad \text{... (3)}$$

$$x_1 - pe^2\cos\alpha = 0 \quad \text{... (4)}$$

$$y_1 - pe^2\sin\alpha = 0 \quad \text{... (5)}$$

The equation (2) then becomes

$$x^2\left(1 - e^2\cos^2\alpha\right) + y^2\left(1 - e^2\sin^2\alpha\right) + \left(x_1^2 + y_1^2 - p^2e^2\right) = 0 \quad \text{... (6)}$$

The ellipse passes through $(a, 0)$ and $(0, b)$.

$$\therefore\ a^2\left(1-e^2\cos^2\alpha\right)+\left(x_1^{\,2}+y_1^{\,2}-p^2e^2\right)=0 \qquad \text{... (7)}$$

$$\text{and } b^2\left(1-e^2\sin^2\alpha\right)+\left(x_1^{\,2}+y_1^{\,2}-p^2e^2\right)=0 \qquad \text{... (8)}$$

$$\therefore\ a^2\left(1-e^2\cos^2\alpha\right)=b^2\left(1-e^2\sin^2\alpha\right) \qquad \text{... (9)}$$

Now, from (3), $\sin\alpha\cos\alpha=0 \Rightarrow \cos\alpha=0$ or $\sin\alpha=0$.

If $\cos\alpha=0$; then (9) gives $a^2=b^2\left(1-e^2\right) \Rightarrow a<b$

This contradicts the hypothesis and therefore $\cos\alpha\neq 0$.

$\therefore \sin\alpha=0$, and $\cos\alpha=\pm 1$.

$\therefore x_1=\pm ep^2, y_1=0$ (from (4) and (5))

Substituting these values in (7),

$$a^2\left(1-e^2\right)+p^2e^4-p^2e^2=\left(1-e^2\right)\left(a^2-p^2e^2\right)=0$$

$$\therefore a=\pm pe\left(\text{or } p=\pm\frac{a}{e}\right).$$

$$\therefore x_1=\pm ae,\ y_1=0.$$

The ellipse has only two foci $(\pm ae, 0)$ and only two directrices having equations $\pm x-p=0$, i.e., $x\mp\frac{a}{e}=0$.

Latus rectum : The chord through a focus parallel to the directrix is called the latus rectum of the ellipse.

If $2l$ is the length of the latus rectum, then the point (ae, l) is on the ellipse.

$$\therefore l^2=b^2\left(1-e^2\right)=a^2\left(1-e^2\right)^2$$

$$\text{or } 2l=2a\left(1-e^2\right)=\frac{2b^2}{a}$$

Theorem. 2: If the equation $S_1=0$ and $S_2=0$ are a pair of intersecting lines, then the equations of the chord joining the points $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ on the ellipse $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $S_1+S_2=S_{12}$.

Proof: If $S_1=0$ and $S_2=0$ are two intersecting lines, then the equation of the chord joining $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ has the form

$$\lambda_1 S_1+\lambda_2 S_2=\lambda_3$$

Since this passes through $A\left(x_1, y_1\right)$.

$$\lambda_1\left(S_1\right)_1+\lambda_2\left(S_2\right)_1=\lambda_3$$

$$\text{or } \lambda_1 S_{11}+\lambda_2 S_{21}=\lambda_3$$

Since $\left(x_1, y_1\right)$ is on the lines $S_1=0$, it follows that $S_{11}=0$.

$\therefore\ \lambda_2 S_{21}=\lambda_3$ or $\lambda_2:\lambda_3=1:S_{21}=1:S_{12}$.

Since the chord passes through $B\left(x_2, y_2\right)$ also, in a similar way, it can be shown that

$$\lambda_1 : \lambda_3 = 1 : S_{12}$$

$$\therefore \lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : S_{12}$$

The equation of the chord joining A and B is therefore

$$S_1 + S_2 = S_{12}.$$

Theorem. 3: The equation of the tangent at $\left(x_1, y_1\right)$ on the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is $S_1 = 0$.

Proof : The equation of the chord joining $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ is

$$S_1 + S_2 = S_{12}.$$

As the point B approaches A along the curve, $\overleftrightarrow{AB}$ becomes the tangent at A. Therefore, the equation of the tangent at A is obtained by taking the limit

$$\underset{\left(x_2, y_2\right) \to \left(x_1, y_1\right)}{\text{Lt}} \left\{S_1 + S_2 = S_{12}\right\}$$

$$S_1 + S_1 = S_{11} = 0 \qquad \left(\because \left(x_1, y_1\right) \text{ is on the ellipse}\right)$$

$\therefore S_1 = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$ is the equation of the tangent at $\left(x_1, y_1\right)$.

Note : (i) The equation of the tangent at $\left(x_1, y_1\right)$ can also be written as

$$\frac{\left(x - x_1\right) x_1}{a^2} + \frac{\left(y - y_1\right) y_1}{b^2} = 0$$

(ii) The equation of the tangent at the point 'θ' is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1.$$

(iii) Since the normal at $\left(x_1, y_1\right)$ is the perpendicular to the tangent $\left(x_1, y_1\right)$, the equation of the normal at $\left(x_1, y_1\right)$ is

$$\frac{\left(x - x_1\right) y_1}{b^2} - \frac{\left(y - y_1\right) x_1}{a^2} = 0$$

(iv) the equation of the normal at 'θ' on the ellipse is

$$\frac{x \sin \theta}{b} - \frac{y \cos \theta}{a} = \left(\frac{a}{b} - \frac{b}{a}\right) \sin \theta \cos \theta.$$

(v) If the tangents at two different points A and B on the ellipse are parallel, then the chord AB passes through the centre.

If the tangents $A\left(x_1, y_1\right)$, $b\left(x_2, y_2\right)$, namely, $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ and $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$ are parallel then $x_2 = kx_1, y_2 = ky_1$.

Since $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$ are on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \text{ and } \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1.$$

$$\therefore \frac{k^2 x_1^2}{a^2} + \frac{k^2 y_1^2}{b^2} = 1 \text{ or } k^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = 1$$

$\therefore$ $k^2 = 1$ or $k = \pm 1$. If $k = 1$, then $x_2 = x_1$ and $y_2 = y_1$. This contradicts the hypothesis. Hence $k = -1$ and $x_2 = -x_1$, $y_2 = -y_1$. The mid point of chord $\overline{AB}$ is $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right) = (0, 0)$.

(vi) Four normals can be drawn from a given point to the ellipse.

If the normal at $(a \cos \theta, b \sin \theta)$ passes through a fixed point (x_1, y_1), then

$$\frac{x_1 \sin \theta}{b} - \frac{y_1 \cos \theta}{a} = \left(\frac{a}{b} - \frac{b}{a}\right) \sin \theta \cos \theta. \qquad \dots (1)$$

Sybstituting

$$\sin \theta = \frac{2 \tan (\theta/2)}{1 + \tan^2 (\theta/2)},$$

$$\cos \theta = \frac{1 - \tan^2 (\theta/2)}{1 + \tan^2 (\theta/2)}$$

and simplifying, equation (1) above reduces to

$$b y_1 \tan^4 (\theta/2) + 2\left(a x_1 + a^2 e^2\right) \tan^3 (\theta/2) + \left(a x_1 - a^2 e^2\right) \tan (\theta/2) - b y_1 = 0.$$

This is a fourth degree equation in tan θ/2 and gives four values for tan θ/2 and hence for θ/2. Thus, four normals can be drawn from a point (x_1, y_1) to the ellipse.

Theorem. 4 : The straight line $x \cos \alpha + y \sin \alpha = p$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if and only if $P = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$.

Proof : Any straight line in the xy plane has the form

$$x \cos \alpha + y \sin \alpha = p \ (p \geq 0) \qquad \dots (1)$$

If this is a tangent at (x_1, y_1) on the ellipse, then its equation is

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1 \qquad \dots (2)$$

Since (1) and (2) represent the same line,

$$\frac{\cos \alpha}{\left(x_1/a^2\right)} = \frac{\sin \alpha}{\left(y_1/b^2\right)} = p.$$

$$\therefore x_1 = \frac{a^2 \cos \alpha}{p}, \quad y_1 = \frac{b^2 \sin \alpha}{p}$$

Since the point (x_1, y_1) lies on the tangent (1),

$$\left(\frac{a^2 \cos \alpha}{p}\right) \cos \alpha + \left(\frac{b^2 \sin \alpha}{p}\right) \sin \alpha = p,$$

$$p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}. \ (\neq 0).$$

Since $p \geq 0$ the positive root is taken.

Conversely, if $p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$, then

$$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.$$

$\therefore$ The point $\left(x_1, y_1\right) = \left(\dfrac{a^2 \cos \alpha}{p}, \dfrac{b^2 \sin \alpha}{p}\right)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The equation of the tangent at $\left(x_1, y_1\right)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

i.e., $\dfrac{x}{a^2}\left(\dfrac{a^2 \cos \alpha}{p}\right) + \dfrac{y}{b^2}\left(\dfrac{b^2 \sin \alpha}{p}\right) = 1$

i.e., $x \cos \alpha + y \sin \alpha = p$

$\therefore$ The given line is a tanget at $\left(x_1, y_1\right)$ on the ellipse.

Theorem. 5 : (i) The feet of perpendiculars from the foci on the tangents to the ellipse lie on a circle, (ii) the product of the distances of the foci from a tangent is a constant and (iii) the points of intersection of pairs of perpendicular tangents to the ellipse lie on a circle.

Proof: (i) Let the equation of a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be

$$x \cos \alpha + y \sin \alpha = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} \qquad \ldots (1)$$

The equations of the perpendiculars from the foci $(\pm ae, 0)$ on the tangent are

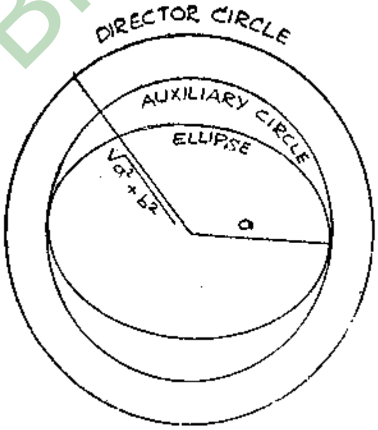


Fig. 3

$$x \sin \alpha - y \cos \alpha = \pm ae \sin \alpha \qquad \ldots (2)$$

$\therefore$ From (1) and (2),

$$(x \cos \alpha + y \sin \alpha)^2 + (x \sin \alpha - y \cos \alpha)^2$$

$$= a^2 \cos^2 \alpha + b^2 \sin^2 \alpha + a^2 e^2 \sin^2 \alpha$$

$$= a^2 \cos^2 \alpha + b^2 \sin^2 \alpha + \left(a^2 - b^2\right) \sin^2 \alpha$$

$$= a^2$$

$\therefore$ The feet of perpendicular from the foci on the tangent to the ellipse lie on the circle $x^2 + y^2 = a^2$.

(ii) If h, h_1 are the perpendicular distances from $(ae, 0)$ and $(-ae, 0)$ on a tangent, then

$$h = \left| ae\cos\alpha - \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \right|$$

$$h_1 = \left| -ae\cos\alpha - \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \right|$$

$$hh_1 = \left| a^2\cos^2\alpha + b^2\sin^2\alpha - a^2e^2\cos^2\alpha \right|$$

$$= a^2\left(1 - e^2\right)\cos^2\alpha + b^2\sin^2\alpha$$

$$= b^2\left(\cos^2\alpha + \sin^2\alpha\right) = b^2 \text{ (a constant)}$$

(iii) Two mutually perpendicular tangents to the ellipse can be taken as

$$x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \text{ and}$$

$$x\cos(\alpha + \pi/2) + y\sin(\alpha + \pi/2) = \sqrt{a^2\cos^2(\alpha + \pi/2) + b^2\sin^2(\alpha + \pi/2)}$$

$$\text{i.e., } -x\sin\alpha + y\cos\alpha = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$$

The point of intersection of these two tangents lies on

$$(x\cos\alpha + y\sin\alpha)^2 + (-x\sin\alpha + y\cos\alpha)^2 = a^2\cos^2\alpha + b^2\sin^2\alpha + a^2\sin^2\alpha + b^2\cos^2\alpha$$

$$\text{i.e., } x^2 + y^2 = a^2 + b^2$$

which represents a circle.

18.6 Auxiliary Circle and Director Circle

The locus of the feet of perpendiculars from the foci on the tangents to the ellipse is a circle defined as the *auxiliary circle* of the ellipse.

This circle has the major axis as the diameter. The circle on minor axis as diameter has the equation $x^2 + y^2 = b^2$. These two circles are usually referred to as the major auxiliary circle and the minor auxiliary circle.

The locus of the point of intersection of mutually perpendicular tangents to an ellipse is a circle defined as the *director circle* of the ellipse.

The director circle has radius $\sqrt{a^2 + b^2}$.

Theorem. 6: If $\left(x_1, y_1\right)$ be the mid point of a chord of an ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, the equation of the chord is $S_1 = S_{11}$

Proof: Let $\left(x_1, y_1\right)$ be the mid point of the chord joining

$A\left(x_2, y_2\right)$ and $B\left(x_3, y_3\right)$. Then $x_2 + x_3 = 2x_1, y_2 + y_3 = 2y_1$.

If $S_2 = 0$ and $S_3 = 0$ are parallel, then $(0, 0)$ is the mid point of $\overline{AB}$ is

$$S_2 + S_3 = S_{23}.$$

$$\text{i.e., } \frac{x\left(x_2 + x_3\right)}{a^2} + \frac{y\left(y_2 + y_3\right)}{b^2} - 2 = S_{23}.$$

$$\frac{xx_1}{b^2}+\frac{yy_1}{b^2}-1 = k \text{ (Constant).}$$

Since this passes through (x_1, y_1).

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 = k.$$

$$\therefore \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1 = k = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$\text{or } S_1 = S_{11}.$$

Theorem. 7: The locusof the mid points of parallel chords of an ellipse is a straight line.

Proof: Consider the system of chords of the ellipse parallel to $lx + my = 0$. The equation of the chord whose middle point is (x_1, y_1) is $S_1 = S_{11}$ or

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1.$$

If this is parallel to $lx + my = 0$, then $\frac{mx_1}{a^2}=\frac{ly_1}{b^2}$.

$\therefore$ The locus of (x_1, y_1) is the straight line $b^2mx - a^2ly = 0$.

Note : The tangents parallel to $lx + my = 0$ touch the ellipse at points which lie on the line $b^2mx - a^2ly = 0$ (see example 10).

Diameter : The locus of the mid points of a system of parallel chords of an ellipse is called a diameter of the ellipse.

Note: (1) Any diameter passes through the centre and hence all diameters of an ellipse bisect each other at the centre.

(ii) Any chord through the centre is a diameter.

The line $l'x + m'y = 0$ through the centre can be written as

$$b^2\left(\frac{l'}{b^2}\right)x - a^2\left(\frac{-m'}{a^2}\right)y = 0$$

This is the diameter which bisects chords parallel to

$$\left(\frac{-m'}{b^2}\right)x + \left(\frac{l'}{b^2}\right)y = 0.$$

Theorem. 8: If two diameters $\overline{AB}$ and $\overline{CD}$ of an ellipse are such that $\overline{AB}$ bisects chords parallel to $\overline{CD}$, then $\overline{CD}$ bisects chords parallel to $\overline{AB}$.

Proof: Let the equation of $\overline{CD}$ be $lx + my = 0$.

The mid points of chords parallel to the diameter $lx + my = 0$ lie on the diameter $\overline{AB}$ whose equation is, by Theorem 7,

$$b^2mx - a^2ly = 0$$

Now, the mid points of chords parallel to $b^2mx - a^2ly = 0$ lie on the diameter

$$b^2\left(-a^2l\right)x - a^2\left(b^2m\right)y = 0$$

$$\text{i.e., } lx + my = 0$$

$\therefore$ $\overline{CD}$ bisects chords parallel to $\overline{AB}$.

Conjugate diameters: If two diameters of an ellipse are such that one diameter bisects chords parallel to the other, they are called conjugate diameters w.r.t. the ellipse.

Principal axis: If a straight line bisects all chords perpendicular to it, it is called a principal axis of the ellipse.

A principal axis is also a diameter.

Theorem. 9: If the diamter $b^2mx - a^2ly = 0$ which bisects chords parallel to $lx + my = 0$ is perpendicular to these chords, then

$$\left(b^2m\right) l + \left(-a^2l\right) m = 0$$

$$\Leftrightarrow lm = 0 \quad (\because a \neq b)$$

$l = 0$ gives the y-axis and $m = 0$ gives x-axis.

Vertices: The points where the principal axes cut the ellipse are called vertices of the ellipse.

The vertices of the ellipse are $A'(a, 0)$, $A(-a, 0)$, $B(0, b)$ and $B'(0, -b)$.

Chord of contact : If the tangents at two points A and B on the ellipse meet at a point P, then $\overline{AB}$ is called the chord of contact w.r.t the point P.

Theorem. 10: The equation of the chord of contact of the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, w.r.t. a point $P\left(x_1, y_1\right)$ is $S_1 = 0$.

Proof: If the tangents at $A\left(x_2, y_2\right)$ and $B\left(x_3, y_3\right)$ meet at a point $P\left(x_1, y_1\right)$, then the equation of the tangents at A and B are $S_2 = 0$ and $S_3 = 0$, Since lies on each of these lines,

$$\left(S_2\right)_1 = \left(S_1\right)_2 = 0.$$

$$\left(S_3\right)_1 = \left(S_1\right)_3 = 0.$$

Hence the points $\left(x_2, y_2\right)$ $B\left(x_3, y_3\right)$ lie on the line $S_1 = 0$.

Theorem. 11: If the chord of contact of the ellipse w.r.t. a point Q passes through a given point $P\left(x_1, y_1\right)$, then Q lies on the line $S_1 = 0$.

Proof: The equation of the chord of contact w.r.t. $Q\left(x_2, y_2\right)$ is $S_2 = 0$.

If this passes through $\left(x_1, y_1\right)$ then $\left(S_2\right)_1 = \left(S_1\right)_2 = 0$. ... (1)

$\therefore$ The point $Q\left(x_2, y_2\right)$ lies on the line $S_1 = 0$.

18.7 Pole and Polar

If the chord of contact of tangents from a point Q passes through a fixed point P, then the locus of points Q is a straight line l called the *polar* of the point P w.r.t. the ellipse. The point P is called the *pole* of the line l w.r.t. the ellipse.

Note: (i) The equation of the polar of $P\left(x_1, y_1\right)$ is $S_1 = 0$. (Theorem. 11)

(ii) If the chord of contact w.r.t. Q passes through an interior point P, the locus of Q is $S_1 = 0$.

(iii) If the chord of contact w.r.t. Q passes through a point P on the ellipse, then the locus of Q is the tangent at P, namely, $S_1 = 0$.

(iv) If the chord of contact of tangents w.r.t Q passes through an exterior point P, then the locus of Q is that part of the line $S_1 = 0$ in the exterior.

(v) If the polar of P w.r.t the ellipse passes through Q, then the polar of Q passes through P. (follows from theorem 11, equation (1).).

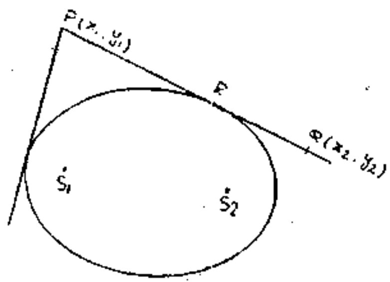
(vi) If two lines l and m are such that the pole of l w.r.t the ellipse lies on line m, then the pole of line m lies on line l.

If P, Q are poles of lines l, m w.r.t the ellipse, then by (v), line m passes through $P \Rightarrow$ line l passes through Q.

18.8 Conjugate points and lines

If two points P and Q are such that the polar of P w.r.t the ellipse passes through Q and the polar of Q passes through P, the points P and Q are called conjugate points w.r.t the ellipse.

If two lines are such that the pole of either lies on the other (w.r.t the ellipse), then the two lines are called conjugate lines w.r.t the ellipse.



(Fig. 4)

Theorem 12 : The equation of the pair of tangents drawn from an exterior point $P\left(x_1, y_1\right)$ of the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is $S_1^2 - SS_{11} = 0$.

Proof : Let $Q\left(x_2, y_2\right)$ be a point on one of the tangents drawn from $P\left(x_1, y_1\right)$. Then the coordinates of any point R which cuts $\overline{PQ}$ in the ratio $1 : \lambda$ are,

$$\left(\frac{\lambda x_1 + x_2}{\lambda + 1}, \frac{\lambda y_1 + y_2}{\lambda + 1}\right)$$

R lies on the ellipse $\Leftrightarrow$

$$\Leftrightarrow \frac{1}{a^2}\left(\frac{\lambda x_1 + x_2}{\lambda + 1}\right)^2 + \frac{1}{b^2}\left(\frac{\lambda y_1 + y_2}{\lambda + 1}\right)^2 = 1$$

$$\Leftrightarrow \lambda^2 S_{11} + 2\lambda S_{12} + S_{22} = 0 \qquad \text{— (1)}$$

$\overleftrightarrow{PQ}$ is a tangent to the ellipse $\Leftrightarrow$ (1) has equal values for λ

$\Leftrightarrow$ the discriminant is zero

$\Leftrightarrow S_{12}^2 - S_{11} S_{22} = 0.$

$\therefore$ The locus of $Q\left(x_2, y_2\right)$ is,

$$S_1^2 - SS_{11} = 0$$

Examples

(1) Find the equation of the ellipse having (1, 2) as focus, $\frac{2}{3}$ as eccentricity and the line $2x - 3y + 6 = 0$ as the directrix.

Sol. : The distance of a point $P(x, y)$ from (1, 2) is $\sqrt{(x-1)^2 + (y-2)^2}$.

The distance of $P(x, y)$ from the line $2x - 3y + 6 = 0$ is $\left|\frac{2x - 3y + 6}{\sqrt{2^2 + 3^2}}\right|$.

Since P is on the ellipse,

$$\sqrt{(x-1)^2 + (y-2)^2} = \frac{2}{3} \cdot \left|\frac{2x - 3y + 6}{\sqrt{13}}\right|$$

Squaring and simplifying, the equation of the ellipse is obtained as

$101x^2 + 48xy + 81y^2 - 330x - 324y + 441 = 0.$

(2) Find the foci, the eccentricity and the latus rectum of the ellipse

$$3x^2 + 4y^2 + 6x - 8y = 5.$$

Sol. : The equation can be written as

$$3(x^2 + 2x) + 4(y^2 - 2y) = 5$$

$$\text{i.e., } 3(x+1)^2 + 4(y-1)^2 = 12.$$

Using the transformations $X = x + 1, Y = y - 1$, the equation reduces to

$$\frac{X^2}{4} + \frac{Y^2}{3} = 1.$$

Comparing with the standard equation of the ellipse,

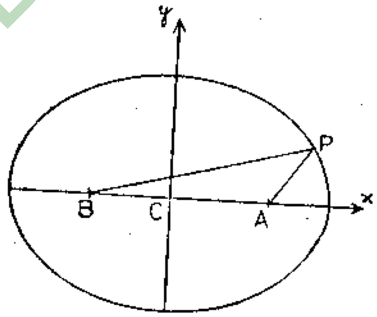
$$a^2 = 4, \ b^2 = 3.$$

$$b^2 = a^2 (1 - e^2) \Rightarrow 3 = 4(1 - e^2) \Rightarrow e = \frac{1}{2}$$

The length of the latus rectum is $\frac{2b^2}{a} = 3.$

The foci are $(ae, 0)$ and $(-ae, 0)$, i.e., $(1, 0)$ and $(-1, 0)$. In the original coordinate system, the foci are $(0, 1)$ and $(-2, 1)$ the directrices are $X = \pm\frac{a}{e}$, i.e., $X = +4$ and $X = -4$. In the original coordinate system, the directrices are $x = 3$, and $x = -5$.

(3) If the distance between two fixed points A and B is $2c$, prove that the locus of a point P given by $\{P \mid PA + PB = 2a, a > c\}$ is an ellipse with A and B as foci.



(Fig. 5)

Sol. : Choose the mid point C of AB as origin, CA as the x – axis and the line Cy perpendicular to Cx as the y-axis. Let the coordinates of A and B are $(c, 0)$ and $(-c, 0)$.

$$PA + PB = 2a \Leftrightarrow \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

$$\Leftrightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$\Leftrightarrow a^2\left\{(x-c)^2 + y^2\right\} = \left(cx - a^2\right)^2.$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{\left(a^2 - c^2\right)} = 1$$

Since $a > c$, this is an ellipse with $2a$ as major axis, $2\sqrt{a^2 - c^2}$ as minor axis, $\frac{c}{a}$ as eccentricity and $(\pm c, 0)$ as foci.

(4) Find the equation of the tangents of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which make an angle $\pi/4$ with the x – axis.

Sol. : The equation of the tangent at 'θ' is

$$\frac{x \cos\theta}{a} + \frac{y \sin\theta}{b} = 1 \qquad \text{— (1)}$$

Since the required tangents are not vertical lines, the slope is

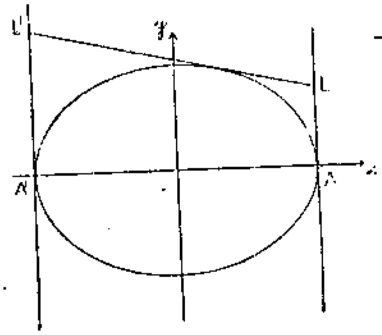
$$\frac{-b}{a} \cot\theta = \tan\frac{\pi}{4} = 1. \qquad \text{— (2)}$$

$$\therefore \ \operatorname{cosec}\theta = \pm \frac{\sqrt{a^2 + b^2}}{b} \qquad \text{— (3)}$$

Dividing (1) by sin θ and substituting for cot θ and cosec θ from (2) and (3), the equation of the tangent reduces to

$$y = x \pm \sqrt{a^2 + b^2}$$

(5) If the tangents at the ends A and A' of the major axis of an ellipse cut any other tangent to the ellipse in points L and L', prove that $AL, A'L'$ = constant.



(Fig. 6)

Sol. : If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the given ellipse, the points A and A' are $(a, 0)$ and $(-a, 0)$. The equation of the tangent at a point 'θ' on the ellipse is

$$\frac{x \cos\theta}{a} + \frac{y \sin\theta}{b} = 1$$

The tangents at A and A' are respectively $x = a$ and $x = -a$.

$$\therefore \ AL = \left|\frac{b(1 - \cos\theta)}{\sin\theta}\right|, A'L' = \left|\frac{b(1 + \cos\theta)}{\sin\theta}\right|$$

$$\therefore \ AL \, . \, A'L' = \left|\frac{b^2 \left(1 - \cos^2\theta\right)}{\sin^2\theta}\right| = b^2 \ \text{(constant)}.$$

(6) If the line $px + qy + r = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that

$$\frac{a^2}{p^2} + \frac{b^2}{q^2} = \frac{\left(a^2 - b^2\right)^2}{r^2}$$

Sol. : If the given line $px + qy + r = 0$ (1) is a normal at a point 'θ' on the ellipse, the equation of the normal at θ is

$$\frac{x \sin\theta}{b} - \frac{y \cos\theta}{a} = \left(\frac{a}{b} - \frac{b}{a}\right) \sin\theta \cos\theta \qquad \text{— (2)}$$

Comparing the coefficients in (1) and (2)

$$\frac{\sin\theta}{bp} = \frac{-\cos\theta}{aq} = \frac{\left(\frac{a}{b} - \frac{b}{a}\right) \sin\theta \cos\theta}{-r}$$

$$\therefore \ \frac{a}{p} = \frac{\left(b^2 - a^2\right) \cos\theta}{r}, \frac{b}{q} = -\frac{\left(b^2 - a^2\right) \sin\theta}{r}$$

$$\therefore \ \frac{a^2}{p^2} + \frac{b^2}{q^2} = \frac{\left(b^2 - a^2\right)^2}{r^2}$$

(7) The line $lx + my + n = 0$ is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if and only if } a^2 l^2 + b^2 m^2 = n^2.$$

Sol. : $lx + my + n = 0$ is a tangent to the ellipse

$\Leftrightarrow$ the line $lx + my + n = 0$ touches the ellipse at a point 'θ'

$\Leftrightarrow \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ and $lx + my + n = 0$ represent the same line

$\Leftrightarrow \dfrac{\cos\theta}{al} = \dfrac{\sin\theta}{bm} = -\dfrac{1}{n}$

$\Leftrightarrow \dfrac{\cos^2\theta + \sin^2\theta}{a^2 l^2 + b^2 m^2} = \dfrac{1}{n^2}$

$\Leftrightarrow a^2 l^2 + b^2 m^2 = n^2$

(8) A chord PQ subtends a right angle at the centre of an ellipse. Show that the locus of the point of intersection of the tangents at P and Q is the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Sol. : Let the tangents at P and Q intersect in (x_1, y_1). Since $\overline{PQ}$ is the chord of contact w.r.t (x_1, y_1) its equation is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Since the centre C is also the origin, the combined equation to CP, CQ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right)^2.$$

Since CP, CQ are perpendicular lines, the coefficient of x^2 + the coefficient of $y^2 = 0$

$$\therefore \frac{1}{a^2} - \frac{x_1^2}{a^4} + \frac{1}{b^2} - \frac{y_1^2}{b^4} = 0$$

i.e., $\dfrac{x_1^2}{a^4} + \dfrac{y_1^2}{b^4} = \dfrac{1}{a^2} + \dfrac{1}{b^2}.$

The locus of (x_1, y_1) is therefore given by

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

(9) If two lines $lx + my + n = 0$; $l'x + m'y + n' = 0$ are conjugate lines w.r.t an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, prove that $a^2 ll' + b^2 mm' = nn'$. Also prove that the conjugate lines through the focus of an ellipse are at right angles.

Sol. : Suppose (x_1, y_1) is the pole of $lx + my + n = 0$ w.r.t the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Then the lines

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

and $lx + my + n = 0$ are the same.

$$\therefore (x_1, y_1) = \left(\frac{-a^2 l}{n}, -\frac{b^2 m}{n}\right), \ (n \neq 0).$$

If $lx + my + n = 0$ and $l'x + m'y + n' = 0$ are conjugate lines, then (x_1, y_1) lies ... 29

on $l'x + m'y + n' = 0$.

$$\therefore \; l'\left(\frac{-a^2 l}{n}\right) + m'\left(\frac{-b^2 m}{n}\right) + n' = 0$$

i.e., $a^2 ll' + b^2 mm' = nn'$. —(1)

If the two given conjugate lines cut at one of the foci $(\pm ae, 0)$, then

$\pm\, lae + n = 0,\ \pm\, l'ae + n' = 0$ ($\because$ $(\pm ae, 0)$ is on both the lines).

(Only one of the signs is to be chosen).

$$\therefore \; nn' = ll'a^2 e^2 \qquad —(2)$$

From (1) and (2),

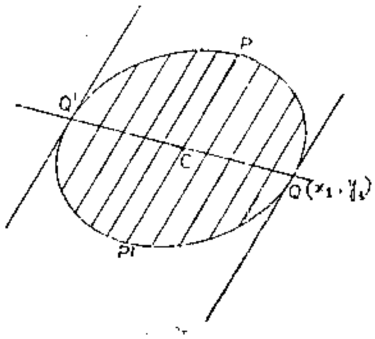
$$a^2 ll' + b^2 mm' = ll'a^2 e^2$$

i.e., $a^2\left(1 - e^2\right)ll' + b^2 mm' = 0$

$$b^2 ll' + b^2 mm' = 0$$

$$ll' + mm' = 0$$

The two given conjugate lines through a focus cut orthogonally.



(Fig. 7)

(10) Prove that the tangents at the extremities of a diameter are parallel to the conjugate diameter.

Sol.: Let PP', QQ' be conjugate diameters w.r.t the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let the equations of $\overleftrightarrow{PP}'$, $\overleftrightarrow{QQ}'$ be

$$lx + my = 0 \quad (1)$$

$$b^2 mx - a^2 ly = 0 \quad (2) \text{ (Theorem 7)}$$

The tangent at Q $\left(x_1, y_1\right)$ has equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad (3)$$

i.e., $\dfrac{x}{a^2}\left(b^2 mx_1\right) + \dfrac{y}{b^2}\left(b^2 my_1\right) = b^2 m$

$\dfrac{x}{a^2}\left(a^2 ly_1\right) + \dfrac{y}{b^2}\left(b^2 my_1\right) = b^2 m$ $\left[\because \text{ (2) is satisfied by } \left(x_1, y_1\right)\right]$

$lx + my = \dfrac{b^2 m}{y_1}\ \left(y_1 \neq 0\right)$.

This line is parallel to $lx + my = 0$.

If $y_1 = 0$, then $x_1 = a$. Equations (1), (2) reduce to $x = 0$ and $y = 0$, The tangents at $(\pm a, 0)$ are $x = \pm a$, parallel to $x = 0$.

(11) Show that the eccentric angles of the ends of semi conjugate diameters of the ellipse differ by $\pi/2$ or $3\pi/2$.

Sol. : Let θ, ϕ be the eccentric angles at the ends P, Q of a pair of semi conjugate diameters. If C is the centre, the tangent at P is parallel to $\overleftrightarrow{CQ}$. The equation of the tangent at P is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \qquad \text{— (1)}$$

The equation of $\overleftrightarrow{CQ}$ is

$$\frac{x \sin \theta}{a} + \frac{y \cos \theta}{b} = 0. \qquad \text{— (2)}$$

Since (1) and (2) are parallel,

$$\cos \theta \cos \phi + \sin \theta \sin \phi = 0$$

$$\therefore \quad \cos (\theta - \phi) = 0$$

$$\theta \sim \phi = \pi/2 \text{ or } 3\pi/2.$$

(12) Prove that the locus of the point of intersection of tangents at the ends of conjugate semi diameters of an ellipse is another ellipse.

Sol. : If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the given ellipse, the eccentric angles of the ends P and Q of semi conjugate diameters can be taken as $\theta, \theta + \pi/2$ or $\theta, \theta + 3\pi/2$. Then the equation of the tangents at P and Q are

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \qquad \text{— (1)}$$

and $$\frac{-x \sin \theta}{a} + \frac{y \cos \theta}{b} = \pm 1. \qquad \text{— (2)}$$

The point of intersection of the tangents in (1) and (2) lie on

$$\left(\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b}\right)^2 + \left(\frac{-x \sin \theta}{a} + \frac{y \cos \theta}{b}\right)^2 = 2$$

i.e., on the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$.

(13) Prove that the locus of poles w.r.t the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, of the tangents to the auxiliary circle $x^2 + y^2 = a^2$ is another ellipse.

Sol. : The equation of the tangent at 'θ' to the auxiliary circle is

$$x \cos \theta + y \sin \theta = a \qquad \text{— (1)}$$

If (x_1, y_1) is the pole of (1), its polar line w.r.t the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{— (2)}$$

Since (1) and (2) represent the same line,

$$\frac{x_1}{a^2 \cos \theta} = \frac{y_1}{b^2 \sin \theta} = \frac{1}{a},$$

$$\therefore \quad \frac{x_1}{a^2} = \frac{\cos\theta}{a}, \ \frac{y_1}{b^2} = \frac{\sin\theta}{a}$$

$$\text{or} \quad \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} = \frac{1}{a^2}$$

$\therefore$ The locus of (x_1, y_1) is $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{a^2}$

18.9 Summary

The standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Parametric equations of an ellipse are $x = a \cos\theta$, $y = b\sin\theta$. An ellipse has only two foci and two directrices corresponding to their foci. Equation to the tangent at any pole (x_1, y_1) to the ellipse is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. The locus of foot of the $\perp r$ drawn from a focus upon a tangent to the ellipse is a circle. It is called an auxiliary circle. The product of the distances of the foci from a tangent is a constant. Locus of the point of intersection of the perpendicular tangents to ellipse is a circle. It is called the director circle. Equation to a chord to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of the coordinates of its mid point is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$. $(S_1 = S_{11})$. Locus of mid points of a system of parallel chords of an ellipse is called a diametre of the ellipse. Equation to the chord of contact of an ellipse with respect to an external point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. Equation of the polar of a point (x_1, y_1) with respect to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. If pole of a line l lies on a line m then pole of m lies on the line l (w.r.t ellipse).

18.10 Sample Examination Questions

I. Answer the following in detail.

(i) (a) Obtain the standard equation of the ellipse and state its properties.

(b) Obtain the equation of an ellipse whose focus is the point (3, 1), whose directrix is the line $x - y + 6 = 0$ and eccentricity $\frac{1}{2}$.

(ii) (a) Show that (i) the feet of perpendiculars from the foci on the tangents to the ellipse lie on a circle, (ii) the product of the distances of the foci from a tangent is a constant and (iii) the points of intersection of pairs of perpendicular tangents to the ellipse lie on a circle.

(b) Find the centre and eccentricity of the ellipse

$$9x^2 + 25y^2 - 18x - 100y - 116 = 0.$$

II. Briefly answer the following.

(i) If the focus, centre and eccentricity of an ellipse are respectively (2, 3), (3, 4) and $\frac{1}{2}$. Find its equation.

(ii) If $\phi - \theta =$ constant, prove that the chord joining 'ϕ' and 'θ' touches a fixed ellipse.

(iii) Show that the locus of the poles of normal chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{a^6}{x^2}+\frac{b^6}{y^2}=\left(a^2-b^2\right)^2.$$

(iv) P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and Q is the corresponding point on the auxiliary circle. If the normals at P and Q meet in R, show that the locus of R is $x^2+y^2=(a+b)^2$.

(v) Show that the locus of the mid points of chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touching the ellipse $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ is $\frac{A^2x^2}{a^4}+\frac{B^2y^2}{b^4}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$

(vi) If P is a point on the director circle show that the locus of mid points of chords in which the polar of P cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}$.

(vii) If P and Q are the ends of a pair of semi conjugate diameters of the ellipse, show that the locus of mid point of PQ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}.$$

(viii) Obtain the parametric equations of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Answers

I. (ib) $7x^2+2xy+7y^2-60x-4y+44=0$

(iib) Centre (1, 2), eccentricity $\frac{4}{5}$.

II. (i) $\frac{(x-y+1)^2}{12}+\frac{(x+y-5)^2}{16}=1.$

UNIT-19 : HYPERBOLA

19.0 Contents

19.1 Aims and Objectives

After going through this unit, you will be able to :

i) derive the standard equation of hyperbola

ii) verify the properties of the hyperbola,

iii) define and obtain the standard equation of the rectangular hyperbola.

19.2 Standard Equation of Hyperbola

Suppose the given conic has eccentricity $e > 1$. Let S be a focus and $\overleftrightarrow{MD}$ be a directrix of a hyperbola. Draw SD perpendicular to the directrix. Choose $\overleftrightarrow{DS}$ as the X-axis and $\overleftrightarrow{DM}$ as the Y -axis. Let $\overline{SD} = c$. From a point $P\ (X, Y)$ on the hyperbola, draw PM perpendicular to the directrix. Then

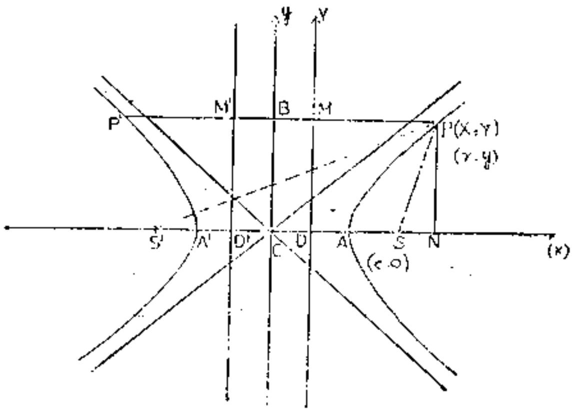


Fig. 1

$$SP = e.\,PM$$

$$SP^2 = e^2.\,PM^2$$

$$\therefore (X-c)^2 + Y^2 = e^2 X^2$$

$$X^2\left(e^2 - 1\right) + 2cX - Y^2 = c$$

$$X^2 + \frac{2c}{e^2-1}X - \frac{Y^2}{e^2-1} = \frac{c^2}{e^2-1}$$

$$\therefore \left(X + \frac{c}{e^2-1}\right)^2 - \frac{Y^2}{e^2-1} = \frac{c^2}{(e^2-1)^2} + \frac{c^2}{(e^2-1)} = \frac{c^2e^2}{(e^2-1)^2}.$$

Denote the point $\left(\frac{-c}{e^2-1}, 0\right)$ by C. Shifting the origin to C with the axes remaining parallel, the equations of the transformation of coordinates are

$$x = X + \frac{c}{e^2-1},\ y = Y.$$

If $a = \frac{ce}{e^2-1}$, the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

Since $e > 1$, there is a real number b such that $a^2(e^2-1) = b^2$. The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Properties : If C is the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, then

(i) $(x, y) \varepsilon C \Rightarrow (x, -y), (-x, y), (-x, -y) \varepsilon C$. Hence the hyperbola is symmetric about the axes. The point C bisects all chords passing through it. C is called the centre of the hyperbola.

(ii) The x-axis cuts the hyperbola in $A\ (a, 0)$ and $A'\ (-a, 0)$. $\overline{AA'} = 2a$.

(iii) The y-axis does not cut the hyperbola.

(iv) If $-a < x < a$, there is no real y satisfying the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Hence there is no point of the hyperbola between $x = -a$ and $x = +a$.

(v) If $x < -a$ or $x > +a$, then $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$. Hence for any such x there are two values of y such that (x, y) is on the curve. The curve extends to infinity on both sides of the x-axis. $(\because x \to \infty \Rightarrow y \to \pm\infty, x \to -\infty \Rightarrow y \to \pm\infty)$.

The hyperbola has two branches and not confined to any bounded region.

(vi) Consider the branch $y = \frac{b}{a}\sqrt{x^2 - a^2}$ and the line $y = \frac{b}{a}x$. If $M\left(x_1, y_1\right)$ is a point on this branch of the curve and $N\left(x_1, y_2\right)$ is on the given line, then $MN = y_2 - y_1$

$$= \frac{b}{a}x_1 - \frac{b}{a}\sqrt{x_1{}^2 - a^2} = \frac{ab}{x_1 + \sqrt{x_1{}^2 - a^2}} > 0.$$

The quantity tends to zero as $x_1 \to \infty$. Hence the straight line nears the curve as $x \to \infty$. Such a line is called an asymptote to the curve. It can be observed that the line $y = \frac{-b}{a}x$ is also an asymptote. Any line parallel to the asymptote meets the hyperbola in only one point. This can be easily observed from the equations of the hyperbola and the line, $y = \frac{-b}{a}x + d$ (d is a constant).

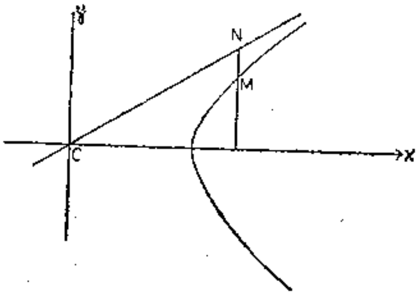


Fig. 2

(vii) The coordinates of the focus S are $(ae, 0)$ and the directrix corresponding to this focus is $x = \frac{a}{e}$ (see Fig. 1).

Since A, A' are points on the hyperbola, $SA = e \cdot AD$, $SA' = e \cdot A'D$.

$$\therefore AS + A'S = e\,(DA + A'D)$$
$$2\,CS = e \cdot 2CA$$
$$\therefore CS = e \cdot CA = e \cdot a$$
$$2CA = A'A = A'S - AS = e\,(A'D - AD) = e\,(A'C + CD) - e\,(CA - CD)$$
$$= 2\,e \cdot CD$$
$$\therefore CA = e \cdot CD$$
$$CD = \frac{a}{e}$$

$\therefore$ S has coordinates $(ae, 0)$ and the equation of the directrix is, $x = \frac{a}{e}$.

(viii) Since the hyperbola is symmetric about the axes, $S'(-ae, 0)$ is another focus and $x = -\frac{a}{e}$ is another directrix corresponding to S'.

(ix) If P is any point on the hyperbola and PM, PM' are perpendicular to the directrices $\overleftrightarrow{DM}$ and $\overleftrightarrow{D'M'}$ (Figure. 1), $S'P - SP = e\,(PM' - PM) = e \cdot MM' = e \cdot DD' = e \cdot \frac{2a}{e} = 2a.$

$\therefore$ For any point P on the curve, $S'P - SP = 2a$ (constant).

19.3 Parametric equtions of the Hyperbola

Let C be the centre of the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw the circles with centre C and radii a and b. The line CT through C making an angle θ with x-axis cuts the circle of radius a in T. The x-axis meets the second circle in $U\,(b, 0)$. The tangent to the circle of radius b at the point U cuts CT in M. The tangent at T to the circle of radius a cuts the x-axis in N. The line through N parallel to y-axis and the line through M parallel to x-axis neet in $P\,(x, y)$. Then

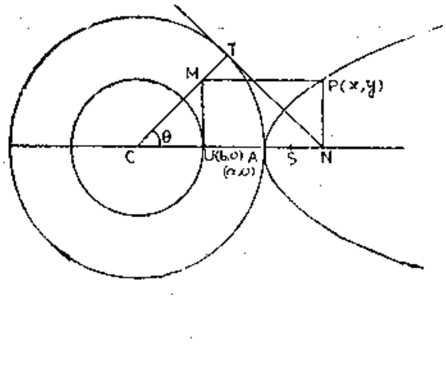


Fig. 3

$$x = CN = CT \sec\theta = a \sec\theta.$$

$$y = NP = UM = b \tan\theta.$$

Since $(a \sec\theta, b \tan\theta)$ satisfies the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, this point is on the hyperbola. As there is no point of the hyperbola between $x = -a$ and $x = +a$, the line $\overleftrightarrow{NP}$ whose equation is $x = a \sec\theta$ (observe that $|x| > a$) definitely cuts the hyperbola in two points $(a \sec\theta, b \tan\theta)$ and $(a \sec(-\theta), b \tan(-\theta))$. Therefore, any point on the hyperbola can be represented as $(a \sec\theta, b \tan\theta)$ for some value of θ

The equations $x = a \sec\theta, y = b \tan\theta$

are called the parametric equations to the hyperbola. The coordinates $(a \sec\theta, b \tan\theta)$ are called parametric coordinates of a point on the hyperbola. Usually the point $(a \sec\theta, b \tan\theta)$ is called the point 'θ' on the hyperbola.

19.4 Classification of points in the plane of a hyperbola

A point $P(x, y)$ in the xy plane is inside the hyperbolic region if and only if $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 > 0$. Such a point is called an interior point w.r.t the hyperbola. The point $P(x, y)$ is outside the hyperbolic region if and only if $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 < 0$. Such a point is called an exterior point w.r.t. the hyperbola. The point (x, y) is on the hyperbola if and only if $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

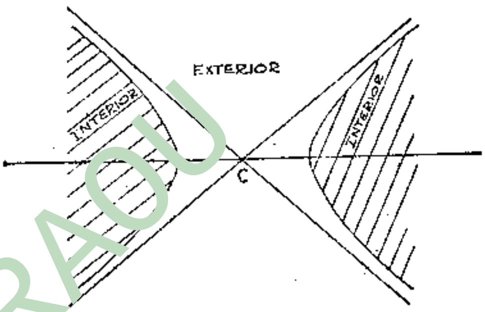


Fig. 4

19.5 A geometric property of the hyperbola

If PN is a perpendicular to the x-axis from a point P on the hyperbola, then

$$\frac{CN^2}{a^2} - \frac{PN^2}{b^2} = 1 \qquad \text{(see figure 1)}$$

$$\therefore \frac{PN^2}{b^2} = \frac{CN^2 - a^2}{a^2} = \frac{CN^2 - CA^2}{CA^2} = \frac{(CN + CA)(CN - CA)}{CA^2}$$

$$= \frac{NA \cdot NA'}{CA^2}$$

$$\frac{PN^2}{NA \cdot NA'} = \frac{b^2}{a^2}$$

Theorem. 1 : The hyperbola has only two foci and two directrices corresponding to the foci.

Proof : Let (x_1, y_1) be a focus and $x \cos\alpha + y \sin\alpha = p$ be a directrix corresponding to this focus. The equation of the hyperbola is,

$$\left(1 - e^2 \cos^2\alpha\right)x^2 - 2e^2 \sin\alpha \cdot \cos\alpha\, xy + \left(1 - e^2 \sin^2\alpha\right) y^2 - 2\left(x_1 - pe^2 \cos\alpha\right) x$$

$$-2\left(y_1 - pe^2 \sin\alpha\right) y + \left(x_1^2 + y_1^2 - p^2e^2\right) = 0 \qquad \ldots (1)$$

Since this is the same as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\sin\alpha\cos\alpha = 0 \qquad \text{... (2)}$$

$$x_1 - pe^2\cos\alpha = 0 \qquad \text{... (3)}$$

$$y_1 - pe^2\sin\alpha = 0 \qquad \text{... (4)}$$

$\therefore$ The equation (1) reduces to

$$x^2\left(1 - e^2\cos^2\alpha\right) + y^2\left(1 - e^2\sin^2\alpha\right) + \left(x_1^{\,2} + y_1^{\,2} - p^2e^2\right) = 0$$

Since this curve passes through $(a, 0)$

$$a^2\left(1 - e^2\cos^2\alpha\right) + \left(x_1^{\,2} + y_1^{\,2} - p^2e^2\right) = 0 \qquad \text{... (5)}$$

(2) gives $\sin\alpha = 0$ or $\cos\alpha = 0$

If $\cos\alpha = 0$, then $\sin\alpha = \pm 1, x_1 = 0, y_1 = \pm pe^2$

From (5),

$$a^2 + p^2e^4 - p^2e^2 = 0$$

$$\text{or } a^2 = p^2e^2(1 - e^2) < 0$$

This is impossible and hence $\sin\alpha = 0$.

$$\therefore \cos\alpha = \pm 1, x_1 = \pm pe^2, y_1 = 0,$$

$$a^2(1 - e^2) + p^2e^4 - p^2e^2 = 0$$

$$\text{or } (e^2 - 1)(p^2e^2 - a^2) = 0$$

$$\therefore a = \pm pe$$

$$\therefore x_1 = \pm ae, y_1 = 0.$$

The hyperbola has only two foci $(ae, 0)$ and $(-ae, 0)$. The equation of the directrices corresponding to these foci are $\pm x - p = 0$, i.e., $x \mp \frac{a}{e} = 0$.

Latus rectum : The chord through a focus parallel to the directrix is called the latus rectum of the hyperbola.

If $2l$ is the length of the latus rectum, then the point (ae, l) is on the hyperbola.

$$\therefore l^2 = b^2(e^2 - 1) = a^2(e^2 - 1)^2$$

$$\therefore 2l = 2a(e^2 - 1) = \frac{2b^2}{a}$$

Theorem. 2 : If $S_1 = 0$ and $S_2 = 0$ are the pair of intersecting lines, then the equation of the chord joining two points $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ on the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is $S_1 + S_2 = S_{12}$.

(The proof follows similar to theorem 2 of Unit 18).

Theorem. 3: The equation of the tangent at (x_1, y_1) on the hyperbola

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \text{ is } S_1 = 0.$$

(The proof follows similar to theorem 3 of Unit 18).

Note : (i) The equation of the tangent at (x_1, y_1) on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0.$$

At the point 'θ' with coordinates ($a \sec\theta, b \tan\theta$) on the hyperbola, the tangent has the equation

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} - 1 = 0.$$

(ii) The equation of the normal at (x_1, y_1) to the hyperbola is

$$\frac{(x - x_1)\, y_1}{b^2} + \frac{(y - y_1)\, x_1}{a^2} = 0.$$

At the point 'θ', the normal has the equation

$$\frac{x \tan\theta}{b} + \frac{y \sec\theta}{a} = \left(\frac{a}{b} + \frac{b}{a}\right) \sec\theta \tan\theta.$$

Theorem. 4 : The line $x \cos\alpha + y \sin\alpha = p$ is a tangent to the hyperbola if and only if $p = \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha}$.

Proof : Any straight line in the xy plane has the form

$$x \cos\alpha + y \sin\alpha = p \ (p > 0). \qquad \dots (1)$$

If this is a tangent at (x_1, y_1) on the hyperbola, then the equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \dots (2)$$

Since (1) and (2) represent the same line

$$\frac{\cos\alpha}{(x_1/a^2)} = \frac{\sin\alpha}{(-y_1/b^2)} = p$$

$$\therefore x_1 = \frac{a^2 \cos\alpha}{p}, \ y_1 = -\frac{b^2 \sin\alpha}{p}.$$

Since the point (x_1, y_1) lies on the tangent (1),

$$\left(\frac{a^2 \cos\alpha}{p}\right) \cos\alpha + \left(\frac{-b^2 \sin\alpha}{p}\right) \sin\alpha = p.$$

$\therefore p = \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha}$ (Since $p \geq 0$, the positive root is taken).

Conversely, if $p = \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha}$, then

$$p^2 = a^2 \cos^2\alpha - b^2 \sin^2\alpha.$$

The point $(x_1, y_1) = \left(\frac{a^2 \cos\alpha}{p}, \frac{-b^2 \sin\alpha}{p}\right)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as can be easily verified. The tangent at this point to the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

i.e., $\frac{x}{a^2}\left(\frac{a^2 \cos\alpha}{p}\right) - \frac{y}{b^2}\left(\frac{-b^2 \sin\alpha}{p}\right) = 1.$

$\therefore$ $x \cos\alpha + y \sin\alpha = p$ is a tangent to the hyperbola.

Theorem. 5 : (i) The feet of perpendiculars from the foci on the tangents to the hyperbola lie on a circle. (ii) The product of the distances of the foci from a tangent is a constant. (iii) If $a > b$, the points of intersection of pairs of perpendicular tangents lie on a circle.

Proof : (i) Let the equation of a tangent to the hyperbola be

$$x \cos\alpha + y \sin\alpha = \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha} \quad \text{... (1)}$$

The equations of the perpendiculars to the tangent from the foci $(\pm ae, 0)$ are

$$x \sin\alpha - y \cos\alpha = \pm ae \sin\alpha. \quad \text{... (2)}$$

From (1) and (2)

$$(x \cos\alpha + y \sin\alpha)^2 + (x \sin\alpha - y \cos\alpha)^2$$
$$= a^2 \cos^2\alpha - b^2 \sin^2\alpha + a^2e^2 \sin^2\alpha$$
$$\therefore x^2 + y^2 = a^2.$$

The feet of perpendiculars from the foci on the tangent lie on the circle $x^2 + y^2 = a^2$.

(ii) If h, h_1 are the perpendicular distances from the foci $(\pm ae, 0)$ on a tangent, then

$$h = \left|ae \cos\alpha - \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha}\right|$$

$$h_1 = \left|-ae \cos\alpha - \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha}\right|$$

$$\therefore hh_1 = \left|a^2e^2 \cos^2\alpha - a^2 \cos^2\alpha + b^2 \sin^2\alpha\right|$$

$$= b^2 \text{ (Constant).}$$

(iii) Two mutually perpendicular tangents to the hyperbola can be taken as

$$x \cos\alpha + y \sin\alpha = \sqrt{a^2 \cos^2\alpha - b^2 \sin^2\alpha} \quad \text{... (1)}$$

and $x \cos(\alpha + \pi/2) + y \sin(\alpha + \pi/2)$

$$= \sqrt{a^2 \cos^2(\alpha + \pi/2) - b^2 \sin^2(\alpha + \pi/2)}$$

i.e., $$-x \sin\alpha + y \cos\alpha = \sqrt{a^2 \sin^2\alpha - b^2 \cos^2\alpha} \quad \text{... (2)}$$

The point of intersection of (1) and (2) lies on

$$(x \cos\alpha + y \sin\alpha)^2 + (-x \sin\alpha + y \cos\alpha)^2$$
$$= a^2 \cos^2\alpha - b^2 \sin^2\alpha + a^2 \sin^2\alpha - b^2 \cos^2\alpha$$

i.e., $$x^2 + y^2 = a^2 - b^2.$$

If $a \le b$, there are no mutually perpendicular tangents to the hyperbola.

19.6 Auxiliary Circle and Director Circle

The locus of the feet of perpendiculars from the foci on the tangents to the hyperbola is a circle defined as the *auxiliary circle* of the hyperbola.

If the locus of the point of intersection of mutually perpendicular tangents to the hyperbola is a circle, it is defined as the *director circle* of the hyperbola.

Theorem. 6 : The equation of the chord of the hyperbola with mid-point $\left(x_1, y_1\right) \neq (0, 0)$, is $S_1 = S_{11}$.

Proof : Let $A\left(x_2, y_2\right)$ and $B\left(x_3, y_3\right)$ be the end points of the chord whose mid-point is $\left(x_1, y_1\right)$. Then $x_2 + x_3 = 2x_1$ and $y_2 + y_3 = 2y_1$. If $S_2 = 0$ and $S_3 = 0$ are parallel, then (0, 0) is mid-point of $\overline{AB}$. If $S_2 = 0$ and $S_3 = 0$ are intersecting, then the equation to $\overline{AB}$ is

$$S_2 + S_3 = S_{23}.$$

$$\frac{x\left(x_2 + x_3\right)}{a^2} - \frac{y\left(y_2 + y_3\right)}{b^2} - 2 = S_{23}$$

$$\text{or} \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = k \text{ (Constant).}$$

Since this line passes through $\left(x_1, y_1\right)$,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = k$$

$$\therefore \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$$\text{i.e., } S_1 = S_{11}.$$

Theorem. 7 : The locus of the mid-points of parallel chords of a hyperbola is a straight line.

Proof : Consider the system of chords of a hyperbola parallel to $lx + my = 0$. The equation of the chord whose mid-point is $\left(x_1, y_1\right)$ is $S_1 = S_{11}$ or

$$\therefore \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1.$$

If this is parallel to $lx + my = 0$, then

$$\frac{mx_1}{a^2} = -\frac{ly_1}{b^2}.$$

$\therefore$ The locus of $\left(x_1, y_1\right)$ is the straight line $b^2mx + a^2ly = 0$.

Note : If the hyperbola has a tangent parallel to $lx + my = 0$, this tangent touches the hyperbola at a point on $b^2mx + a^2ly = 0$.

Diameter : A straight line which is the locus of the mid-points of a system of parallal chords of a hyperbola is called a diameter of the hyperbola.

It can be observed that all diameters of a hyperbola bisect each other at the centre.

Any line through the centre is a diameter. But all diameters need not cut the hyperbola. The straight line $y = mx$ cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in two points only, when $|m| < \frac{b}{a}$. If $|m| > \frac{b}{a}$, it will not meet the hyperbola.

Theorem. 8 : If two diameters of a hyperbola are such that one of them bisects chords parallel to the second diameter, then the second diameter bisects chords parallel to the first.

Proof : Let $lx + my = 0$ be a diameter. Then by Theorem 5 the other diameter has the equation

$b^2mx + a^2ly = 0$ lie on

$$b^2(a^2l)x + a^2(b^2m)y = 0$$

$$\text{i.e., } lx + my = 0.$$

Conjugate diameters : If two diameters of a hyperbola are such that either bisects chords parallel to the other, they are called conjugate diameters w.r.t. the hyperbola.

(i) Only one of these conjugate diameters cuts the hyperbola (if the absolute value of the slope $lx + my = 0$ is less than b/a, then the absolute value of the slope of $b^2mx + a^2ly = 0$ is greater than b/a.)

(ii) If a diamter meets a hyperbola, then the tangents at the two ends of this diameter are parallel to the conjugate diameter.

(If $lx + my = 0$ is a diameter meeting the hyperbola in (x_1, y_1) and (x_2, y_2), then $lx_1 + my_1 = 0$ and $lx_2 + my_2 = 0$. The tangent at (x_1, y_1) has slope

$$\frac{b^2x_1}{a^2y_1} = \frac{b^2}{a^2}\left(\frac{-m}{l}\right) = \frac{-mb^2}{la^2}.$$

This is also the slope of the conjugate diameter

$$b^2mx + a^2ly = 0).$$

Principal axis : A straight line which bisects all chords of the hyperbola perpendicular to it is called a principal axis of the hyperbola.

The principal axes are also diameters of a hyperbola.

Theorem. 9 : The hyperbola has only two principal axes.

Proof : If the diameter $b^2mx + a^2ly = 0$ which bisects chords parallel to $lx + my = 0$ is perpendicular to the chords, then

$$\left(b^2m\right)l + \left(a^2l\right)m = 0$$

$$\Leftrightarrow lm = 0$$

If $l = 0$, then $x = 0$ (y-axis) and if $m = 0$, then $y = 0$ (x-axis) are principal axes of the hyperbola. Out of these two, only the x-axis cuts the hyperbola.

Vertices : The points where the principal axes cut the hyperbola are called the vertices of the hyperbola.

These are the points $A(a, 0)$ and $A'(-a, 0)$.

Transverse axis : If A, A' are the vertices of the hyperbola, then $\overrightarrow{AA'}$ is called the transverse axis of the hyperbola.

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the length of the transverse axis $\overline{AA'} = 2a$.

Theorem. 10 : (i) The product of distances of point P from the asymptotes l and l' of a hyperbola C is a constant k.

(ii) The point P lies on the hyperbola C or on another hyperbola C'. (C' is called conjugate hyperbola of C).

Proof : Suppose the hyperbola C is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then the equations of the asymptotes l and l' are

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

If θ is any point on C, the product of distances of θ from l and l' is

$$= \left|\frac{\frac{x_1}{a} - \frac{y_1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| \cdot \left|\frac{\frac{x_1}{a} + \frac{y_1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| \qquad \text{where } x_1 = a\sec\theta, \; y_1 = b\tan\theta.$$

$$= \left|\frac{\left(\sec^2\theta - \tan^2\theta\right) a^2b^2}{a^2 + b^2}\right| = \frac{a^2b^2}{a^2 + b^2} = k \text{ (a constant)}.$$

(ii) The product of distances of $P\left(x_1, y_1\right)$ from the asymptotes l and l' is k

$$\Leftrightarrow \left|\frac{\frac{x_1}{a} - \frac{y_1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| \left|\frac{\frac{x_1}{a} + \frac{y_1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = k = \frac{a^2b^2}{a^2 + b^2}$$

$$\Leftrightarrow \left|\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right| = 1$$

$$\Leftrightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \pm 1.$$

∴ The locus of $P\left(x_1, y_1\right)$ is either the given hyperbola C, namely $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, or the hyperbola C', $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ i.e., $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. The hyperbolas C and C' are shown in the following figure.

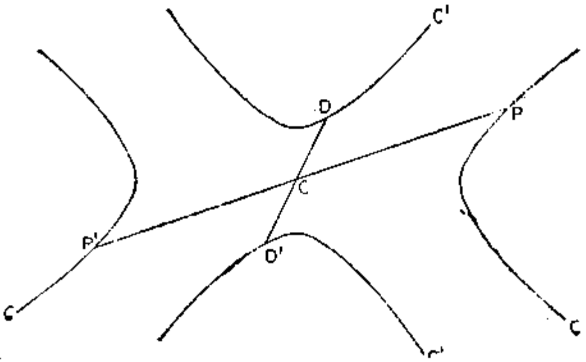


Fig. 5

Note : If the hyperbola C' is the conjugate of C, then the hyperbola C is the conjugate of C'.

If the C, C' are conjugate hyperbolas, then the transverse axis of C' is known as the conjugate axis of C.

The length of conjugate axis of C is $2b$.

The hyperbolas C, C' have the same asymptotes.

Chord of contact : If the tangents at two points A and B on the hyperbola meet at a point P, then $\overline{AB}$ is called the chord of contact w.r.t the point P.

Theorem. 11 : The equation of the chord of contact of the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, w.r.t. a point $P\left(x_1, y_1\right)$ is $S_1 = 0$.

(The proof follows on lines similar to Theorem 10 of Unit 18).

Theorem. 12 : If the chord of contact of the hyperbola w.r.t. a point Q passes through a given (fixed) point $P\left(x_1, y_1\right)$, then the point Q lies on the straight line $S_1 = 0$.

(The Proof follows on lines similar to Theorem 11 of Unit 18).

19.7 Pole and Polar

If the chord of contact of a hyperbola w.r.t. Q passes through a fixed point P, then the locus of points Q is a straight line l called the polar of point P w.r.t. the hyperbola. The point P is called the pole of the line l w.r.t. the hyperbola.

Note : (i) The equation of the polar line of $P\left(x_1, y_1\right)$ w.r.t. the hyperbola is $S_1 = 0$.

(ii) If the polar of a point P w.r.t. a hyperbola passes through a point Q, then the polar of Q passes through P. The P and Q are called conjugate points w.r.t the hyperbola.

(iii) If two lines are such that the pole of any one of them w.r.t. the hyperbola lies on the second line, then the pole of the second line lies on the first. The two lines are called conjugate lines w.r.t. the hyperbola.

(The proofs of these statements follow on lines similar to those in Unit 18).

Theorem. 13 : The equation of the pair of tangents from an exterior point $P\left(x_1, y_1\right) \neq (0, 0)$ to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is

$$S_1^2 - S\,S_{11} = 0$$

(The proof is similar to that of Theorem 12 of Unit 18).

19.8 Rectangular Hyperbola

Definition : If a hyperbola is such that the lengths of its transverse axis and conjugate axis are equal, then it is called a rectangular hyperbola.

Therefore, the equation of a rectangular hyperbola with $(a, 0)$ and $(-a, 0)$ as vertices is $x^2 - y^2 = a^2$. The asymptotes are $x - y = 0$ and $x + y = 0$ and are mutually perpendicular. The eccentricity of the rectangular hyperbola is $\sqrt{2}$ ($\because b^2 = a^2(e^2 - 1) \Rightarrow a^2 = a^2(e^2 - 1) \Rightarrow e^2 = 2$).

19.9 Standard equation of the rectangular hyperbola

The equation of the rectangular hyperbola having vertices $(a, 0)$ and $(-a, 0)$, referred to OX, OY is

$$X^2 - Y^2 = a^2.$$

The asymptotoes are $X + Y = 0$ and $X - Y = 0$.

Choosing a new set of axes, ox, oy got by rotating the original axes OX ,OY through an angle $-\pi/4$, the transformations are

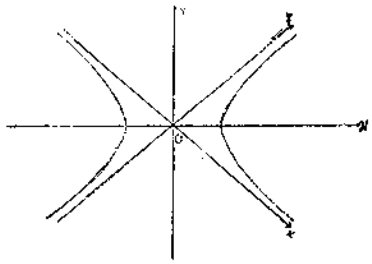


Fig. 6

$$X = x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} = \frac{x+y}{\sqrt{2}}$$

$$Y = -x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} + \frac{-x+y}{\sqrt{2}}$$

The equation of the curve now becomes

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{-x+y}{\sqrt{2}}\right)^2 = a^2$$

$$\text{i.e., } xy = a^2/2.$$

Writing $c = a/\sqrt{2}$ the rectangular hyperbola in its standard form is

$$xy = c^2$$

The asymptotes $X + Y = 0$ and $X - Y = 0$ now become $x = 0$ and $y = 0$. Thus, the x and y axes are now asymptotes to the rectangular hyperbola $xy = c^2$.

19.10 Parametric representation

If $t \neq 0$ is a real number, the point $\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. Also, given any point $\left(x_1, y_1\right)$ on the curve, choosing $t_1 = \frac{x_1}{c}$, the point $\left(x_1, y_1\right)$ on the curve can be represented as $\left(ct_1, \frac{c}{t_1}\right)$, $t_1 \neq 0$. Threfore, the equations $x = ct$, $y = \frac{c}{t}$ are called parametric equations of the rectangular hyperbola $xy = c^2$. The coordinates $\left(ct, \frac{c}{t}\right)$ are called parametric coordinates. This point on the rectangular hyperbola is also denoted by the parameter 't'.

The following results on rectangular hyperbola are obtained as in Units 17 and 18.

If $S \equiv xy - c^2 = 0$ is the equation of the rectangular hyperbola, then

(i) If $S_1 = 0$ and $S_2 = 0$ are intersecting lines, the equation of the chord joining $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ on the rectangular hyperbola $S \equiv xy - c^2 = 0$ is $S_1 + S_2 = S_{12}$. In parametric representation, the chord joining points t_1 and t_2 on the rectangular hyperbola is

$$x + t_1 t_2 y = c\left(t_1 + t_2\right).$$

(ii) If $(x_1, y_1) \neq (0, 0)$ is the mid-point of a chord of a rectangular hyperbola $S \equiv xy - c^2 = 0$, the equation of the chord is $S_1 = S_{11}$.

(iii) The equation of the tangent at (x_1, y_1) to

$$S \equiv xy - c^2 = 0 \text{ is } S_1 = 0$$

i.e., $\dfrac{xy_1 + x_1 y}{2} - c^2 = 0.$

In parametric representation, the equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is $x + yt^2 = 2ct$.

(iv) The tangents at the points t_1 and t_2 on the rectangular hyperbola, namely

$$x + yt_1^2 = 2ct_1$$

$$\text{and } x + yt_2^2 = 2ct_2$$

intersect in $\left(\dfrac{2ct_1t_2}{t_1 + t_2}, \dfrac{2c}{t_1 + t_2}\right)$

(v) The equation of the normal at (x_1, y_1) on the rectangular hyperbola is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(vi) The polar of $P(x_1, y_1)$ w.r.t. the rectangular hyperbola is $S_1 = 0$.

(vii) The equation of the pair of tangents from $P(x_1, y_1)$ is

$$S_1^2 - S\,S_{11} = 0.$$

Examples

1. Find the foci, eccentricity and latus rectum of the hyperbola $x^2 - 4y^2 = 4$.

Sol : The equation of the hyperbola is $\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1.$

Comparing with standard form,

$a = 2, b = 1.$

$\therefore b^2 = a^2(e^2 - 1) \Rightarrow e = \sqrt{5/4}.$

The foci are $(\pm ae, 0)$, i.e., $(\pm \sqrt{5}, 0)$.

The length of the latus rectum is $2l = 2\dfrac{b^2}{a} = 1.$

2. Find the centre, eccentricity, foci and directrices of the hyperbola

$$9x^2 - 16y^2 + 72x - 32y - 16 = 0.$$

Sol : The equation of the hyperbola can be wirtten as

$$9(x^2 + 8x) - 16(y^2 + 2y) = 16$$

$$9\left\{(x + 4)^2 - 16\right\} - 16\left\{(y + 1)^2 - 1\right\} = 16$$

$$9(x + 4)^2 - 16(y + 1)^2 = 144.$$

$$\frac{(x + 4)^2}{16} - \frac{(y + 1)^2}{9} = 1.$$

Shifting the origin to $(-4, -1)$, the equation for transformation of coordinates are

$X = x + 4,\ Y = y + 1,$

$$\therefore \frac{X^2}{16} - \frac{Y^2}{9} = 1.$$

The centre of the hyperbola is (0, 0) in the XY coordinate system and $(-4, -1)$ in the original coordinate system

$a = 4, b = 3.$

$$\therefore e = 5/4.$$

The foci are $(\pm ae, 0)$, i.e., $(\pm 5, 0)$ in the new system. In the original system, the foci are

$(5 - 4, 0 - 1); (-5 - 4, 0 - 1)$

i.e., $(1, -1)$; $(-9, -1)$.

The directrices are $X = \pm \frac{a}{e} = \pm \frac{16}{5}$.

In the original system, the directrices are $x = \frac{-4}{5}, x = \frac{-36}{5}$.

3. Prove that the locus of the poles of the tangents to $x^2 - y^2 = a^2$ w.r.t. the parabola $y^2 = 4ax$ is the ellipse $4x^2 + y^2 = 4a^2$.

Sol : The equation of the tangent at $(a \sec\theta, b \tan\theta)$ to the hyperbola $x^2 - y^2 = a^2$ is

$$x \sec\theta - y \tan\theta = a \qquad \ldots (1)$$

If (x_1, y_1) is the pole of this line w.r.t. $y^2 = 4ax$, then the polar of (x_1, y_1) w.r.t. $y^2 = 4ax$ is

$$yy_1 - 2a(x + x_1) = 0 \qquad \ldots (2)$$

Since (1) and (2) represent the same line,

$$\frac{y_1}{-\tan\theta} = \frac{2a}{-\sec\theta} = \frac{2ax_1}{a}$$

$$\therefore x_1 = -a\cos\theta,\ y_1 = 2a\sin\theta.$$

The locus of (x_1, y_1) is therefore given by

$$\frac{x^2}{a^2} + \frac{y^2}{4a^2} = 1$$

$$\text{or } 4x^2 + y^2 = 4a^2.$$

4. If the tangent at a point P on a hyperbola (with centre C) cuts the transverse axis in T and N is the foot of perpendicular from P on the transverse axis, show that $CT \cdot CN = CA^2$.

Sol : Suppose the given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If P is the point $(a \sec\theta, b \tan\theta)$, then the equation of $\overleftrightarrow{PT}$ is

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1.$$

This line cuts the transverse axis in T $(a \cos\theta, 0)$.

$$CT = a\cos\theta$$

$$CN = x\text{-coordinates of } P = a\sec\theta.$$

$$\therefore CT \cdot CN = a^2 = CA^2$$

5. Show that the locus of the poles w.r.t. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, of tangents to the circle on the line joining the foci as diameter is the conic $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$

Sol : The equation of the circle on the line joining the foci ($\pm ae$, 0) as the diameter is $x^2 + y^2 = a^2e^2$.

Let (x_1, y_1) be the coordinates of a point, the polar of which with respect to the hyperbola touches the circle.

The Polar of (x_1, y_1) w.r.t. the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \ldots (1)$$

If this line touches the circle, $x^2 + y^2 = a^2e^2$ at (x_2, y_2), then the tangent at (x_2, y_2) to the circle is

$$xx_2 + yy_2 = a^2e^2. \qquad \ldots (2)$$

Since (1) and (2) represent the same line

$$\frac{x_2}{x_1/a^2} = \frac{y_2}{-y_1/b^2} = a^2e^2$$

$$\therefore x_2 = x_1e^2, y_2 = -\frac{y_1e^2a^2}{b^2}.$$

Since (x_2, y_2) lies on the circle $x_2^2 + y_2^2 = a^2e^2$

$$\therefore \quad x_1^2e^4 + \frac{y_1^2e^4a^4}{b^4} = a^2e^2$$

$$\text{i.e.,} \quad \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} = \frac{1}{a^2e^2} = \frac{1}{a^2 + b^2}$$

The locus of (x_1, y_1) is therefore given by

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$$

6. Tangents from a point P on the circle $x^2 + y^2 = a^2$ are drawn, to the hyperbola to meet it at Q and R. Show that the locus of the mid-point of QR is

$$(x^2 - y^2)^2 = a^2(x^2 + y^2).$$

Sol : Let (x_1, y_1) be the mid-point of QR. Then the equation of QR is

$$xx_1 - yy_1 = x_1^2 - y_1^2. \qquad \ldots (1)$$

Let P be the point (x_2, y_2). Since it lies on the circle $x^2 + y^2 = a^2$,

$$x_2^2 + y_2^2 = a^2. \qquad \ldots (2)$$

Since QR is the chord of contact of tangents from P (x_2, y_2), to the hyperbola, the equation of QR is

$$xx_2 - yy_2 = a^2 \qquad \ldots (3)$$

Since (1) and (3) represent the same straight line,

$$\frac{x_2}{x_1} = \frac{y_2}{y_1} = \frac{a^2}{x_1^{\,2} - y_1^{\,2}}$$

$$x_2 = \frac{x_1 a^2}{x_1^{\,2} - y_1^{\,2}},\; y_2 = \frac{y_1 a^2}{x_1^{\,2} - y_1^{\,2}}.$$

Equation (2) then gives

$$x_1^{\,2} a^4 + y_1^{\,2} a^4 = a^2 \left(x_1^{\,2} - y_1^{\,2}\right)^2.$$

$\therefore$ The locus of (x_1, y_1) is given by

$$\left(x^2 + y^2\right) a^2 = \left(x^2 - y^2\right)^2.$$

7. Find the condition for the straight line $lx + my + n = 0$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Sol : Suppose that the given line

$$lx + my + n = 0 \qquad \ldots (1)$$

touches the hyperbola at $(a \sec \theta, b \tan \theta)$. Then the tangent at $(a \sec \theta, b \tan \theta)$ is

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1. \qquad \ldots (2)$$

Since (1) and (2) represent the same line.

$$\frac{\sec \theta}{al} = \frac{-\tan \theta}{bm} = -\frac{1}{n}.$$

$\therefore \sec \theta = -al/n, \tan \theta = bm/n.$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\frac{a^2 l^2}{n^2} - \frac{b^2 m^2}{n^2} = 1$$

$a^2 l^2 - b^2 m^2 = n^2$ is the required condition.

8. If a straight line cuts a hyperbola in P and Q and its asymptotes in R and S, then show that $PR = QS$.

Sol :

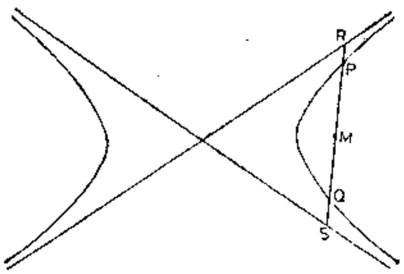


Fig. 7

Let $M\ (x_1, y_1)$ be the mid-point of PQ. Then the equation of PQ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^{\,2}}{a^2} - \frac{y_1^{\,2}}{b^2} \qquad \ldots (1)$$

The asymptotes of the hyperbola are

$$\frac{x}{a} - \frac{y}{b} = 0$$

$$\frac{x}{a} + \frac{y}{b} = 0 \qquad \dots (3)$$

Solving (1) and (2), the coordinates of R are obtained as

$$\left\{a\left(\frac{x_1}{a}+\frac{y_1}{b}\right),\ b\left(\frac{x_1}{a}+\frac{y_1}{b}\right)\right\}.$$

Solving (1) and (3), the coordinates of S are obtained as

$$\left\{a\left(\frac{x_1}{a}-\frac{y_1}{b}\right),\ -b\left(\frac{x_1}{a}-\frac{y_1}{b}\right)\right\}.$$

The mid point of $R\,S$ is therefore

$$\left\{\frac{a\left(\frac{x_1}{a}+\frac{y_1}{b}\right)+a\left(\frac{x_1}{a}-\frac{y_1}{b}\right)}{2},\ \frac{b\left(\frac{x_1}{a}+\frac{y_1}{b}\right)-b\left(\frac{x_1}{a}-\frac{y_1}{b}\right)}{2}\right\} = \{x_1, y_1\}.$$

$\therefore$ The mid point of $\overline{RS}$ is the same as the mid point of $\overline{PQ}$.

$$\therefore PR = QS$$

9. If e, e_1 are the eccentricities of a hyperbola and its conjugate show that

$$\frac{1}{e^2} + \frac{1}{e_1^2} = 1$$

Sol : The equation of a hyperbola C and its conjugate C' are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \dots (1)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad \dots (2)$$

$$\text{where } b^2 = a^2(e^2-1). \qquad \dots (3)$$

The transverse axis of C' is the conjugate axis of C.

$$\therefore a^2 = b^2\left(e_1^2 - 1\right). \qquad \dots (4)$$

From (3) and (4),

$$a^2b^2 = a^2b^2(e^2-1)\left(e_1^2-1\right)$$

$$e^2e_1^2 - e^2 - e_1^2 = 0$$

$$\therefore \frac{1}{e^2} + \frac{1}{e_1^2} = 1$$

10. Prove that the tangent to a rectangular hyperbola terminated by its asymptotes is bisected at the point of contact and encloses a triangle of constant area.

Sol : Taking the asymptotes of the rectangular hyperbola as coordinate axes, its equation will be

$$xy = c^2.$$

Let P be the point (x_1, y_1) on it.

The equation of the tangent at P is $xy_1 + yx_1 = 2c^2$.

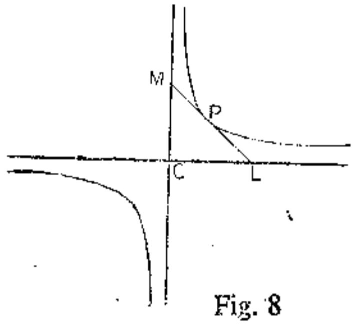


Fig. 8

Since (x_1, y_1) lies on the hyperbola,

$$x_1 y_1 = c^2$$

$\therefore$ The tangent at $P\ (x_1, y_1)$ is

$$xy_1 + yx_1 = 2x_1y_1$$

$$\therefore \frac{x}{2x_1} + \frac{y}{2y_1} = 1.$$

The intercepts CL, CM are $2x_1$ and $2y_1$

The points L and M have coordinates $(2x_1, 0)$ and $(0, 2y_1)$.

$\therefore$ The mid point of LM is

$$\left(\frac{2x_1 + 0}{2}, \frac{0 + 2y_1}{2}\right) \text{ i.e., } (x_1, y_1).$$

Area of $\Delta\, CLM = \frac{1}{2} \cdot CL \cdot CM = \frac{1}{2} \cdot 2x_1 \cdot 2y_1 = 2x_1y_1 = 2c^2$ (a contant).

11. If a hyperbola be rectangular and its equation be $xy = c^2$, prove that the locus of mid points of chords of constant length $2l$ is

$$\left(x^2 + y^2\right)\left(xy - c^2\right) = l^2xy.$$

Sol : Let the chord joining the points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the rectangular hyperbola be of length $2l$ and let it's mid point be (x_1, y_1). Then

$$2x_1 = ct_1 + ct_2 \qquad \ldots (1)$$

$$2y_1 = \frac{c}{t_1} + \frac{c}{t_2} \qquad \ldots (2)$$

$$(2l)^2 = \left(ct_1 - ct_2\right)^2 + \left(\frac{c}{t_1} - \frac{c}{t_2}\right)^2 \qquad \ldots (3)$$

Eliminating t_1 and t_2 from (1), (2) and (3),

$$4l^2 = c^2\left(t_1 - t_2\right)^2 + c^2\left(\frac{1}{t_1} - \frac{1}{t_2}\right)^2$$

$$= c^2 \left(t_1 - t_2\right)^2 + c^2 \frac{\left(t_1 - t_2\right)^2}{t_1^2 t_2^2}$$

$$= c^2 \left(t_1 - t_2\right)^2 \left(1 + \frac{1}{t_1^2 t_2^2}\right)$$

$$= c^2 \left\{ \frac{\left[\left(t_1 + t_2\right)^2 - 4t_1 t_2\right]\left[t_1^2 t_2^2 + 1\right]}{t_1^2 t_2^2} \right\} \qquad \dots (4)$$

From (1), $\quad t_1 + t_2 = \dfrac{2x_1}{c}$

From (2), $\quad \dfrac{t_1 + t_2}{t_1 t_2} = \dfrac{2y_1}{c}$

$$\therefore t_1 t_2 = \frac{x_1}{y_1}.$$

Substituting the values of $t_1 + t_2$ and $t_1 t_2$ in (4).

$$4l^2 = \frac{c^2\left[\frac{4x_1^2}{c^2} - \frac{4x_1}{y_1}\right]\left[1 + \frac{x_1^2}{y_1^2}\right]}{\left(\frac{x_1^2}{y_1^2}\right)}$$

Simplifying,

$$l^2 x_1 y_1 = \left(x_1 y_1 - c^2\right)\left(x_1^2 + y_1^2\right).$$

$\therefore$ The locus of $\left(x_1, y_1\right)$ is

$$l^2 xy = \left(xy - c^2\right)\left(x^2 + y^2\right).$$

12. Prove that the orthocentre of a triangle inscribed in a rectangular hyperbola lies on the rectangular hyperbola.

Sol. : Let PQR be the triangle inscribed in the rectangular hyperbola $xy = c^2$ and let the coordinates of P, Q and R be

$$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right)$$

The equation of PQ is

$$x + yt_1 t_2 = c\left(t_1 + t_2\right).$$

The equation of the perpendicular from R on PQ is

$$y - \frac{c}{t_3} = t_1 t_2 \left(x - ct_3\right).$$

$$\text{i.e., } xt_1 t_2 - y = c\left(t_1 t_2 t_3 - \frac{1}{t_3}\right) \qquad \dots (1)$$

Similarly, the equation of the perpendicular from P on QR is

$$xt_2 t_3 - y = c\left(t_1 t_2 t_3 - \frac{1}{t_1}\right) \qquad \text{... (2)}$$

The orthocentre of the triangle is the point of intersection of (1) and (2), namely

$$\left(\frac{-c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$$

This point again lies on the given rectangular hyperbola.

13. Show that four normals can be drawn from a point to the rectangular hyperbola $xy = c^2$ and that the feet form a triangle and its orthocentre.

Sol. : The equation of the normal at 't' is

$$xt - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$$

If it passes through (x_1, y_1), then

$$x_1 t - \frac{y_1}{t} = c\left(t^2 - \frac{1}{t^2}\right)$$

i.e., $ct^4 - x_1 t^3 + y_1 t - c = 0$ ... (1)

This is biquadratic in t and has four roots. If these roots are t_1, t_2, t_3, t_4, corresponding to each of them, there is a normal passing through the fixed point (x_1, y_1). The feet of the normals are the points 't_1', 't_2', 't_3' and 't_4' on the rectangular hyperbola. From equation (1),

$$t_1 + t_2 + t_3 + t_4 = \frac{x_1}{c}$$

$$\Sigma\, t_1 t_2 = 0$$

$$\Sigma\, t_1 t_2 t_3 = -\frac{y_1}{c}$$

$$t_1 t_2 t_3 t_4 = -1$$

If the points t_1, t_2, t_3, t_4 on the rectangular hyperbola are denoted by P, Q, R, T, then the orthocentre of ΔPQR is

$$\left(\frac{-c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$$

i.e., $\left(ct_4, \frac{c}{t_4}\right)$ since $t_1 t_2 t_3 t_4 = -1$

This is the point T.

19.11 Summary

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the standard equation of a hyperbola. $x = a \sec\theta$, $y = b \tan\theta$ are the parametric equations of the hyperbola. The hyperbola has only two foci and two directrices corresponding to them. $x^2 + y^2 = a^2$ is the auxiliary circle and $x^2 + y^2 = a^2 - b^2$ is the director circle of the above hyperbola;

Locus of mid points of a system of parallel chords is a straight line. It is called a diameter. If m and m' denote the slopes of the diameters to a hyperbola, and if $mm' = \frac{-b^2}{a^2}$, then they are said to be conjugate diameters. Equation of the tangent at a point (x_1, y_1) of the hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. Equation of the polar of a point (x_1, y_1) with respect to the hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. Equation to the chord of contact of the hyperbola with respect to an external point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

The equation of a the conjugate hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. If $a = b$, the hyperbola is called rectangular hyperbola.

19.12 Sample Examination Questions

I. Answer the following in detail

(i) a) Obtain the standard equation of the hyperbola and state its properties.

b) Find the axis, the coordinates of the foci, the eccentricity and the latus rectum of the hyperbola $4x^2 - 9y^2 = 36$.

(ii) a) Obtain the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

b) Obtain the equations of tangent and normal at point (x_1, y_1) of a hyperbola.

II. Briefly answer the following

(i) If the lines $l_1 x + m_1 y = 0$ and $l_2 x + m_2 y = 0$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, show that $a^2 l_1 l_2 - b^2 m_1 m_2 = 0$.

(ii) Prove that the locus of the mid points of chords of the hyperbola $x^2 - y^2 = a^2$ which touch $y^2 = 4ax$ is $y^2 (x - a) = x^3$.

(iii) Prove that the locus of the poles of the tangents to the rectangular hyperbola $xy = c^2$ w.r.t the circle $x^2 + y^2 = a^2$ is the rectangular hyperbola $4c^2 xy = a^4$.

(iv) If P is any point on the parabola $x^2 + 16ay = 0$, prove that the polar of P w.r.t the rectangular hyperbola $xy = 2a^2$ will touch the parabola $y^2 = ax$.

(v) Prove that the locus of the poles of all normal chords of the rectangular hyperbola $xy = c^2$ is the curve.

$$\left(x^2 - y^2\right)^2 + 4c^2 xy = 0.$$

(vi) Obtain the standard equation of the rectangular hyperbola.

Answers

I. (ib) foci $\left(\pm\sqrt{13}, 0\right)$, eccentricity $\frac{\sqrt{13}}{3}$, latus rectum $\frac{8}{3}$.

BLOCK – 5 : CONICOIDS

Introduction

If a surface in space is represented by a second degree equation alone, the surface is called a second degree surface or a conicoid. A second degree equation can also represent a striaght line or a pair of planes. These are called degenerate conicolds. Conicoids which are not degenerate are called non-degenerate conicoids. A plane section of a conicoid can be entire plane, an empty set or a curve of degree not more than two. Thus certain plane sections of certain conicoids give conics. A surface of revolution of a curve around an axis of symmetry (if it exists) can also be a conicoid. The properties of the surfaces of revolution in general and the properties of second degree surfaces of revolution in particular are studied in unit 20. Standard equations of various conicoids and their geometric properties are studied in unit 21. The properties of tangent planes and normals are studied in unit 22. The concept of a ruled surface is introduced in unit 23. The properties of ruled surfaces, cone and cylinder, are studied in unit 24. It is learnt that using a suitable coordinate transformation, any second degree equation can be reduced to one of the standard forms introduced in the preceeding chapters. Thus a conicoid is a cone. a cylinder (these can be degenerate), an ellipsoid, a hyperboloid or a paraboloid.

This is yet another attempt to study structure of space as manifested in geometrical forms.

UNIT- 20 : SURFACES OF REVOLUTION

20.0 Contents

20.1Aims and Objectives

After going through this unit, you will be able to :

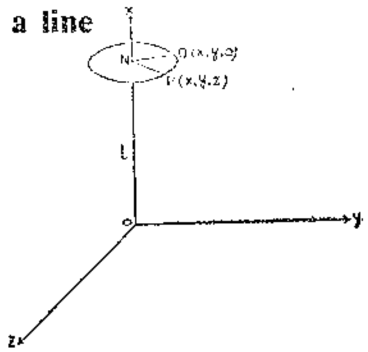
i) define a surface of revolution generated by revolving a plane curve around a line of symmetry.

ii) show that a homogeneous equation in three variables represents a cone with vertex at the origin.

20.2 Locus of revolution of a point about a line

Definition : Suppose P is a point in space and PN is the perpendicular from P on the line l. The circle with centre N and radius PN is called the locus of revolution of P about line the l.

If P is on the x – axis itself, then P and N coincide. In the case the locus of revolution of P about l is a point circle.



(Fig. 1)

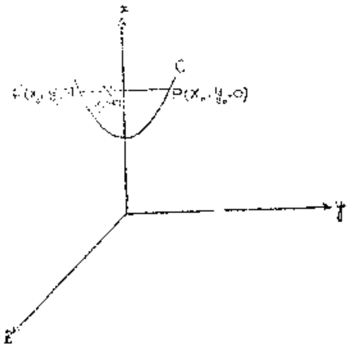
In the above figure, if $PN = a$ and $ON = k$ (constant), then the equation of the locus of revolution of $P\ (x, y, z)$ about the x – axis is $y^2 + z^2 = a^2$, $x = k$. This is a circle in the plane $x = k$.

20.3 Surface of revolution of a curve about a line

Definition : Suppose l is a given line and C is a given curve,, both in the same plane. Then the union of the loci of revolution of all points P on the curve C, about line l, is called the surface of revolution of curve C about line l. The line l is called the axis of revolution of the surface.

Theorem 1 : If $f\left(x , y^2\right) = 0$ is the equation of a curve C in the xy plane with the x – axis as the axis of symmetry, the equation of the surface of revolution of the curve C about the x – axis is $f\left(x, y^2 + z^2\right) = 0$.

Proof : The given curve lies in the xy plane ($z = 0$). It's equation is $f\left(x , y^2\right) = 0$, $z = 0$. If $P\left(x_0 , y_0 , 0\right)$ is a point on the curve C, then the point $\left(x_0, - y_0 , 0\right)$ is also on the curve C. Hence the x – axis is an axis of symmetry. If N is the foot of perpendicular from P on the x – axis, then N has coordinates $\left(x_0, 0, 0\right)$. The locus of revolution of P about the x–axis is $R\ (P) = \left\{P\left(x_0 , y , z\right) : y^2 + z^2 = NP^2\right\}$



(Fig. 2)

P is a point on the curve $\Leftrightarrow f\left(x_0, NP^2\right) = 0$

$$\Leftrightarrow R\ (P) = \left\{\left(x_0 , y , z\right) : f\left(x_0, y^2 + z^2\right) = 0\right\}$$

Since this is true for all points $P\ \left(x_0, y_0, 0\right)$ on C, the required surface of revolution of the curve C about x – axis is the union of all sets of points $R\ (P)$, for P on the curve C.

i.e., $\left\{(x , y , z) : f\left(x, y^2 + z^2\right) = 0\right\}$.

$\therefore$ The surface of revolution of the curve C about x – axis is

$$f\left(x, y^2 + z^2\right) = 0.$$

Examples :

1. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in the xy plane has the x – axis as the axis of symmetry. Hence by theorem 1 the equation of the surface of revolution of the ellipse about the x – axis is

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1. \qquad \text{... (1)}$$

Since the y – axis is also an axis of symmetry of the ellipse, the equation to the surface of revolution of the ellipse about the y – axis is

$$\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{... (2)}$$

Note: If $a = b$, the surfaces (1) and (2) are one and the same, namely a sphere.

If $a > b$, equation (1) represents the surface of revolution of the ellipse about the major axis. It is called a prolate spheroid (see Fig. 4). The equation (2) represents the surface of revolution of the ellipse about its minor axis. It is called an oblate spheroid (See Fig. 3).

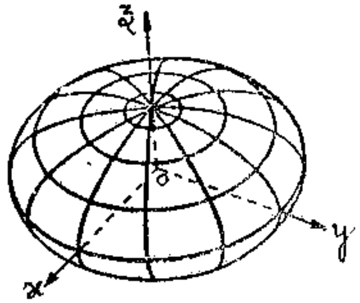


Fig. 3

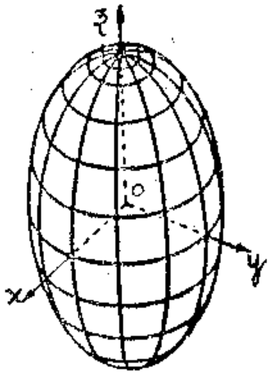


Fig. 4

2. The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a hyperbola in the xy – plane . The x and y – axis are axes of symmetry. The surface of of revolution about the x – axis (transverse axis) is

$$\frac{x^2}{a^2} - \frac{y^2 + z^2}{b^2} = 1. \qquad \text{... (1)}$$

This is a hyperboloid of revolution of two sheets. The surface of revolution about the y – axis (conjugate axis) is

$$\frac{x^2 + z^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \text{... (1)}$$

This is a hyperboloid of revolution of one sheet. These two surfaces are generally known as hyperboloids and are shown below in figure 5 and figure 6.

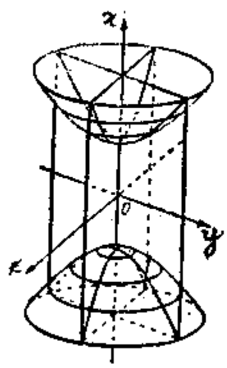


Fig. 5

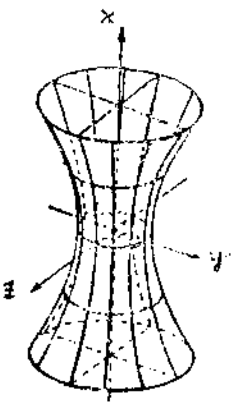


Fig. 6

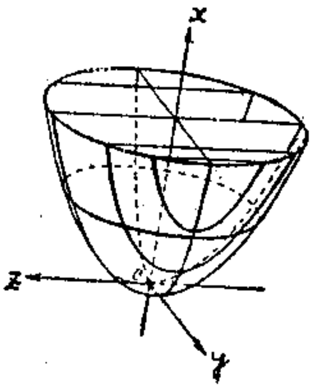


Fig. 7

3. The equation $y^2 = 4ax$ in the xy – plane represents a parabola which has x – axis as the only axis of symmetry. The equation of the surface of revolution of the parabola about the x – axis is $y^2 + z^2 = 4ax$. This is the paraboloid of revolution and is shown in figure 7.

The equation of the surface of revolution of the parabola $x^2 = 4ay$ about the y axis is $x^2 + z^2 = 4ay$

20.4 Cone

Definition : If α is a set of points in space and there exists a point V in space such that

$$P \in \alpha \Rightarrow \overleftrightarrow{VP} \subset \alpha,$$

then the set α of points is called a cone. V is called the vertex of the cone.

Examples :

1. The equation $x^2 + y^2 = z^2$ represents a cone with the origin $V(0, 0, 0)$ as vertex.

If a point P on the surface $x^2 + y^2 = z^2$ has coordinates (x, y, z), then the coordinates of any point on $\overleftrightarrow{VP}$ are (kx, ky, kz) for some real k. This point also lies on the given surface.

2. If two planes intersect, the set of all points on the planes is a cone. Any point on the line of intersection is a vertex.

3. The set of points on the plane is a cone with every point on it as a vertex.

4. The set of all points in space is a cone with any point therein as vertex.

5. A straight line is a cone. Any point on the line is a vertex.

6. The union of concurrent straight lines is a cone. The point of concurrence is the vertex.

7. If $\alpha = \{P\}$, then α is a cone with P as vertex.

8. Two parallel planes is not a cone.

Theorem 2 : (i) If $f(x, y, z)$ is a homogeneous polynomial, then $f(x, y, z) = 0$ represents a cone with the origin as vertex.

(ii) The line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on the cone $\Leftrightarrow f(l, m, n) = 0$

Proof : (i) Suppose $f(x, y, z)$ is a homogeneous polynomial of degree n. Then for any real number t,

$$f(tx, ty, tz) = t^n f(x, y, z).$$

If α is the surface represented by the equation $f(x, y, z) = 0$, then

$$\begin{aligned} P(x, y, z) \in \alpha &\Rightarrow f(x, y, z) = 0 \\ &\Rightarrow f(tx, ty, tz) = 0 \text{ for any real number } t. \\ &\Rightarrow (tx, ty, tz) \in \alpha. \\ &\Rightarrow \overleftrightarrow{OP} \text{ is contained in } \alpha. \ (O - \text{origin}) \end{aligned}$$

$\therefore$ α is a cone with the origin O as vertex

(ii)

$$\begin{aligned} f(l, m, n) = 0 &\Leftrightarrow f(tl, tm, tn) = 0 \text{ for any real number } t. \\ &\Leftrightarrow f(x, y, z) = 0 \text{ for every point } (x, y, z) \text{ on the line } \frac{x}{l} = \frac{y}{m} = \frac{z}{n} \\ &\Leftrightarrow \text{The line } \frac{x}{l} = \frac{y}{m} = \frac{z}{n} \text{ lies on the cone } \alpha. \end{aligned}$$

20.5 Cylinder

Definition : Denote the straight line through a point P and having direction cosines l, m, n by $\overleftrightarrow{P ; (l,m,n)}$.

Then given a set α of points in space, if there exists a direction l, m, n satisfying the condition

$$P \,\varepsilon\, \alpha \;\Rightarrow\; \overleftrightarrow{P ; (l,m,n)} \;\varepsilon\; \alpha,$$

the set α of points is called a cylinder. A straight line with direction ratios l, m, n is called an axis of the cylinder.

Examples :

1. $2x^2 + 3y^2 = 1$ represents a cylinder. Any straight line parallel to z – axis is an axis of this cylinder.

2. Every plane is a cylinder. Any straight line parallel to the plane is an axis.

3. Two parallel planes is a cylinder. Any line parallel to these planes is an axis.

4. Two intersecting planes is a cylinder. The line of intersection is an axis.

5. The set of all points in space is a cylinder. Any straight line is an axis.

Theorem 3 : The equation $f(x, y) = 0$ is a cylinder with z – axis as axis.

Proof : Let α be the set of points satisfying the equation $f(x, y) = 0$.

Then

$$P\left(x_0, y_0, z_0\right) \varepsilon\, \alpha \;\Rightarrow\; f\left(x_0, y_0\right) = 0$$

$$\Rightarrow \left(x_0, y_0, z\right) \varepsilon\, \alpha \text{ for every real number } z.$$

$$\Rightarrow \overleftrightarrow{P ; (0,0,1)} \subset \alpha$$

$\left(\because\right.$ the line joining $\left(x_0, y_0, z_0\right)\Big)$

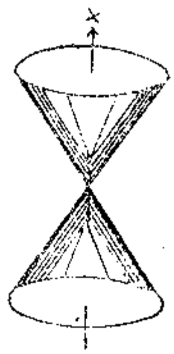
and $\left(x_0, y_0, z\right)$ has direction ratios $(0, 0, 1)$.

20.6 Circular cone

The surface of revolution of the pair of straight lines $y^2 = m^2 x^2$ about the x – axis (by Theorem 1), $y^2 + z^2 = m^2 x^2$.

This is a cone with the origin as vertex. It is called the cone of revolution of the straight lines $y^2 = m^2 x^2$ or the circular cone.

If $m = 0$, the cone reduces to the x – axis.



(Fig. 8)

20.7 Circular cylinder

The surface of revolution of the pair of parallel straight lines $y^2 = c^2$ ($c \neq 0$) in the xy plane, about the x – axis, is $y^2 + z^2 = c^2$ (Theorem 1). This surface is called the cylinder of revolution of the parallel straight lines $y^2 = c^2$. It is also called a circular cylinder.

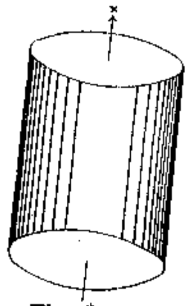


Fig. 9

Examples :

1. Find the surface of revolution of the parabola $y^2 = 4\,ax$ around the y – axis.

The y – axis is not an axis of symmetry of the given parabola. Hence Theorem 1 cannot be applied to obtain the surface of revolution about the y – axis. But the surface of revolution of the parabola $y^2 = 4ax$ about y – axis is same as the surface of revolution of the parabola $y^2 = -4ax$. Therefore, the surface of revolution of the pair of parabolas $\left(y^2 + 4ax\right)\left(y^2 - 4ax\right) = y^4 - 16a^2x^2 = 0$ is the same as the surface of revolution of $y^2 = 4ax$. The y – axis is an axis of symmetry for $y^4 = 16a^2x^2$ and therefore by Theorem 1, the surface of revolution about the y – axis is $y^4 = 16\,a^2\left(x^2 + z^2\right)$.

2. Find the surface of revolution of the rectangular hyperbola $xy = 1$ about the x – axis.

As in example 1, the required surface of revolution is the same as the surface of revolution of the pair of hyperbolae $x^2\,y^2 = 1$, which is symmetric about the x – axis. Hence the required surface of revolution is $x^2\left(y^2 + z^2\right) = 1$.

3. Find the surface of revolution of the circle $x^2 + y^2 = 1$ about the line $x = 2$.

If we make the transformations $x = X + 2$, $y = Y$, the equation to the circle in the XY – coordinate system is $(X + 2)^2 + Y^2 = 1$.

$$\text{i.e., } X^2 + Y^2 + 3 = -4\,X \qquad \text{... (1)}$$

The straight line $x = 2$ transforms to $X = 0$ (new Y axis). The surface of revolution of (1) about $X = 0$ is the same as that of the pair of circles

$$\left(X^2 + Y^2 + 3\right)^2 = 16X^2 \qquad \text{... (2)}$$

Since (2) has the Y – axis as the axis of symmetry, the required surface of revolution is

$$\left(X^2 + Y^2 + Z^2 + 3\right)^2 = 16\left(X^2 + Z^2\right)$$

Transforming back to the original axis (using $x = X + 2$, $y = Y$, $z = Z$) the equation to the surface of revolution is

$$\left[(x-2)^2 + y^2 + z^2 + 3\right]^2 = 16\left[(x-2)^2 + z^2\right]$$

This surface is called the Anchor ring of Torus.

SAQ. (i) What are degenerate conicoids?

SAQ (ii) Define a circular cone and a circular cylinder.

20.8 Summary

Let C be a point on a given line l, the locus of a point with centre C and radius PC so that PC is perpendicular to l is called the locus of revolution of P about the line l. Let l be a given line. Let C be a given curve. Then the union of the loci of revolution of all points P as the curve C is called surface of revolution of the curve C about the line l. Examples (i) The surface of revolution of the curve C about x – axis is $f\left(x,\ y^2 + z^2\right) = 0$.

If C is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then the surface is $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$ is the surface of revolution about Z–axis. Equation of the circular cone is $y^2 + z^2 = m^2 x^2$ obtained when lines $y^2 = m^2 x^2$ are rotated about X–axis.

The surface of revolution of the pair of parallel straight lines $y^2 = c^2$ ($c \neq 0$) in the xy plane about x - axis is $y^2 + z^2 = c^2$. It is called a circular cylinder.

20.9 Sample Examination questions

I. Answer the following in detail.

(i) What is a surface of revolution of a curve about a line ? If $f\left(x,\ y^2\right) = 0$ is the equation of a curve C in the xy plane with the x–axis as the axis of symmetry, show that the equation of the surface of revolution of the curve C about the x–axis is $f\left(x,\ y^2 + z^2\right) = 0$.

(ii) Define a cone. If $f(x, y, z)$ is a homogeneous polynomial then show that (i) $f(x, y, z) = 0$ represents a cone with the origin as vertex. (ii) The line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on the cone $\Leftrightarrow f(l, m, n) = 0$.

II. Briefly answer the following

Find the surface of revolution of the following curves in the xy – plane about the straight line shown against them.

(i)	$x^2 + y^2 = 1,$	$x + 1 = 0$
(ii)	$x^2 - y^2 = 1,$	$y + 1 = 0$
(iii)	$y^2 = x,$	$x = y$
(iv)	$4x^2 + 9y^2 = 36$	$x = 3$

Answers

(i) $\left(x^2 + y^2 + z^2 + 2x + 1\right)^2 = 4\left(x^2 + z^2 + 2x + 1\right)$

(ii) $\left(x^2 - y^2 - z^2 - 2y - 3\right)^2 = 4\ \left(y^2 + z^2 + 2y + 1\right)$

(iii) $\left(x^2 + y^2 + z^2 - x - y\right)^2 = \left(x^2 + y^2 + 2z^2 - 2xy\right)\ (1 + x + y)^2$

(iv) $\left\{4\ \left[(x - 3)^2 + z^2\right] + 9y^2\right\}^2 = 576\ \left[(x - 3)^2 + z^2\right].$

20.10 Answers to SAQ's

SAQ-1 : Pair of straight lines, pair of planes are called degenerate conicoids.

If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents a pair of lines and the equation

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2ux + 2vy + 2wz + d = 0$

represents a pair of planes. Therefore these are called the degenerate conicoids.

SAQ-2 : (i) If a moving line passes through a fixed point and rotates about a fixed line (through the fixed line) then its locus is a cone. It is called a right circular cone.

(ii) If a line rotates parallel to a fixed line about the fixed line at a constant distance then the surface obtained is called a cylinder. It is called a circular cylinder.

UNIT–21 : STANDARD EQUATION OF CONICOIDS

21.0 Contents

21.1 Aims and objectives

After going through this unit, you will be able to :

(i) derive the standard equations of various conicoids,

(ii) study the nature of conicoids and their plane sections,

(iii) conclude that the ellipsoid is a conicoid which is a bounded point set in space and other conicoids are not bounded.

21.2 Standard equation of conicoids

Ellipsoid : The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ represents an ellipsoid and is shown in figure 1. Without loss of generality we assume that a, b, c are real and positive constants. If two of a, b, c are equal, the surface is called a spheroid. If $a = b = c$, the surface is a sphere.

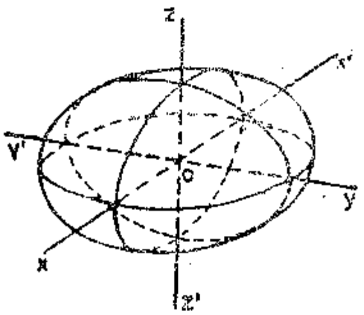


Fig. 1

Properties :

(i) If the coordinates (x, y, z) of any point satisfy the equation then $(-x, -y, -z)$ also satisfy the equation. These two are the points in which a line through the origin meets the ellipsoid. The origin is the middle point of the chord joining (x, y, z) and $(-x, -y, -z)$. Hence the origin bisects every chord which passes through it and is called the centre of the ellipsoid. The chords through the centre are called diameters.

(ii) If the coordinates (x, y, z) of any point satisfy the equation then $(x, y, -z)$ also satisfy the equation. The chord joining these two points is bisected by the xy plane. The xy plane bisects every chord perpendicular to it. Hence the surface is symmetric about this plane. If (x, y, z) is on the ellipsoid, then $(x, -y, z)$ is also on the ellipsoid. Similar argument shows that the surface is symmetric about the zx plane. Similarly it can be observed that the ellipsoid is symmetric about the xy plane. The three coordinate planes are such that each bisects chords perpendicular to it. Such planes are called the principal planes of the ellipsoid. The three lines of intersection of the principal planes, taken two at a time, are called the principal axes of the ellipsoid. For the given ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the coordinate planes are the principal planes and the coordinate axes are principal axes (more about the principal axes will be studied in unit 24. General definition for centre of a conicoid will be given in unit 22).

(iii) If (x, y, z) is a point on the ellipsoid, then x cannot take a value greater than a. This is because if $x > a$, then

$$1<\frac{x^2}{a^2} = 1-\frac{y^2}{b^2}-\frac{z^2}{c^2}$$

$$\therefore \quad 0< -\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

This is impossible for any real values of y and z. Similarly y can not take value greater than b and z can not take value greater than c. Hence the ellipsoid lies between the planes

$x=+a, x=-a, y=+b, y=-b, z=+c, z=-c$

and so the ellipsoid is a set of points bounded in space.

The ellipsoid meets the x–axis in the points $A(a, 0, 0)$ and $A'(-a, 0, 0)$. It meets the y–axis in $B(0, b, 0)$ and $B'(0, -b, 0)$. It meets the z–axis in $C(0, 0, c)$ and $C'(0, 0, -c)$. The lengths of the chords of the ellipsoid along the principal axes are $AA'=2a$, $BB'=2b$, $CC'=2c$. These lengths are called lengths of the principal axes or principal diameters. If O is the centre, then $OA=a$, $OB=b$ and $OC=c$ are called principal semi axes. The points A, A', B, B', C, C' are called the vertices of the ellipsoid.

(iv) The set of points common to ellipsoid and a plane Π (i.e., $\alpha \cap \Pi$) is called the section of α by the plane Π, or a plane section of α by Π.

The section of the ellipsoid by the plane $z=k$ parallel to the xy plane, is an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1-\frac{k^2}{c^2},\ z=k \qquad \dots (1)$$

if $|k|<c$. This elliptic section has centre on the z–axis and diminishes in area as k increases from 0 to c or decreases from 0 to $-c$. The set of all these ellipses knit the total surface of the ellipsoid. This also means that the ellipse (1) generates the ellipsoid as k varies from $-c$ to $+c$. Similarly it can be observed that the sections of the ellipsoid by planes parallel to other coordinate planes are also ellipses and the ellipsoid may be supposed to have been generated by them.

21.3 The hyperboloid of one sheet

The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1 \qquad \dots (1)$$

is called the hyperboloid of one sheet.

Without loss of generality, assume $z>0, b>0, c>0$.

If $a=b$, the equation (1) represents a surface of revolution of the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ about the z–axis.

The chords of the surface (1) through the origin are bisected at the origin and are called diameters. The origin is the centre. The coordinate planes are the principal planes and the coordinate axes are the principal axes. The xy plane cuts the surface in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$. This is called the principal elliptic section or principal ellipse. The section of the surface by the plane $z = k$ (constant) parallel to xy plane is the ellipse

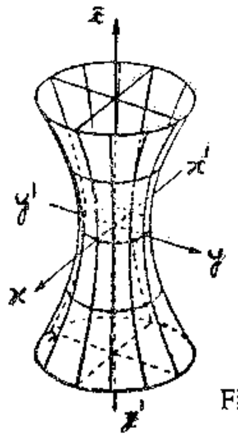


Fig. 2

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1+\frac{k^2}{c^2} = 1; z = k$$

with this centre on the z–axis. This can be written as

$$\frac{x^2}{a^2\left(1+\frac{k^2}{c^2}\right)} + \frac{y^2}{b^2\left(1+\frac{k^2}{c^2}\right)} = 1; z = k \qquad \dots (2)$$

and has $a\left(1+\frac{k^2}{c^2}\right)^{1/2}$ and $b\left(1+\frac{k^2}{c^2}\right)^{1/2}$ as principal semi axes which increase with $|k|$. There is no limit to the increase or decrease of k $(-\infty < k < \infty)$ in contrast to the case of ellipsoid. Thus, the surface is generated by the variable ellipse (2) as k varies from $-\infty$ to $+\infty$.

The x–axis meets the surface in $(a, 0, 0)$ and $(-a, 0, 0)$ and intercepts a length $2a$ on it. The surface intercepts a length $2b$ on the y–axis. The z–axis does not meet the surface in real points. The sections by the planes $x = k$ or $y = k$ are hyperbolas.

21.4 The Hyperboloid of two sheets

The surface represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ is called a hyperboloid of two sheets.}$$

If $b = c$, the surface is a surface of revolution about the x–axis. The origin is the centre, the coordinate planes are principal planes and the coordinate axes are principal axes.

The cross section of the surface by the xy plane and the zx plane are the hyperbolas,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1+\frac{k^2}{c^2}\ ; z = k$$

and $\frac{x^2}{a^2}-\frac{z^2}{c^2}= 1 + \frac{k^2}{b^2}; y = k$ respectively.

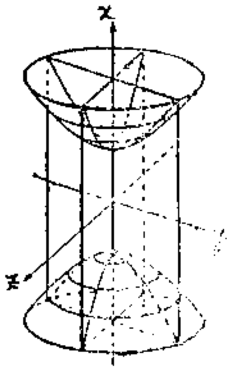


Fig. 3

The cross section by the yz plane is an empty set of points.

Let the plane $x = k$ cut the surface. Then

$$\frac{y^2}{b^2}+\frac{z^2}{c^2} = \frac{k^2}{a^2}-1, x = k.$$

This is an ellipse if $k < -a$ or $k > a$, that is, if $|k| > a$. There is no portion of the surface between the planes $x = -a$ and $x = +a$. The size of the ellipse increases if k^2 increases. The surface has two detached portions. The points $(a, 0, 0)$ and $(-a, 0, 0)$ are called the vertices of the hyperboloid of two sheets.

Theorem 1 : The hyperboloid of one sheet and the hyperboloid of two sheets are not bounded point sets in space.

Proof : (i) Consider the hyperboloid of one sheet. For any real λ, the point $P(a, b\lambda, c\lambda)$ lies on it. If O is the origin, the length $\overline{OP} = \sqrt{a^2 + \lambda^2\left(b^2 + c^2\right)}$. Given any real number k, choose a real number λ such that $\lambda > \frac{k}{\sqrt{b^2 + c^2}}$. Then $OP > k$ for the point P on the surface given by this λ.

(ii) Consider the hyperboloid of two sheets. For any real λ, the point $P\left(\sqrt{2\lambda^2 + 1}\, a, \lambda b, \lambda c\right)$ lies on the hyperboloid of two sheets. Then $OP^2 = \left(2\lambda^2 + 1\right)a^2 + \lambda^2 b^2 + \lambda^2 c^2 > \lambda^2 a^2$. Given any real number k, choose $a\lambda$ such that $\lambda > k/a$. Then $\overline{OP} > k$ for the point P on the surface given by this λ.

Theorem 2 : If a, b, c are not zero, the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ is

(i) an ellipse, or a point circle or an empty set if $bcl^2 + cam^2 + abn^2 > 0$

(ii) a parabola or a pair of parallel lines if $bcl^2 + cam^2 + abn^2 = 0$

(iii) a hyperbola or a pair of intersecting lines if $bcl^2 + cam^2 + abn^2 < 0$.

Proof : Suppose the given plane is not parallel to a coordinate plane. Let l, m, n be the direction cosines of the normal to the given plane where $l^2 + m^2 \neq 0$. Choose a new coordinate system $Ox'y'z'$ in which the given plane is a coordinate plane

$$x = \frac{mx' + lny'}{\sqrt{l^2 + m^2}} + lz'$$

$$y = + \frac{lx' - mny'}{\sqrt{l^2 + m^2}} + mz'$$

$$z = \sqrt{l^2 + m^2}\, y' + nz'.$$

The point $P(x, y, z)$ in the frame $Oxyz$ is now represented by $P(x', y', z')$ in the new frame $Ox'y'z'$. The equation of the given plane in the new frame is $z' = p$ (parallel to $x'y'$ plane). The equation of the conicoid transforms to,

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2f'y'z' + 2g'z'x' + z'^2\left(al^2 + bm^2 + cn^2\right) - 1 = 0 \qquad \ldots (1)$$

where,

$$a' = \frac{am^2 + bl^2}{l^2 + m^2},$$

$$b' = \frac{(al^2 + bm^2)\, n^2 + c\,(l^2 + m^2)^2}{(l^2 + m^2)}$$

$$h' = \frac{(a - b)\, lmn}{l^2 + m^2}$$

$$f' = \frac{-n\,(m^2 b + l^2 a)}{\sqrt{l^2 + m^2}} + cn\sqrt{l^2 + m^2}$$

$$g' = lm\,(b - a) / \sqrt{l^2 + m^2}$$

It can be verified that $a'b' - h'^2 = bcl^2 + cam^2 + abn^2$

The section by the conicoid (1) by the plane $z' = p$ is of the form

$$a'x'^2 + 2h'x'y' + b'y'^2 + \ldots y' + \ldots' = 0. \qquad \ldots (2)$$

This equation has the properties mentioned in (i), (ii) or (iii) of the statement of the theorem according as $a'b' - h'^2$ is greater than, equal to or less than zero (see appendix at the end of the Unit), i.e., according as $bcl^2 + cam^2 + abn^2$ is greater than, equal to or less than zero.

Note: If the equation of the conicoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then a, b, c in the statement of the theorem should be replaced by $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$. Since $\frac{l^2}{b^2c^2} + \frac{m^2}{c^2a^2} + \frac{n^2}{a^2b^2} > 0$ always, the intersection of the ellipsoid by a given plane is an ellipse, a point circle or an empty set.

21.5 Cylinders

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents a cylinder whose axis is the z–axis and whose cross section by any plane $z = k$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = k$.

The cylinder is called an elliptic cylinder. In the case $a = b$ it is a circular cylinder.

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a cylinder whose axis is the z–axis and whose section by any plane parallel to the xy plane ($z = k$ for some real k) is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ z = k.$$

Fig. 4

This cylinder is called hyperbolic cylinder.

The equation $\frac{x^2}{a^2} = 2y$ is a cylinder with the z– axis as the axis. In any plane $z = k$, the section is the parabola $x^2 = 2a^2y$. The cylinder is called parabolic cylinder.

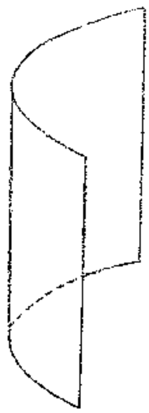


Fig. 5

21.6 Elliptic paraboloid

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 2z$ represents an elliptic paraboloid.

Without loss of generality, assume $a > 0, b > 0$. If $a = b$ the surface represents a surface of revolution (paraboloid of revolution) with z–axis as axis. The section of the surface by the yz and zx plane are parabolas. The section by the xy plane is a point. The section by the plane $z = k$, parallel to the xy plane, is an ellipse if $k > 0$ and an empty set if $k < 0$. The surface is entirely on the positive side of the plane $z = 0$.

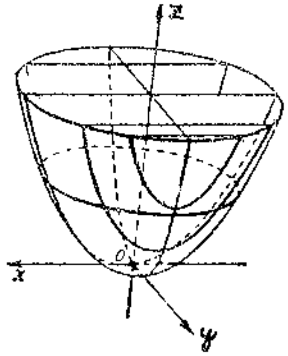


Fig. 6

The surface $\frac{x^2}{a^2}+\frac{y^2}{b^2} = -2z$ is also an elliptic paraboloid and lies entirely on the negative side of the plane $z = 0$. It is a mirror image in the plane $z = 0$ of the elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 2z$.

21.7 Hyperbolic paraboloid

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ represents a hyperbolic paraboloid.

The cross section of the surface by the yz, zx planes are parabolas. The cross section by the xy plane is a pair of straight lines

$$\frac{x}{a} = \frac{y}{b} \text{ and } \frac{x}{a} = -\frac{y}{b} \quad (z = 0).$$

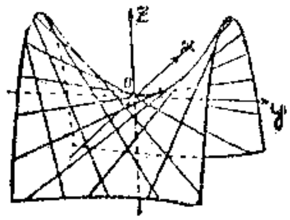


Fig. 7

The cross section by a plane $z = k$, parallel to xy plane, is a hyperbola if $k \neq 0$.

SAQ 1: Define parabolic, hyperbolic cylinders.

SAQ 2: Write down the standard form of ellipsoid, hyperboloids, paraboloids.

SAQ 3: Classify the conicoids given by the equation $ax^2 + by^2 + cz^2 = 1$ on the basis of values of a, b, c.

Diameter

Definition : The origin is the vertex, the z–axis is the axis and lines parallel to the axis are called diameters of the paraboloid $ax^2 + by^2 = 2z$ $(ab \neq 0)$.

A diameter meets the paraboloid in only one point ($\because$ a line parallel to z–axis is $x = \alpha$ (constant), $y = \beta$ (constant) and hence meets the paraboloid in (α, β, γ) where $\gamma = \frac{a\,\alpha^2 + b\beta^2}{2}$)

Theorem 3 : The section of a paraboloid by a plane parallel to the axis is a parabola. The sections by two parallel planes, parallel to the axis, are parabolas of the same latus rectum.

Proof : Let the paraboloid be $ax^2 + by^2 = 2z$ and the given plane be $x \cos \theta + y \sin \theta = k$. Rotate the axes to a new set of axes $Ox'y'z'$ with the transformations

$$x = x' \cos \theta + y' \sin \theta$$
$$y = x' \sin \theta - y' \cos \theta$$
$$z = z'.$$

The equation of the given plane transforms to $x' = k$.

The given paraboloid transforms to

$$a'x'^2 + b\ y^2 + 2h\,x\,y = 2z'$$

The section of this paraboloid by $x' = k$ is

$$b'y'^2 + 2h'ky' + ak^2 = 2z', \ x' = k.$$

From geometry of conics, we infer that this represents a parabola with latus rectum $2/b'$ which does not depend on k. (see appendix).

21.8 General equation of second degree

If the locus of a point in space is represented by an equation of second degree in x, y, z and cannot be reduced to an equation of first degree, the locus is called a second degree surface or a quadric or a conicoid.

It can be noticed that the equations (i) $y^2 = 0$ and $y = 0$ and (ii) $x^2 + 2xy + y^2 = 0, x + y = 0$; represent the same loci. The surfaces represented by them are first degree surfaces, namely; (i) the xz plane ($y = 0$) and (ii) the plane $x + y = 0$, respectively.

Theorem 4 : The section of a coincoid by a plane is the entire plane, an empty set or a curve of not more than second degree.

Proof : Transform the equation to a new set of axes in which the given plane is the xy–plane. Take the equation of the conicoid in the new coordinate system as

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

The intersection of this surface by the plane $z = 0$ is

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0, \ z = 0.$$

If all of a, b, h, u, v, d are zero, the intersection is the entire plane $z = 0$. Otherwise the intersection is an empty set, a straight line or a conic (see appendix).

Note: (i) If the given surface is an ellipsoid (a bounded point set in space), the section is either empty or a bounded conic (a point or an ellipse).

(ii) If the intersection contains a straight line and a point not on the straight line, then the section is the given plane or a pair of straight lines.

(iii) If the intersection has three different straight lines, then the section is the given plane (see appendix).

The surfaces considered so far are represented by simple equations of second degree. The simplicity is achieved because of a suitable choice of coordinate axes. These equations when referred to a general coordinate system may contain more terms.

The most general equation of second degree is

$$S(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

This equation represents either a cone or a surface classified in the above articles.

21.9 Points of intersection of a conicoid and a straight line

Consider the general equation of second degree

$$S(x, y, z) = 0 \qquad \ldots (1)$$

Consider the straight line

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} = t \text{ (say)}.$$

Any point on this line is $\left(x_0 + lt, y_0 + mt, z_0 + nt\right)$

If this point is common to the conicoid,

$$S\left(x_0 + lt, y_0 + mt, z_0 + nt\right) = 0.$$

On simplification, this reduces to the form

$$At^2 + Bt + C = 0 \qquad \ldots (2)$$

Depending on the values of A, B, C the nature of points of intersection can be determined.

21.10 Notation

$$\xi = ax + hy + gz + u$$
$$\eta = hx + by + fz + v$$
$$\zeta = gx + fy + cz + w$$
$$\delta = ux + vy + wz + d$$
$$\xi_i = ax_i + hy_i + gz_i + u$$
$$\eta_i = hx_i + by_i + fz_i + v$$
$$\zeta_i = gx_i + fy_i + cz_i + w$$
$$\delta_i = ux_i + vy_i + wz_i + d$$
$$S_i = \xi x_i + \eta y_i + \zeta z_i + \delta$$
$$= axx_i + byy_i + czz_i + f(yz_i + zy_i) + g(zx_i + xz_i) + h(xy_i + yx_i)$$
$$+ u(x + x_i) + v(y + y_i) + w(z + z_i) + d,$$
$$S_{ij} = \xi_j x_i + \eta_j y_i + \zeta_j z_i + \delta_j$$
$$H = H(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

In view of these symbols, it can be verified that

$$S_i = \xi_i x + \eta_i y + \zeta_i z + \delta_i$$
$$S_{ij} = S_{ji}$$
$$S_{ii} = S(x_i, y_i, z_i)$$
$$S = x\xi + y\eta + z\zeta + \delta$$
$$H = x\xi + y\eta + z\zeta - \delta + d$$

The equation (2) of 21.9 now can be written as

$$H(l, m, n)\, t^2 + 2(l\xi_0 + m\eta_0 + n\zeta_0)\, t + S_{00} = 0 \qquad \ldots (3)$$

Equation (3) is quadratic in t if $H(l, m, n) \neq 0$ and a first degree equation if $H(l, m, n) = 0$, $l\xi_0 + m\eta_0 + n\zeta_0 \neq 0$. (3) is an impossibility if $H(l, m, n) = 0$, $l\xi_0 + m\eta_0 + n\zeta_0 = 0$ and $S_{00} \neq 0$. This is an identity if all the three coefficients are zero.

Case (i) : If $H(l, m, n) \neq 0$ and $(l\xi_0 + m\eta_0 + n\zeta_0)^2 > S_{00} \cdot H(l, m, n)$, then the equation (3) has two real roots. There are two distinct values for t. These two values of t will determine the two points of intersection of the conicoid and the straight line. The line segment intercepted by these two points is called the chord of the conicoid. The straight line is called a chord line.

(ii) If $H(l, m, n) \neq 0$ and $(l\xi_0 + m\eta_0 + n\zeta_0)^2 < S_{00} \cdot H(l, m, n)$, the equation (3) has no real roots. There are no points common to the line and the conicoid. Such line is called a virtual chord of the conicoid.

Example 1 : The line $x = y$, $z = 2$ is a virtual chord of the sphere $x^2 + y^2 + z^2 = 1$. [The equation (3) is $2t^2 + 4 = 0$]

Example 2 : The line $x = y = 0$ is a virtual chord of the hyperboloid $x^2 - y^2 - z^2 = 1$. [The equation (3) is $t^2 + 1 = 0$]

(iii) If $H(l, m, n) \neq 0$, $(l\xi_0 + m\eta_0 + n\zeta_0)^2 = S_{00} \cdot H(l, m, n)$, then equation (3) determine two equal values for t. There is only one point common to the line and the conicoid. Such a line is called a tangent to the conicoid.

Example : The straight line $z = 0$, $2x = 3y$ is a tangent to the paraboloid $x^2 + y^2 = 2z$ at $(0, 0, 0)$ [$x = 3t$, $y = 2t$, $z = 0$ is the parametric representation of the line. Equation (3) reduces to $13t^2 = 0$.]

(iv) If $h(l, m, n) = 0$, $l\xi_0 + m\eta_0 + n\zeta_0 = 0$, $S_{00} = 0$ then the equation (3) is an identity. Any point on the line is common to the line and the conicoid. Such a line is called a generating line or a generator of the conicoid.

Example 1 : The line $x = y = 1$ is a generator of the cylinder $x^2 + y^2 = 2$.

Example 2 : The line $x = 1$, $y = z$ is a generator of the hyperboloid $x^2 + y^2 - z^2 = 1$.

(v) If $H(l, m, n) = 0$, $l\xi_0 + m\eta_0 + n\zeta_0 = 0$, $S_{00} \neq 0$, then the equation (3) is an impossibility. No point on the straight line is on the conicoid. Such a line is called an asymptote.

Example 1 : The line $x = y = z$ is an assymptote of the hyperboloid $x^2 + y^2 - 2z^2 = 1$ (t, t, t are the parametric coordinates of the line).

Example 2 : The line $x = y = 0$ is an assymptote of the cylinder $x^2 + y^2 = 1$ ($0, 0, t$ are the parametric coordinates of the line).

(vi) If $H(l, m, n) = 0$, $l\xi_0 + m\eta_0 + n\zeta_0 \neq 0$, then the equation (3) is a first degree equation. Hence there is only one root for equation (3). The given straight line cuts the surface in only one point.

Example 1 : The line $x = y = z + 1$ cuts the hyperboloid $x^2 + y^2 - 2z^2 = 1$ in only one point (equation (3) is $4t = 3$).

Example 2 : The z–axis, or any line parallel to z–axis cuts the paraboloid $ax^2 + by^2 = 2z$ in only one point. (If x_0, y_0, t are the parametric coordinates of the line, then $(x_0, y_0, 0)$ is a point on it and the direction cosines of the line are 0, 0, 1. The equation (3) is $ax_0^2 + by_0^2 = 2t$).

Note 1 : If a conicoid and a straight line have three points in common, then the straight line is a generator (since there are three common points, the equation (3) has three distinct roots. Since (3) has at most two distinct roots, it must be an identity).

Note 2 : It can be proved that the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid have no generators.

Examples :

1. Show that the equation $8x^2 + 4y^2 + 4z^2 + 16x - 16y + 4z + 13 = 0$ represents an ellipsoid. Find the centre and the principal semiaxes.

The given equation can be written as

$$8(x+1)^2 + 4(y-2)^2 + (2z+1)^2 = 12$$

$$\text{or}\quad 2(x+1)^2 + (y-2)^2 + \left(z + \frac{1}{2}\right)^2 = 3.$$

Shifting the origin to $\left(-1, 2, -\frac{1}{2}\right)$, using the transformations

$$X = x + 1, Y = y - 2, Z = z + \frac{1}{2},$$

the equation reduces to $2X^2 + Y^2 + Z^2 = 3$, or

$$\frac{X^2}{(3/2)} + \frac{Y^2}{3} + \frac{Z^2}{3} = 1.$$

This is an ellipsoid whose centre in $OXYZ$ frame is $(0, 0, 0)$ and in the $Oxyz$ frame $\left(-1, 2, -\frac{1}{2}\right)$.

The principal semiaxes are $a = \sqrt{\frac{3}{2}}, b = \sqrt{3}, c = \sqrt{3}$. Also, the ellipsoid is an ellipsoid of revolution of the ellipse $\frac{X^2}{(3/2)} + \frac{Y^2}{3} = 1$ about the Y axis (i.e., $x = -1$).

2. Find out the conicoid represented by the equation

$$72x^2 + 144y^2 - 72z^2 - 72x + 72y - 48z - 53 = 0$$

The given equation can be written as

$$18\left(4x^2 - 4x\right) + 9\left(16y^2 + 8y\right) - 8\left(9z^2 + 6z\right) - 53 = 0$$

$$18(2x-1)^2 + 9(4y+1)^2 - 8(3z+1)^2 - 53 - 18 - 9 + 8 = 0$$

$$18 \times 4\left(x - \frac{1}{2}\right)^2 + 9 \times 16\left(y + \frac{1}{4}\right)^2 - 8 \times 9\left(z + \frac{1}{3}\right)^2 - 72 = 0$$

Dividing by 72,

$$\left(x - \frac{1}{2}\right)^2 + 2\left(y + \frac{1}{4}\right)^2 - \left(z + \frac{1}{3}\right)^2 = 1.$$

Using the transformations

$$X = x - \frac{1}{2}, Y = y + \frac{1}{4}, Z = z + \frac{1}{3}$$

the equation reduces to $X^2 + 2Y^2 - Z^2 = 1$. This is a hyperboloid of one sheet whose centre is $\left(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{3}\right)$

3. Find the conicoid represented by the equation

$$11025x^2 + 7350y^2 - 6300x + 8820y - 14700z - 1354 = 0.$$

The equation can be written as

$$225\left(49x^2-28x\right)+294\left(25y^2+30y\right)-4\times 49\times 75z-1354 = 0$$

$$225\times(7x-2)^2+294\,(5y+3)^2-4\times 49\times 75z-4900 = 0$$

$$225\times 49\left(x-\frac{2}{7}\right)^2+294\times 25\left(y+\frac{3}{5}\right)^2-4\times 49\times 75z-4900 = 0$$

Dividing by 75×49, $\quad 3\left(x-\frac{2}{7}\right)^2+2\left(y+\frac{3}{5}\right)^2-4\left(z+\frac{1}{3}\right) = 0$

Using the transformations

$$X=x-\frac{2}{7},\ Y=y+\frac{3}{5},\ Z=z+\frac{1}{3},$$

the equation reduces to $\quad 3X^2+2Y^2 = 4Z.$ This is a paraboloid.

Appendix

Theorem: The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ ... (1)

(a, h, b not all zero) represents

(i) an ellipse or a point circle or an empty set if $ab-h^2>0$

(ii) a parabola or a pair of parallel lines if $ab-h^2=0$.

(iii) a hyperbola or a pair of intersecting lines if $ab-h^2<0$.

Proof : If the axes are turned through an angle θ, without shifting the origin, the transformations are

$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\cos\theta$$

Then (1) gives

$$AX^2+2HXY+BY^2+2GX+2FY+C = 0$$

where

$$A = a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta$$

$$H = -(a-b)\sin\theta\cos\theta+h\left(\cos^2\theta-\sin^2\theta\right)$$

$$B = a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta$$

$$G = g\cos\theta+f\sin\theta$$

$$F = f\cos\theta-g\sin\theta$$

$$C = c$$

It can be verified that $\quad a+b = A+B$

$$ab-h^2 = AB-H^2$$

If θ is chosen as $\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$, then $H=0$.

The equation (1) then transforms to

$AX^2+BY^2+2GX+2FY+C=0$ ...(2)

Case (i) : If $ab - h^2 = 0$, then $AB = 0$

If $A = 0, B \neq 0$, then the equation (2) can be written as

$$B\left(Y + \frac{F}{B}\right)^2 = -2GX - C + \frac{F^2}{B}$$

If $G = 0$, $F^2 > BC$, this represents a pair of parallel lines.

If $G = 0$, $F^2 < BC$, this represents an empty set.

If $G \neq 0$, this equation can be further written as

$$\left(Y + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(X - \frac{F^2}{2BG} + \frac{C}{2G}\right)$$

Shifting the origin to $\left(\frac{F^2}{2BG} - \frac{C}{2G}, -\frac{F}{B}\right)$ the equation reduces to

$$Y'^2 = -\frac{2G}{B}X'$$

which is a parabola.

Case (ii) : If $ab - h^2 > 0$, then

$$ab - h^2 > 0 \iff AB > 0$$

$$\iff A, B \text{ are both positive or both negative.}$$

Equation (2) can be written as

$$A\left(X + \frac{G}{A}\right)^2 + B\left(Y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C = K \text{ (say)}$$

Shifting the origin to $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, the equation reduces to

$$AX'^2 + BY'^2 = K; \qquad \ldots(3)$$

$$\left(X' = X + \frac{G}{A}, Y' = Y + \frac{F}{B}\right)$$

If $A > 0, B > 0, K > 0$ or $A < 0, B < 0, K < 0$ then this is an ellipse.

If $A > 0, B > 0, K < 0$ or $A < 0, B < 0, K > 0$ then this is an empty set.

If $K = 0$, then it is a point circle.

Case (iii) : If $ab - h^2 < 0$, then

$$ab - h^2 < 0 \iff AB < 0$$

$$\iff A > 0, B < 0 \text{ or } A < 0, B > 0.$$

If $K = 0$, then (3) represents a pair of intersecting lines.

If $K \neq 0$, then (3) represents a hyperbola.

Note 1 : If $a = b = h = 0$, (then $A = B = 0$), then the equation (1) represents a line.

Note 2 : The section of the conicoid $S(x, y, z) = 0$ by the plane $z = k$ is of the form (1). If this section has three straight lines, then the section in any case cannot be represented by (1). Hence the section is the entire plane $z = k$.

21.11 Summary

The equation $ax^2 + by^2 + cz^2 = 1$ represents

(i) an ellipsoid if all of a, b, c are positive

(ii) a hyperboloid of one sheet if one of a, b, c is negative and the other two are positive

(iii) a hyperboloid of two sheets if one of a, b, c is positive and the other two are negative.

(iv) an elliptic cylinder if one of a, b, c is zero and the other two are positive.

(v) a hyperbolic cylinder if one of a, b, c is positive, one is zero and the other negative.

(vi) a pair of parallel planes if two of a, b, c are zero and the other positive.

(vii) an empty set if a, b, c are all zero.

The equation $ax^2 + by^2 = 2z$ represents

(i) an elliptic paraboloid if $ab > 0$.

(ii) a hyperbolic paraboloid if $ab < 0$.

(iii) a parabolic cylinder if only either of a, b is zero.

(iv) xy plane if $a = b = 0$.

The most general equation of second degree is

$$S(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

This equation represents either a cone or a surface classified above.

21.12 Sample Examination Questions

I. *Answer the following in detail.*

i) State the standard equation of the ellipsoid and discuss its properties.

ii) Discuss the nature of hyperboloid of one sheet and two sheets.

iii) Discuss the nature of elliptic paraboloid and hyperbolic paraboloid.

Transform the following equations into the form $aX^2 + bY^2 + cZ^2 = d$ and find the conicods represented by them.

(i) $3x^2 + 4y^2 + z^2 - x + 12y - 4z + 13 = 0$

(ii) $2x^2 - 7y^2 + 4z^2 + 6x + 7y - 12z = 0$

(iii) $9x^2 + 36y^2 + 4z^2 + 18x - 144y + 150 = 0$

(iv) $x^2 - 2y^2 - 3z^2 - 4x + 5y - 6z - 2 = 0$

(v) $6x^2 - 5y^2 - z^2 + 12x + 10y - 2z = 0$

(vi) $x^2 + 2y^2 - z^2 + 6x - 4y + 10 = 0$

(vii) $x^2 + yz = 1$

Hint : Use the transformations $y = \dfrac{Y + Z}{\sqrt{2}}, \; z = \dfrac{Y - Z}{\sqrt{2}}, x = X.$

Answers

(i) $36X^2 + 48Y^2 + 12Z^2 = 1$, ellipsoid

(ii) $2X^2 - 7Y^2 + 4Z^2 = \frac{47}{7}$, hyperboloid of one sheet

(iii) $9X^2 + 36Y^2 + 4Z^2 = 3$, ellipsoid

(iv) $X^2 - 2Y^2 - 3Z^2 + \frac{1}{8} = 0$, hyperboloid of two sheets

(v) $6X^2 - 5Y^2 - Z^2 = 0$, cone

(vi) $X^2 + 2Y^2 - Z^2 + 3 = 0$, hyperboloid of two sheets

(vii) $X^2 + \frac{Y^2}{2} - \frac{Z^2}{2} = 1$ hyperboloid of one sheet.

21.13 Answers to SAQ's

(1) If the plane sections of a cylinder, perpendicular to its axis, are parabolas it is called parabolic cylinder. If these sections are hyperbolas then the cylinder is called hyperbolic cylinder.

(2) Standard from of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Standard form of hyperboloids.

i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (hyperboloid of one sheet)

ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (hyperboloid of two sheets).

(3) Standard form of paraboloid

i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ (Elliptic paraboloid)

This will lie in the positive side of the plane $z = 0$

ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -2z$ (Elliptic paraboloid). This will lie in the negative side of the plane $z = 0$

(3) For the classification of $ax^2 + by^2 + cz^2 = 1$ see the summary 21.11.

UNIT-22 : TANGENT PLANES AND NORMALS

22.0 Contents

22.1 Aims and Objectives

After going through this unit, you will be able to

i) verify the properties of tangent planes, normals, polar planes and diametral planes of a conicoid.

22.2 Tangent planes and normals

Theorem : If $P\left(x_0, y_0, z_0\right)$ is a point on the conicoid $S(x, y, z) = 0$ and if one of ξ_0, η_0, ζ_0 is different from zero (in the notation of 21.10) then (i) the tangents and generators of the conicoid at P are in the plane $S_0 = 0$ and (ii) any line through P and in the plane $S_0 = 0$ is a tangent or generator of the conicoid $S = 0$.

Proof : Let the given line L through $P\left(x_0, y_0, z_0\right)$ be

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}.$$

Since $P\left(x_0, y_0, z_0\right)$ is on the conicoid, $S_{00} = 0$.

$$\therefore \quad \left(l\xi_0 + m\eta_0 + n\zeta_0\right)^2 - S_{00} H(l, m, n) = 0 \Leftrightarrow l\xi_0 + m\eta_0 + n\zeta_0 = 0$$

$\therefore$ L is a tangent or a generator $\Leftrightarrow l\xi_0 + m\eta_0 + n\zeta_0 = 0$

$\Leftrightarrow \left(x - x_0\right)\xi_0 + \left(y - y_0\right)\eta_0 + \left(z - z_0\right)\zeta_0 = 0$ for every (x, y, z) on L.

$\Leftrightarrow x\xi_0 + y\eta_0 + z\zeta_0 = x_0\xi_0 + y_0\eta_0 + z_0\zeta_0$ for every (x, y, z) on L.

$\Leftrightarrow S_0 - \delta_0 = S_{00} - \delta_0$ for every $(x, y, z) \varepsilon L$.

$\Leftrightarrow S_0 = 0$ for every $(x, y, z) \varepsilon L$ $\left(\because S_{00} = 0\right)$.

$\Leftrightarrow L$ lies on the plane $S_0 = 0$.

Definition : If $P\left(x_0, y_0, z_0\right)$ is a point on the conicoid and if one of ξ_0, η_0, ζ_0 is not zero, the plane through P containing the tangents at P and the generators through P to the conicoid, is called the tangent plane to the conicoid at P. The normal to the tangent plane at P is called the normal to the conicoid at P.

If $P\ (x_0, y_0, z_0)$ is a point on the conicoid and if $\xi_0 = \eta_0 = \zeta_0$, then P is called a singular point of the conicoid.

If a conicoid α is not a single point set, then it can be proved that a plane Π is a tangent plane of $\alpha \Leftrightarrow \alpha \cap \Pi$ has only one point.

Theorem 1 : If $P\ (x_0, y_0, z_0)$ is a singular point of the conicoid $S = 0$, then

(i) any straight line through P is a tangent or generator of the conicoid.

(ii) The conicoid is a cone with P as vertex.

Proof : (i) Since P is a singular point on $S = 0$,

$$l\xi_0 + m\eta_0 + n\zeta_0 = 0, \ S_{00} = 0$$

If l, m, n are the direction cosines of a line through P, then

$$H(l, m, n) \neq 0 \iff L \text{ is a tangent line}$$

$$H(l, m, n) = 0 \iff L \text{ is a generator.}$$

(ii) If Q is a point on the conicoid, $Q \neq P$, then $\overleftrightarrow{PQ}$ has more than one point in common with the conicoid. So it is not a tangent. Hence $\overleftrightarrow{PQ}$ is a generator. The conicoid is a cone with P as vertex.

Note : (i) If $P\ (x_0, y_0, z_0)$ is a non – singular point on the conicoid $S = 0$, the tangent plane at P is,

$$x\xi_0 + y\eta_0 + z\zeta_0 = x_0\xi_0 + y_0\eta_0 + z_0\zeta_0$$

Then equation of the normal at (x_0, y_0, z_0) is,

$$\frac{x - x_0}{\xi_0} = \frac{y - y_0}{\eta_0} = \frac{z - z_0}{\zeta_0}.$$

Theorem 2 : The conicoids $ax^2 + by^2 + cz^2 = 1$ and $ax^2 + by^2 = 2z$ have no singular points.

Proof : (i) Consider $S = ax^2 + by^2 + cz^2 - 1 = 0$

Then $\xi_0 = ax_0, \eta_0 = by_0, \zeta_0 = cz_0$ If $P\ (x_0, y_0, z_0)$ is a singular point,

then

$$\xi_0 = \eta_0 = \zeta_0 = 0 \Rightarrow a{x_0}^2 + b{y_0}^2 + c{z_0}^2 = 0$$

$$\Rightarrow P\ (x_0, y_0, z_0) \notin S.$$

(ii) Consider $S = ax^2 + by^2 - 2z = 0$ ... (2)

Then $\xi_0 = ax_0, \eta_0 = by_0, \zeta_0 = -1$. Since $\zeta_0 \neq 0$ for any (x_0, y_0, z_0), there are no singular points of the conicoid.

Note : (i) The equation of the tangent plane at (x_0, y_0, z_0) to the conicoid $S = 0$ is $S_0 = 0$.

i.e., $ax_0x + by_0y + cz_0z = 1$

Then normal to $S = 0$ at (x_0, y_0, z_0) is

$$\frac{x - x_0}{ax_0} = \frac{y - y_0}{by_0} = \frac{z - z_0}{cz_0}$$

(ii) The tangent plane at (x_0, y_0, z_0) to the conicoid (2) is

$$ax_0x + by_0y = z + z_0.$$

The normal at (x_0, y_0, z_0) to the conicoid is

$$\frac{x - x_0}{ax_0} = \frac{y - y_0}{by_0} = \frac{z - z_0}{-1}.$$

(iii) If S is a homogeneous polynomial, the conicoid $S = 0$ has the origin as singular point.

Theorem 3 : If $abc \neq 0$, the necessary and sufficient condition for the plane $lx + my + nz = p\ (\neq 0)$ to be a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} > 0.$$

Proof : Suppose the given plane is a tangent plane to the conicoid at (x_0, y_0, z_0). Then the equation of the tangent plane at (x_0, y_0, z_0) is

$$(ax_0)x + (by_0)y + (cz_0)z = 1.$$

Since this is same as the given plane $lx + my + nz = p$,

$$\frac{l}{ax_0} = \frac{m}{by_0} = \frac{n}{cz_0} = \frac{p}{1}.$$

Since $p \neq 0$,

$$x_0 = \frac{l}{ap}, y_0 = \frac{m}{bp}, z_0 = \frac{n}{cp}.$$

Since (x_0, y_0, z_0) is a point on the conicoid,

$$1 = ax_0^2 + by_0^2 + cz_0^2 = \frac{l^2}{ap^2} + \frac{m^2}{bp^2} + \frac{n^2}{cp^2}$$

$$\therefore\ 0 < p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}.$$

Conversely, suppose $p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} > 0$. It can be verified that the point $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$ lies on the conicoid. The equation of the tangent at this point is

$$a \cdot \frac{l}{ap} \cdot x + b \cdot \frac{m}{bp} \cdot y + c \cdot \frac{n}{cp} \cdot z = 1$$

or $lx + my + nz = p\ (p \neq 0)$.

Note (i) : If $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} > 0$, then $p = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$ is the perpendicular distance from the origin on the tangent plane. Hence there are two tangent planes in this case. For an ellipsoid, a, b, c are always greater than zero and there are two tangent planes to the ellipsoid perpendicular to any straight line.

(ii) If $ab \neq 0$, $c = 0$, then the plane $lx + my + nz = p$ is a tangent plane to the cylinder $ax^2 + by^2 = 1$ if and only if $\frac{l^2}{a} + \frac{m^2}{b} = p^2 > 0, n = 0$.

All such tangent planes are parallel to the axis of the cylinder.

Theorem 4 : If $abc \neq 0$, the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$ is a sphere.

Proof : Let
$$l_1x + m_1y + n_1z = p_1$$
$$l_2x + m_2y + n_2z = p_2$$
$$l_3x + m_3y + n_3z = p_3$$

be three mutually perpendicular tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$. Choosing l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 as the direction cosines of the three planes, (l_1, l_2, l_3); (m_1, m_2, m_3); (n_1, n_2, n_3) are also three mutually perpendicular unit vectors.

$$\therefore \quad l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1$$
$$m_1n_1 + m_2n_2 + m_3n_3 = n_1l_1 + n_2l_2 + n_3l_3 = l_1m_1 + l_2m_2 + l_3m_3 = 0.$$

Substituting x_0, y_0, z_0 for x, y, z in the equations of the three planes, squaring them and adding, the following equation is obtained.

$$x_0^2 + y_0^2 + z_0^2 = p_1^2 + p_2^2 + p_3^2.$$

Noting that $p_r^2 = \frac{l_r^2}{a} + \frac{m_r^2}{b} + \frac{n_r^2}{c}$; $r = 1, 2, 3$ the above equation gives

$$x_0^2 + y_0^2 + z_0^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$\therefore$ The point (x_0, y_0, z_0) which is the common point of intersection of three mutually perpendicular tangent planes lies on the sphere $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Note : (i) If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 0$, then there is no point satisfying $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Hence there do not exist three mutually perpendicular tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$.

(ii) If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$, origin is the only point satisfying $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. But through the origin there is no tangent plane to the conicoid. Hence in this case also there do not exist three mutually perpendicular tangent planes to the given conicoid.

22.3 Director sphere

Definition : If $abc \neq 0$ and if the locus of the point of intersection of three mutually perpendicular tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ is a sphere, then the sphere is called the Director sphere of the conicoid.

Theorem 5 : The plane $lx + my + nz = p$ is a tangent plane to the paraboloid $ax^2 + by^2 = 2z$ iff $n \neq 0$, $\frac{l^2}{a} + \frac{m^2}{b} + 2np = 0$.

Proof : Since the given surface is a paraboloid, $a \neq 0$, $b \neq 0$.

(i) Suppose $lx + my + nz = p$ is a tangent plane to the paraboloid at the point (x_0, y_0, z_0), then the equation to the tangent plane is

$$(ax_0)\,x + (by_0)\,y - z = z_0$$

$\therefore\ n \neq 0;\ \dfrac{ax_0}{l} = \dfrac{by_0}{m} = -\dfrac{1}{n} = \dfrac{z_0}{p}$

$\therefore\ x_0 = -\dfrac{l}{an},\ y_0 = -\dfrac{m}{bn},\ z_0 = -\dfrac{p}{n}$

Since (x_0, y_0, z_0) is on the paraboloid,

$$a\frac{l^2}{a^2 n^2} + b\frac{m^2}{b^2 n^2} = -2\frac{p}{n}.$$

$$\therefore\ \frac{l^2}{a} + \frac{m^2}{b} + 2pn = 0.$$

(ii) Suppose $n \neq 0, \dfrac{l^2}{a} + \dfrac{m^2}{b} + 2pn = 0.$

Then the point $\left(\dfrac{-l}{an}, \dfrac{-m}{bn}, \dfrac{-p}{n}\right)$ satisfies the equation of the paraboloid and hence is on the paraboloid. The tangent plane at this point can be easily verfied to be $l\cdot + my + nz = p$.

Theorem 6 : The locus of the point of intersection of three mutually perpendicular tangent planes to the paraboloid is a plane perpendicular to the axis of the paraboloid.

Proof : Let the given paraboloid be $ax^2 + by^2 = 2z$. Let the three mutually perpendicular tangent planes

$$l_r x + m_r y + n_r z = p_r \ (r = 1, 2, 3)$$

meet at (x_0, y_0, z_0). Then

$$n_r \neq 0,\ \frac{l_r^2}{a} + \frac{m_r^2}{b} + 2n_r p_r = 0 \ (r = 1, 2, 3)$$

$$\therefore\ 2n_r\left(l_r x + m_r y + n_r z\right) + \frac{l_r^2}{a} + \frac{m_r^2}{b} = 0.$$

Substituting (x_0, y_0, z_0) for (x, y, z) and adding,

$$2z_0 + \frac{1}{a} + \frac{1}{b} = 0.$$

Hence all points like (x_0, y_0, z_0) satisfy $2z + \dfrac{1}{a} + \dfrac{1}{b} = 0$. This is a plane perpendicular to the axis of the paraboloid.

Definition : If the locus of the point of intersection of three mutually perpendicular tangent planes to the paraboloid is a plane, then th [redacted]e is called a *director plane*.

Theorem 7 : If $abc \neq 0$, through a given point in space, at most six normals can be drawn to the conicoid.

Proof : Let (α, β, γ) be the given point and let (x_0, y_0, z_0) be the foot of a normal from (α, β, γ) to the conicoid. Since ax_0, by_0, cz_0 are the direction ratios of the normal at (x_0, y_0, z_0) on the conicoid, the equation of the normal is

$$\frac{x - x_0}{ax_0} = \frac{y - y_0}{by_0} = \frac{z - z_0}{cz_0} = \lambda \text{ (say)}.$$

Since (α, β, γ) is on this normal,

$\alpha = x_0 + \lambda ax_0,\ \beta = y_0 + \lambda by_0,\ \gamma = z_0 + \lambda cz_0$

for some real λ. Since (x_0, y_0, z_0) lies on the conicoid,

$$ax_0^2 + by_0^2 + cz_0^2 = 1.$$

Eliminating x_0, y_0, z_0 the following equation in λ results.

$$a\alpha^2 (1 + b\lambda)^2 (1 + c\lambda)^2 + b\beta^2 (1 + c\lambda)^2 (1 + a\lambda)^2 + c\gamma^2 (1 + a\lambda)^2 (1 + b\lambda)^2$$
$$= (1 + a\lambda)^2 (1 + b\lambda)^2 (1 + c\lambda)^2 \qquad \text{... (1)}$$

This sixth degree equation in λ has $a^2 b^2 c^2 (\neq 0)$ as coefficient of λ^6. The equation has maximum of six real roots. Hence there are not more than six normals to the conicoid passing through the fixed point (α, β, γ).

Note : If $a = b$, $a \neq c$, the conicoid is a surface of revolutin with the z-axis as the axis of symmetry. If $\lambda = -\frac{1}{a}$, then $\alpha = x_0 (1 + a\lambda) = 0$, $\beta = y_0 (1 + b\lambda) = 0$, $\gamma = z_0 (1 + c\lambda) = z_0 (a - c)/a$.

Now consider the section of the conicoid by the plane $z = \frac{a\gamma}{a - c}$.

This section may be an empty set, a single point of a circle.

At any point (x_0, y_0, z_0) of this section, the normal is

$$\frac{x - x_0}{ax_0} = \frac{y - y_0}{by_0} = \frac{z - z_0}{cz_0} = \lambda \left(= -\frac{1}{a}\right)$$

This line passes through the fixed point $(0, 0, \gamma)$.

Hence corresponding to this λ, either there is no normal passing through $(0, 0, \gamma)$ or there is one normal or there are an infinite number of normals to the conicoid passing through $(0, 0, \gamma)$.

If $a = b = c$, the conicoid is a sphere and every line through the origin is normal to the sphere.

Theorem 8 : Through a given point in space, at most five normals can be drawn to the paraboloid.

Proof : Suppose the normal at a point (x_0, y_0, z_0) on the paraboloid $ax^2 + by^2 = 2z$ passes through a fixed point (α, β, γ). Then the direction ratios of the normal at (x_0, y_0, z_0) are $(ax_0, by_0, -1)$. Hence

there is a real number λ such that

$\alpha = x_0 (1 + a\lambda)$, $\beta = y_0 (1 + b\lambda)$, $\gamma = z_0 - \lambda$,

Since (x_0, y_0, z_0) is on the paraboloid, $ax_0^2 + by_0^2 = 2z_0$.

Eliminating (x_0, y_0, z_0), the following equation in λ results.

$$a\alpha^2 (1 + b\lambda)^2 + b\beta^2 (1 + a\lambda)^2 = 2(\gamma + \lambda)(1 + a\lambda)^2 (1 + b\lambda)^2$$

This is a fifth degree equation in λ. Hence there are at most five normals of the paraboloid passing through (α, β, γ).

Note : If $a = b$ and $\lambda = -1/a$, the normal at any point of the circular section of the paraboloid by the plane $z = \gamma - 1/a$ passes through $(0, 0, \gamma)$.

There are infinite number of normals through $(0, 0, \gamma)$.

22.4 Enveloping Cone

Definition : The locus of the tangent lines or asymptotes to a conicoid from a point not on the conicoid is called the enveloping cone or tangent cone of the concoid from that point.

Therorem 9: The enveloping cone of the concoid $S = 0$ from a point $p\ (x_0, y_0, z_0)$ has the equation $S_0^2 = S \cdot S_{00}$.

Proof : Since P is not on the conicoid $S_{00} \neq 0$.

if the straight line L given by,

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

is a tangent or an asymptote of the conicoid, then using 21.10.

$$(l\xi_0 + m\eta_0 + n\zeta_0)^2 = S_{00} H\ (l, m, n). \qquad \ldots(1)$$

Conversely, if (1) is true, then the straight line L is a tangent line if $H\ (l, m, n) \neq 0$ and an asymptote if $H\ (l, m, n) = 0$

($\because$ Then (1) implies that $l\xi_0 + m\eta_0 + n\zeta_0 = 0$).

$\therefore L$ is a tangent of asymptote of the given conicoid

$$\Leftrightarrow (l\xi_0 + m\eta_0 + n\zeta_0)^2 = S_{00} H\ (l, m, n)$$

$\Leftrightarrow L$ is a generator of the cone.

$$\left[(x - x_0)\,\xi + (y - y_0)\,\eta + (z - z_0)\,\xi\right]^2 = S_{00} H\ (x - x_0, y - y_0, z - z_0)$$

This is a homogeneous equation of not more than second degree. The equation can also be written as,

$$\left(S_0 - S_{00}\right)^2 = S_{00}\left\{H\ (x, y, z) + H\left(x_0, y_0, z_0\right) - 2S_0 + 2\delta + 2\delta_0 - 2d\right\}$$

$$= S_{00}\left\{S - 2\delta + d + S_{00} - 2\delta_0 - 2S_0 + 2\delta + 2\delta_0 - 2d\right\}$$

$$= S_{00}\left\{S + S_{00} - 2S_0\right\}$$

i.e., $$S_0^2 = S\, S_{00}$$

Note: At the origin (0, 0, 0), $\xi_0 = \eta_0 = \zeta_0 = 0$ and $S_{00} \neq 0$ for the conicoid $S \equiv ax^2 + by^2 + cz^2 - 1 = 0$ and hence there is no tangent to the conicoid through (0, 0, 0).

If there are asymptotes through (0, 0, 0), they are the generators of $S_0^2 = S \cdot S_{00}$, which on simplification gives

$$ax^2 + by^2 + cz^2 = 0 \qquad \ldots(2)$$

(a) If $bc > 0$, $ca > 0$, this is a point cone.

(b) If one of bc, ca is not greater than zero, all the generators of the cone (2) can be shown to be asympototes of the given conicoid. Hence the envoloping cone from the centre is also called asymptotic cone.

(c) The asymptotic cone of the ellipsoid is a point cone.

(d) The asymptotic cone of an elliptic cylinder is a straight line.

(e) The asymptotic cone of a hyperbolic cylinder is a pair of planes.

(f) The asymptotic cone of the hyperboloid of one sheet, namely

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and the a symptotic cone of the hyperboloid of two sheets, namely

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

are the same (non - degenerate cone).

22.5 Enveloping cylinder

Definition : If $H(l, m, n) \neq 0$, the locus of the tangent lines to the conicoid having direction ratios l, m, n, is called the enveloping cylinder of the conicoid, in the direction (l, m, n)

Theorem 10 : If $H(l, m, n) \neq 0$, the enveloping cylinder of the conicoid $S = 0$ in the direction (l, m, n) is,

$$(l\xi + m\eta + n\zeta)^2 = S \cdot H(l, m, n).$$

Proof : $H(l, m, n) \neq 0$.

$\therefore (x_0, y_0, z_0)$ is a point on the enveloping cylinder in the direction (l, m, n)

$\Leftrightarrow (x_0, y_0, z_0)$ is a point on tangent to $S = 0$ whose DR's are (l, m, n)

$\Leftrightarrow \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$ is a tangent to $S = 0$

$\Leftrightarrow (l\xi_0 + m\eta_0 + n\zeta_0)^2 = S_{00} \cdot H(l, m, n).$

$\therefore$ The points (x_0, y_0, z_0) determine the cone

$$(l\xi + m\eta + n\zeta)^2 = S \cdot H(l, m, n).$$

Examples

1. If $lx + my + nz + p = 0$ is a tangent plane of the conicoid $S(x, y, z) = 0$, show that

$$\begin{vmatrix} a & h & g & u & l \\ h & b & f & v & m \\ g & f & c & w & n \\ u & v & w & d & p \\ l & m & n & p & 0 \end{vmatrix} = 0.$$

If the given plane is a tangent plane at (x_0, y_0, z_0) on $S = 0$, then

$$S_0 = \xi_0 x + \eta_0 y + \zeta_0 z + \delta_0 = 0,$$

and

$$lx + my + nz + p = 0,$$

both represent the same plane. Hence there exists a real number λ such that $\xi_0 + l\lambda = \eta_0 + m\lambda = \zeta_0 + n\lambda = \delta_0 + p\lambda = 0.$

Also $lx_0 + my_0 + nz_0 + p = 0$. These give the following five equations in the five unknowns x, y, z, t, s.

$$\begin{aligned} ax + hy + gz + ut + ls &= 0 \\ hx + by + fz + vt + ms &= 0 \\ gx + fy + cz + wt + ns &= 0 \\ ux + vy + wz + dt + ps &= 0 \\ lx + my + nz + pt &= 0 \end{aligned}$$

These equations have a nontrival solution

$x = x_0, y = y_0, z = z_0, t = 1, s = \lambda$.

Hence the determinant in the example is zero.

2. If the normal at a point P, not on the principal planes, of the conicoid $ax^2 + by^2 + cz^2 = 1, (abc \neq 0)$ cuts the principal planes at G_1, G_2, G_3, then prove that

(i) $PG_1 : PG_2 : PG_3 = \frac{1}{|a|} : \frac{1}{|b|} : \frac{1}{|c|}$

(ii) If $PG_1^2 + PG_2^2 + PG_3^2 + K^2$ (constant), the locus of such points P is an ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = \frac{K^2}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

(iii) If P has coordinates (x_0, y_0, z_0), then normal at P has direction ratios ax_0, by_0, cz_0. If $a^2x_0^2 + b^2y_0^2 + c^2z_0^2 = \frac{1}{p^2}$, the direction cosines are pax_0, pby_0, pcz_0. The coordinates of a point Q on the normal at P are

$x_0 + tpax_0, y_0 + tpby_0, z_0 + tpcz_0$

Then $PQ = |t|$. If this normal cuts the principal planes $x = 0, y = 0, z = 0$ in G_1, G_2, G_3, then the values of t corresponding to these points are $-\frac{1}{pa}, -\frac{1}{pb}, -\frac{1}{pc}$ respectively.

$\therefore PG_1 = \frac{1}{p|a|}, PG_2 = \frac{1}{p|b|}, PG_3 = \frac{1}{p|c|}$.

$PG_1 : PG_2 : PG_3 = \frac{1}{|a|} : \frac{1}{|b|} : \frac{1}{|c|}$

(ii) If $PG_1^2 + PG_2^2 + PG_3^2 = K^2$, then

$$\begin{aligned} K^2 &= \frac{1}{p^2a^2} + \frac{1}{p^2b^2} + \frac{1}{p^2c^2} = \frac{1}{p^2}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \\ &= \left(a^2x_0^2 + b^2y_0^2 + c^2z_0^2\right)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \end{aligned}$$

The locus of the point (x_0, y_0, z_0) is

$$a^2x^2 + b^2y^2 + c^2z^2 = K^2 / \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

22.6 Centre of a conicoid

Definition : (i) If P is the mid point of $\overline{QQ'}$, then Q' is called the image of Q in P.

(ii) Let α be a conicoid. Let P be a point in space such that the image of any point $Q \varepsilon \alpha$ in P is again on α. Then P is called the centre of the conicoid α.

1. A conicoid may not have a centre. It may have only one centre. It may have more than one centre also, as illustrated by the examples bellow.

Examples:

(i) (0,0, 0) is the centre of the conicoid $ax^2 + by^2 + cz^2 = 1$.

(ii) Any point on the z-axis is a centre of the cylinder $ax^2 + by^2 = 1$

(iii) Any point on the yz plane is a centre of the pair of parallel planes $x^2 =$ (i.e. $x = \pm 1$).

(iv) Any vertex of a cone is a centre.

(v) Paraboloids, and parabolic cylinders have no centre.

2. The following result may be noted without proof.

(x_0, y_0, z_0) is a centre of a conicoid $S = 0 \Leftrightarrow \xi_0 = \eta_0 = \zeta_0 = 0$

3. If the centre of conicoid is on the conicoid itself it can be proved that the conicoid is a cone with this centre as vertex.

(If P is the centre of conicoid α, then $Q \varepsilon \alpha$, $Q \neq P \Rightarrow$ The point Q', which is the image of Q in P, is on α.

$\therefore$ P, Q, Q′ are in $\alpha \cap \overleftrightarrow{PQ}$

$\therefore \overleftrightarrow{PQ} \subset \alpha$.)

Examples :

1. Find the centre of the conicoid represented by the equation $3z^2 - 2yz - 2zx - 2xy + 8x - y - 3z + 11 = 0$

The coordinates of centre are obtained by solving $\xi = \eta = \zeta = 0$.

$$\xi = ax + hy + gz + u = y - z + 4 = 0$$

$$\eta = hx + by + fz + v = 2x - 2z - 1 = 0$$

$$\zeta = gx + fy + cz + w = 2x + 2y - 6z + 3 = 0$$

On solving, the centre is (–3/2, –6, –2).

2. Find the centre of the conicoid given by the equation

$$2z^2 - 2yz - 2zx + 2xy + 3x - y - 2z + 1 = 0.$$

The equations $\xi = 0, \eta = 0, \zeta = 0$ give

$$2y - 2z = -3$$

$$2x - 2z = 1$$

$$2x + 2y - 4z = -2$$

$\therefore$ Any point on the straight line $2x - 1 = 2y + 3 = 2z$ is a centre.

22.7 Polar plane

Definition : If $P(x_0, y_0, z_0)$ is not a centre of the conicoid $S = 0$, then the plane $S_0 = 0$ is called the polar plane of P w.r.t. the conicoid. The $P(x_0, y_0, z_0)$ is called a pole of the plane $S_0 = 0$.

Examples :

(i) The polar plane of (x_0, y_0, z_0) w.r.t. $ax^2 + by^2 + cz^2 = 1$ is $ax_0x + by_0y + cz_0z = 1$.

(ii) The polar plane of (x_0, y_0, z_0) w.r.t. $ax^2 + by^2 = 2z$ is $ax_0x + by_0y = z + z_0$

(iii) The polar plane of (x_0, y_0, z_0) w.r.t. the cone $ax^2 + by^2 + cz^2 = 0$ is $ax_0x + by_0y + cz_0z = 0$. Any point $(tx_0, ty_0, tz_0) \neq (0, 0, 0)$ is a pole of this plane w.r.t. the given cone.

Note : If a point P is on the conicoid $S = 0$, then the polar plane of P and the tangent plane at P are the same.

Theorem 11 : If a point Q (other than the centre) is on the polar plane of a point P, then the point P is on the polar plane of Q w.r.t. a given conicoid. Equivalently, if the poles of a plane Π w.r.t. a conicoid are on a plane Σ, then the poles of Σ are on Π.

Proof : The equation of the polar plane of $P(x_0, y_0, z_0)$ w.r.t $S = 0$ is

$$0 = S_0 = x\,\xi_0 + y\,\eta_0 + z\,\zeta_0 + \delta_0 = x_0\xi + y_0\eta + z_0\zeta + \delta.$$

Since $Q(x_1, y_1, z_1)$ is on this plane,

$$x_0\xi_1 + y_0\eta_1 + z_0\zeta_1 + \delta_1 = 0$$

$\therefore$ The point (x_0, y_0, z_0) is on the plane

$$x\,\xi_1 + y\,\eta_1 + x\,\zeta_1 + \delta_1 = 0$$

which is the polar plane of (x_1, y_1, z_1).

Theorem 12 : The polar planes of points (other than the centre) on a straight line are

(i) parallel to each other, or

(ii) intersect in a straight line,

In case (ii), the polar planes of points on the line of intersection pass through the given line.

Proof : Let $S = 0$ be a given conicoid. Let $P_t = (x_0 + lt, y_0 + mt, z_0 + nt)$ be a point on the given straight line L. If P_t is not the centre, the polar plane of P_t is

$$(x_0 + lt)\,\xi + (y_0 + mt)\,\eta + (z_0 + nt)\,\zeta + \delta = 0$$

This can be written as

$$S_0 + t\,(l\xi + m\eta + n\zeta) = 0 \qquad \ldots(1)$$

The following cases arise.

(a) If the centre is a point on the given straight line, choose this point for (x_0, y_0, z_0). Then, corresponding to this (x_0, y_0, z_0), $\xi_0 = \eta_0 = \zeta_0 = 0$. (See definition of centre, result (2)).

$$\therefore S_0 \equiv \xi_0x + \eta_0y + \zeta_0z + \delta_0 \equiv \delta_0$$

$\therefore$ The equation (1) represents parallel planes for different values of t.

(b) If the centre is not a point on the given line, and if $al + hm + gn = hl + bm + fn = gl + fm + cn = 0$ then it can be verified that $l\xi + m\eta + n\zeta \equiv ul + vm + wn$. $\therefore$ The equation (1) reduces to $S_0 + t\,(ul + vm + wn) = 0$. This again represents parallel planes as t varies.

(c) If the planes $S_0 = 0$ and $l\xi + m\eta + n\zeta = 0$ are parallel planes, then also (1) represents parallel planes as t varies.

(d) If $S_0 = 0$ and $l\xi + m\eta + n\zeta = 0$ are a pair of intersecting planes, let them intersect in a line M. Then (1) represents a plane through this line. Thus the polar plane of any point P on the given line L passes through the line M.

If $P \in L$, $Q \in M$, then the point Q lies on the polar plane of P. Hence by theorem 11 the point P lies on the polar plane of Q.

From this the last part of the theorem follows.

22.8 Polar line

Definite : If the polar planes of points on a straight line intersect, the line of intersection is called the polar line of the given line.

Note : (i) If M is the polar line of L, then L is the polar line of M by theorem 12.

(ii) If $\overline{PQ}$ is the chord of a conicoid and if the tangent planes at P and Q intersect, then the line of intersection is the polar line of $\overleftrightarrow{PQ}$.

Examples :

(i) The polar line $x = y = 1$ w.r.t the conicoid $ax^2 + by^2 + cz^2 = 1$ $(abc \neq 0)$ is the line of intersection of

$ax + by - 1 = 0$; $cz = 0$

(ii) The principal axes have no polar lines w.r.t $ax^2 + by^2 + cz^2 = 1, abc \neq 0$.

(iii) The y-axis is the polar line of x-axis w.r.t the paraboloid $ax^2 + by^2 = 2z$.

Examples :

1. If a straight line through P meets the conicoid in two different points A, B and meets the polar plane of P w.r.t the conicoid in Q, prove that the points A, B divide $\overline{PQ}$ harmonically).

Let $P = (x_0, y_0, z_0)$; $Q = (x_0 + l, y_0 + m, z_0 + n)$. Then l, m, n are the direction ratios of line PQ. Hence let

$$A = (x_0 + t_1 l, y_0 + t_1 m, z_0 + t_1 n)$$
$$B = (x_0 + t_2 l, y_0 + t_2 m, z_0 + t_2 n).$$

Then t_1, t_2 are the roots of

$$H(l, m, n)\, t^2 + 2(l\xi_0 + m\eta_0 + n\zeta_0)\, t + S_{00} = 0.$$

$$\therefore t_1 + t_2 = -2(l\xi_0 + m\eta_0 + n\zeta_0) / H(l, m, n)$$
$$t_1 t_2 = S_{00} / H(l, m, n).$$

Since the point Q is on the polar plane P,

$$(x_0 + l)\,\xi_0 + (y_0 + m)\,\eta_0 + (z_0 + n)\,\zeta_0 + \delta_0 = 0.$$

$$\therefore l\zeta_0 + m\eta_0 + n\zeta_0 = -S_{00}$$

$$\therefore t_1 + t_2 = 2 t_1 t_2$$

Since $PQ = \sqrt{l^2 + m^2 + n^2}$, $PA = \sqrt{t_1^2 (l^2 + m^2 + n^2)}$,

$PB = \sqrt{t_2^2 (l^2 + m^2 + n^2)}$, we gave $PA = t_1 PQ$, $PB = t_2 \cdot PQ$

$$\therefore \frac{1}{PA} + \frac{1}{PB} = \frac{1}{PQ}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) = \frac{2}{PQ}$$

2. Prove that the locus of the poles of the tangent planes of $ax^2 + by^2 + cz^2 = 1$, $(abc \neq 0)$ w.r.t $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is the conicoid

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1$$

Let Σ be a tangent plane of $ax^2 + by^2 + cz^2 = 1$ and let (x_0, y_0, z_0) be the pole of Σ w.r.t $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$. Then the equation of Σ as the polar plane of (x_0, y_0, z_0) is

$$\alpha x_0 x + \beta y_0 y + \gamma z_0 z = 1.$$

Since this is a tangent plane of $ax^2 + by^2 + cz^2 = 1$

$$\frac{\alpha^2 x_0^2}{a} + \frac{\beta^2 y_0^2}{b} + \frac{\gamma^2 z_0^2}{c} = 1.$$

$\therefore (x_0, y_0, z_0)$ is on the conicoid $\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1.$

3. Prove that the polar planes of P, Q w.r.t. the conicoid $ax^2 + by^2 + cz^2 = 1$ $(abc \neq 0)$ are parallel iff P, O, Q are collinear. (O is the origin).

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. The equations of the polar planes of P and Q are

$$ax_1 x + by_1 y + cz_1 z = 1$$

$$ax_2 x + by_2 y + cz_2 z = 1$$

Since $abc \neq 0$, these planes are parallel iff $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

Hence the points $(x_1, y_1, z_1) : (x_2, y_2, z_2) : (0, 0, 0)$ are collinear.

22.9 Diameter and diametral planes

System of parallel chords : A set of all chords of a conicoid, parallel to a given straight line is called a system of parallel chords of the conicoid.

Theorem 13 : The mid points of chords of a system of parallel chords of a conicoid determine a plane.

Proof : Let (x_0, y_0, z_0) be the mid point of a chord PQ of the conicoid given by $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$. Then the points

$P\,(x_0 - lr, y_0 - mr, z_0 - nr)$ and Q $(x_0 + lr, y_0 + mr, z_0 + nr)$ are on the conicoid for a fixed real number r, then $r, -r$ are the roots of

$$H(l, m, n)\, t^2 + 2(l\xi_0 + m\eta_0 + n\zeta_0)\, t + S_{00} = 0 \qquad \text{...(1)}$$

$$\therefore l\xi_0 + m\eta_0 + n\zeta_0 = 0$$

$\therefore$ The point (x_0, y_0, z_0) determines the plane

$$l\xi + m\eta + n\zeta_0 = 0 \qquad ...(2)$$

which can also be written as

$$(al + hm + gn)\,x + (hl + bm + fn)\,y + (gl + fm + cn)\,z + (ul + vm + wn) = 0$$

If the coefficients of x, y and z are zero in this equation, then

$H(l, m, n) = 0$ and then (1) shows that there are no chords of conicoid parallel to (l, m, n). Hence if there is a system of parallel chords then the coefficients of x, y and z are not all zero and (2) determines a plane.

Definition : The plane determined by the middle points of system of chords of a conicoid parallel to a given straight line is called the diametral plane of that straight line w.r.t. the given conicoid.

Note: (i) The equation of the diametral plane of $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ w.r.t. a conicoid $ax^2 + by^2 + cz^2 = 1$ is $alx + bmy + cnz = 0$. If $H(l, m, n) = al^2 + bm^2 + cz^2 = 0$, there is no diametral plane for the straight line l, m, n. The diametral planes of this conicoid pass through the centre. If $abc \neq 0$, the diametral planes of two different straight lines through the origin are different.

(ii) The diametral plane of the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ w.r.t. a paraboloid $ax^2 + by^2 = 2z$ is $alm + bmy - n = 0$. Hence all the diametral planes are parallel to the axis of the paraboloid. If $al^2 + bm^2 = 0$, there is no diametral plane.

(iii) The diametral plane of $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ w.r.t. a pair of intersecting planes $y^2 - k^2x^2 = 0$ is $my - k^2lx = 0$. There is no diametral plane if $m^2 = k^2l^2$.

All the diametral planes pass through the line of intersection of the given planes, namely, the z–axis.

Theorem 14 : If $a\,b\,c \neq 0$ and L, M are two straight lines through the origin such that the diametral plane of L passes through M, the diametral plane of M passes through L, w.r.t. the conicoid $ax^2 + by^2 + cz^2 = 1$.

Proof : Let l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of L and M. The diametral plane of L is $al_1x + bm_1y + cn_1z = 0$. If the line M is on this plane, then $al_1l_2 + bm_1m_2 + cn_1n_2 = 0$. But this is also the conditions for the diametral plane of M, namely $al_2x + bm_2y + cn_2z = 0$ to contain the line L.

Theorem 15 : For the conicoid $ax^2 + by^2 + cz^2 = 1$, $a\,b\,c \neq 0$, if three lines L, M, N through the origin are such that the diametral plane of L w.r.t. the conicoid passes through M, N and the diametral plane of M passes through N, then the diametral planes of any two lines pass through the third line.

Proof : By hypothesis,

diametral plane of L passes through M ... (1)

diametral plane of L passes through N ... (2)

diametral plane of M passes through N ... (3)

Then using theorem 14, Corresponding to these results, we get

diametral plane of M passes through L ... (4)

diametral plane of N passes through L ... (5)

diametral plane of N passes through M ... (6)

$\therefore$ From (2) and (3),

diametral planes of L and M pass through N.

From (1) and (6)

diametral planes of L and N pass through M

From (4) and (5),

diametral planes of M and N pass through L.

Definition : If three straight lines through the origin are such that the diametral planes of any two will intersect in the third line w.r.t. the conicoid $ax^2 + by^2 + cz^2 = 1$, $a\,b\,c \neq 0$, then the three lines are called a triad of conjugate diametral lines. The three diametral planes are called a triad of conjugate diametral planes.

Theorem 16 : If POP', QOQ', ROR' are a triad of conjugate diameters of an ellipsoid with O as centre P, P', Q, Q', R, R' are on the ellipsoid, then

(i) $OP^2 + OQ^2 + OR^2 = \text{constant}$

(ii) The volume of the tetrahedran $OPQR$ is a constant.

Proof : Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ be the given ellipsoid. Let the coordinates of P, Q and R be (x_1, y_1, z_1); (x_2, y_2, z_2) and (x_3, y_3, z_3). Then the direction ratios of OP, OQ, OR are (x_1, y_1, z_1); (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Since P, Q, R are on the ellipsoid

$$\frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2} = 1 \,; \; (i = 1, 2, 3) \qquad \ldots (1)$$

The diametral planes of POP', QOQ' will intersect in ROR'.

$\therefore$ R lies on the diametral plane of POP' and on the diametral plane of QOQ' ... (2)

The diametral planes of POP' and ROR' will intersect in QOQ'.

$\therefore$ Q lies on the diametral plane of POP' ... (3)

from (2) and (3)

$$\frac{x_i x_j}{a^2} + \frac{y_i y_j}{b^2} + \frac{z_i z_j}{c^2} = 0 \qquad \ldots (4)$$

$$(i = 1, 2, 3;\; j = 1, 2, 3;\; i \neq j)$$

Equations (1) and (4) show that the vectors

$\left(\dfrac{x_1}{a}, \dfrac{y_1}{b}, \dfrac{z_1}{c}\right)$; $\left(\dfrac{x_2}{a}, \dfrac{y_2}{b}, \dfrac{z_2}{c}\right)$; $\left(\dfrac{x_3}{a}, \dfrac{y_3}{b}, \dfrac{z_3}{c}\right)$ are mutually perpendicular unit vectors.

$\therefore$ The vectors $\left(\dfrac{x_1}{a}, \dfrac{x_2}{a}, \dfrac{x_3}{a}\right)$; $\left(\dfrac{y_1}{b}, \dfrac{y_2}{b}, \dfrac{y_3}{b}\right)$; $\left(\dfrac{z_1}{c}, \dfrac{z_2}{c}, \dfrac{z_3}{c}\right)$ are also mutually perpendicular unit vectors.

$$\therefore \quad x_1^2 + x_2^2 + x_3^2 = a^2$$

$$y_1^2 + y_2^2 + y_3^2 = b^2$$

$$z_1^2 + z_2^2 + z_3^2 = c^2$$

$$OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2 \;\text{(constant)}$$

(ii) If $r_i = \left(\frac{x_i}{a}, \frac{y_i}{b}, \frac{z_i}{c}\right)$, $i = 1, 2, 3$, then r_1, r_2, r_3 are three mutually perpendicular unit vectors. Hence $\left[r_1, r_2, r_3\right] = \pm 1$. The volume of the tetrahedran $OPQR$

$$= \frac{1}{6}\left[\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}\right]$$

$$= \frac{1}{6}\left\|\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}\right\|$$

$$= \frac{|abc|}{6}\left\|\begin{matrix} \frac{x_1}{a} & \frac{y_1}{b} & \frac{z_1}{c} \\ \frac{x_2}{a} & \frac{y_2}{b} & \frac{z_2}{c} \\ \frac{x_3}{a} & \frac{y_3}{b} & \frac{z_3}{c} \end{matrix}\right\|$$

$$= \frac{|abc|}{6}\left|\left[\bar{r}_1, \bar{r}_2, \bar{r}_3\right]\right|$$

$$= \frac{|abc|}{6} \text{ (Constant).}$$

Theorem 17: For a paraboloid

(i) No diameter has a diametral plane.

(ii) Any diametral plane is parallel to the axis of the paraboloid

(iii) The diametral planes of straight lines perpendicular to the axis of the paraboloid intersect on the axis;

Proof : Suppose $ax^2 + by^2 = 2z$ be the given paraboloid. (i) By definition, the diameter of a paraboloid is a line parallel to the axis of the paraboloid and hence the direction ratios (0, 0, 1). Hence (0, 0, 1) = 0 and consequently, there are no chords parallel to a diameter.

∴ No diameter has a diametral plane.

(ii) The diametral plane of a line L with direction ratios (l, m, n) has equation $alx + bmy - n = 0$ and is parallel to the axis of the paraboloid.

(iii) If L is a line perpendicular to the axis, then $l.0 + m.0 + n.1 = n = 0$.

∴ The diametral plane of L is $alx + bmy = 0$

∴ The axis is on the diametral plane of L.

Theorem 18: If the diametral plane of line L_1 is parallel to a line L_2 w.r.t a paraboloid, then the diametral plane of L_2, if it exists, is parallel to line L_1.

Proof : Let the given paraboloid be $ax^2 + by^2 = 2z$.

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of line L_1 and L_2 respectively, the diametral planes of L_1 and L_2 are

$$al_1x + bm_1y - n_1 = 0 \quad \text{...(1)}$$

$$al_2x + bm_2y - n_2 = 0 \quad \text{...(2)}$$

Since the plane (1) is parallel to line L_2, the condition is

$$(al_1)\, l_2 + (bm_1)\, m_2 + 0.n_2 = 0$$

$$\Rightarrow \quad (al_2)\, l_1 + (bm_2)\, m_1 + 0.n_1 = 0$$

$\therefore$ The plane (2) is parallel to line L_1.

Conjugate diametral planes : If L and M are two lines such that the diametral plane of either line is parallel to the other line w.r.t a paraboloid, then the diametral planes of L and M are called conjugate diametral planes.

Examples :

1. If POP', QOQ', ROR' are a set of conjugate diameters of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, prove that the plane PQR is the tangent plane of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{3}$ at the centroid of triangle PQR.

Let $P = (x_1, y_1, z_1)$; $Q = (x_2, y_2, z_2)$; $R = (x_3, y_3, z_3)$. Let the equation of the plane PQR be $lx + my + nz = p$. Since P, Q, R lie on this plane, $lx_r + my_r + nz_r = p \ (r = 1, 2, 3)$. The vectors $\left(\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}\right)$; $\left(\frac{y_1}{b}, \frac{y_2}{b}, \frac{y_3}{b}\right)$; $\left(\frac{z_1}{c}, \frac{z_2}{c}, \frac{z_3}{c}\right)$ are unit vectors.

$$\begin{aligned} p\,(x_1 + x_2 + x_3) &= (lx_1 + my_1 + nz_1)\, x_1 + (lx_2 + my_2 + nz_2)\, x_2 + \\ &\quad + (lx_3 + my_3 + nz_3)\, x_3 \\ &= l\,(x_1^2 + x_2^2 + x_3^2) + m\,(y_1 x_1 + y_2 x_2 + y_3 x_3) \\ &\quad + n\,(z_1 x_1 + z_2 x_2 + z_3 x_3) \\ &= la^2. \end{aligned}$$

Similarly, $p\,(y_1 + y_2 + y_3) = mb^2$

and $p\,(z_1 + z_2 + z_3) = nc^2$.

These equations determine l, m, n. The equation of the plane PQR is

$$\frac{x_1 + x_2 + x_3}{a^2}\, x + \frac{y_1 + y_2 + y_3}{b^2}\, y + \frac{z_1 + z_2 + z_3}{c^2}\, z = 1$$

If (x_0, y_0, z_0) is the centroid of the triangle PQR, then

$$x_0 = \frac{x_1 + x_2 + x_3}{3},\ y_0 = \frac{y_1 + y_2 + y_3}{3},\ z_0 = \frac{z_1 + z_2 + z_3}{3}$$

The equation of the plane PQR is

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0^2}{c^2} = \frac{1}{3} \quad \text{(1)}$$

Since (x_0, y_0, z_0) is in the plane PQR,

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2} = \frac{1}{3} \quad \text{...(2)}$$

Equations (1) and (2) show that the plane PQR is a tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = \frac{1}{3}$ at (x_0, y_0, z_0).

2. If a triad of conjugate diametral planes of an ellipsoid are tangent planes of a sphere with its centre P on α, then prove that the radius of this sphere does not depend on P.

Let the equation of the given ellipsoid α be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Let $P_1 = (x_1, y_1, z_1)$; $P_2 = (x_2, y_2, z_2)$; $P_3 = (x_3, y_3, z_3)$ be points on the ellipsoid such that OP_1, OP_2, OP_3 are semi conjugate diameters, whose diametral planes satisfy the hypothesis. Then the equations of these diametral planes are

$$\frac{xx_r}{a^2}+\frac{yy_r}{b^2}+\frac{zz_r}{c^2} = 0 \quad (r = 1, 2, 3). \quad \text{...(1)}$$

If these planes (1) are tangent planes of sphere, centre $P(\alpha, \beta, \gamma)$ on the given ellipsoid, then the distance of P from these planes are all equal to the radius k of the required sphere.

$$\therefore k^2 = \left(\frac{\alpha x_r}{a^2}+\frac{\beta y_r}{b^2}+\frac{\gamma z_r}{c^2}\right)^2 \Bigg/ \left(\frac{x_r^2}{a^4}+\frac{y_r^2}{b^4}+\frac{z_r^2}{c^4}\right) \quad r = 1, 2, 3.$$

$$\therefore k^2\left(\frac{x_r^2}{a^4}+\frac{y_r^2}{b^4}+\frac{z_r^2}{c^4}\right) = \left(\frac{\alpha x_r}{a^2}+\frac{\beta y_r}{b^2}+\frac{\gamma z_r}{c^2}\right)^2 ; r = 1, 2, 3.$$

Adding these three equations and simplifying,

$$k^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}$$

$$= 1 \quad (\because (\alpha, \beta, \gamma) \text{ is on the given ellipsoid})$$

$\therefore k$ does not depend on $P(\alpha, \beta, \gamma)$.

3. If POP', QOQ', ROR' is a triad of conjugate diametral lines of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ such that $OP = OQ = OR$ (traid of equi-conjugate diameters) and if S is the pole of plane PQR, show that $SP \perp QR$, $SQ \perp PQ$.

Since $OP = OQ = OR$ and $OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$;

$$OP^2 = OQ^2 = OR^2 = (a^2 + b^2 + c^2)/3.$$

If $P = (x_1, y_1, z_1)$; $Q = (x_2, y_2, z_2)$; $R = (x_3, y_3, z_3)$, the equation of the plane PQR is,

$$\frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z = 1 \text{ (example 1).}$$

Since this is the polar plane of S, the coordinates of S are,

$$(x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$$

$$\vec{PS} = (x_2 + x_3, y_2 + y_3, z_2 + z_3)$$

$$\vec{QR} = (x_3 - x_2, y_3 - y_2, z_3 - z_2)$$

$$\therefore \vec{PS} \cdot \vec{QR} = x_3^2 - x_2^2 + y_3^2 - y_2^2 + z_3^2 - z_2^2.$$

$$= OR^2 - OQ^2 = 0$$

$$\therefore \vec{PS} \perp \vec{QR}$$

in a similar way, the other results can be verified.

4. Prove that the sum of squares of the areas OPQ, OQR, ORP of faces of tetrahedran $OPQR$ formed with any three conjugate semidiameters OP, OQ, OR as coterminous edges is constant.

Let (x_1, y_1, z_1) ; (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of P, Q and R respectively. Let A_1, A_2 and A_3 be the areas of the triangles OQR, ORP and OPQ.

Since OP, OQ, OR are semi conjugate diameters.

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1, \qquad \frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 1,$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1, \qquad \frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2} = 1,$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1, \qquad \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} = 1,$$

From these equations,

$$\frac{x_1/a}{\left(\frac{y_2z_3 - y_3z_2}{b\ c}\right)} = \frac{y_1/b}{\left(\frac{z_2x_3 - z_3x_2}{c\ a}\right)} = \frac{z_1/c}{\left(\frac{x_2y_3 - x_3y_2}{a\ b}\right)}$$

$$= \frac{\sqrt{\Sigma\,(x_1^2/a^2)}}{\sqrt{\Sigma\left(\frac{y_2z_3 - y_3z_2}{bc}\right)^2}} = \pm 1$$

(because $\sqrt{\Sigma\left(\frac{y_2z_3 - y_3z_2}{bc}\right)^2} = \sin\theta$, where θ is the angle between the lines whose direction cosines are $\left(\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}\right)$ and $\left(\frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}\right)$. Since these lines are perpendicular, $\sin\theta = 1$). Let l_1, m_1, n_1 be the direction cosines of the normal to the plane OQR. The projection of $\Delta\, OQR$ on the yz plane is the triangle whose vertices are $(0, 0, 0)$; $(0, y_2, z_2)$; $(0, y_3, z_3)$. The area of this projection is

$$A_1 l_1 = \frac{1}{2}(y_2z_3 - y_3z_2) = \pm\frac{bc\,x_1}{2a}$$

$$\text{Similarly } A_1 m_1 = \pm\frac{ca\,y_1}{2b}, \quad A_1 n_1 = \pm\frac{ab\,z_1}{2c}$$

Squaring and adding,

$$A_1^2 = \frac{b^2c^2x_1^2}{4a^2} + \frac{c^2a^2y_1^2}{4b^2} + \frac{a^2b^2z_1^2}{4c^2}$$

Similarly, projecting the areas ORP and OPQ on the coordinate planes,

$$A_2^2 = \frac{b^2c^2x_2^2}{4a^2} + \frac{c^2a^2y_2^2}{4b^2} + \frac{a^2b^2z_2^2}{4c^2}$$

$$A_3^2 = \frac{b^2c^2x_3^2}{4a^2} + \frac{c^2a^2y_3^2}{4b^2} + \frac{a^2b^2z_3^2}{4c^2}$$

Adding,

$$A_1^2 + A_2^2 + A_3^2 = \frac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2) = \text{constant.}$$

5. Prove that the sum of the squares of the projections of three semi conjugate diameters on any line or plane is constant.

Let (l, m, n) be the direction cosines of any given line so that the sum of the squares of the projections of OP, OQ and OR on this line is

$$(lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$

$$= l^2 \Sigma x_1^2 + m^2 \Sigma y_1^2 + n^2 \Sigma z_1^2 + 2lm \Sigma x_1y_1 + 2mn \Sigma y_1z_1 + 2nl \Sigma z_1x_1$$

$$= a^2l^2 + b^2m^2 + c^2n^2 \text{ (constant).}$$

Let (l, m, n) be the direction cosines of the normal to any given plane so that the sum of the squares of the projections of OP, OQ, OR on this plane is

$$= OP^2 - (lx_1 + my_1 + nz_1)^2 + OQ^2 - (lx_2 + my_2 + nz_2)^2 + OR^2 - (lx_3 + my_3 + nz_3)^2$$

$$= a^2 + b^2 + c^2 - (a^2l^2 + b^2m^2 + c^2n^2)$$

$$= a^2(m^2 + n^2) + b^2(n^2 + l^2) + c^2(l^2 + m^2) = \text{constant.}$$

6. Find the locus of the equi conjugate semi diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let OP, OQ, OR be equi conjugate semi diameters.

$OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$; $OP^2 = OQ^2 = OR^2$

Let P be (x_1, y_1, z_1). The locus of the OP, namely

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}, \text{ is required.}$$

Since $x_1^2 + y_1^2 + z_1^2 = \frac{1}{3}(a^2 + b^2 + c^2)$ and $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = \frac{3(x_1^2 + y_1^2 + z_1^2)}{(a^2 + b^2 + c^2)}$$

Using $\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$ and eliminating x_1, y_1, z_1, the locus of OP is obtained as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{3(x^2 + y^2 + z^2)}{(a^2 + b^2 + c^2)}.$$

If the cone $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ has three of its generators along conjugate diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, show that $Aa^2 + Bb^2 + Cc^2 = 0$.

Let OP, OQ, OR be generators of the given cone where $P(x_1, y_1, z_1)$; $Q(x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$ are the extermities of the given conjugate semi diameters of the ellipsoid. Since these points lie on the cone,

$Ax_i^2 + By_i^2 + Cz_i^2 + 2F\, y_i z_i + 2G\, z_i x_i + 2H\, x_i y_i = 0 \ (i = 1, 2, 3)$

Adding these three equations and using the relations,

$\Sigma x_i^2 = a^2 \quad \Sigma y_i^2 = b^2, \ \Sigma z_i^2 = c^2,$

$\Sigma y_i z_i = \Sigma z_i x_i = \Sigma x_i y_i = 0$, the required relation $Aa^2 + Bb^2 + Cc^2 = 0$ is obtained.

22.10 Summary

Let $P(x_0, y_0, z_0)$ be a point on the conicoid $S = 0$, $\xi_0 = ax_0 + hy_0 + gz_0 + u$, $\eta_0 = hx_0 + by_0 + fz_0 + v$, $\zeta_0 = gx_0 + fy_0 + cz_0 + w$. If any one of these ξ_0, η_0, ζ_0 is zero then the tangent lines, generating line through P will lie on a plane called the tangent plane at $P(x_0, y_0, z_0)$ to the conicoid. If $\xi_0 = \eta_0 = \zeta_0 = 0$ then $P(x_0, y_0, z_0)$ is called a singular point of the conicoid. Then every line through P is either a tangent line or a generator to the surface. In this case surface is a cone with vertex $P(x_0, y_0, z_0)$ If $abc \neq 0$, then the plane $lx + my + nz = p$ touches, the surface $ax^2 + by^2 + cz^2 = 1$ if and only if

$$p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} > 0$$

If $abc \neq 0$ and if the locus of the point of intersection of three mutually perpendicular tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ is a sphere, then the sphere is called the director sphere of the conicoid surface. The unnecessary and sufficient condition for the contact of the plane $lx + my + nz = p$ to the paraboloid $ax^2 + by^2 = 2z$ is $n \neq 0, \frac{l^2}{a} + \frac{m^2}{b} + 2np = 0$. If abc $\neq 0$ then one can draw at the most 6 normals from a given point to the surface $ax^2 + by^2 + cz^2 = 1$.

One can draw at the most 5 normals to the surface of paraboloid from a given point. We cannot draw diametral plane to any diameter of the paraboloid.

22.11 Sample Examination Questions

I. Answer the following Questions in detail

1. Show that the tangent plane at any point of the cone $H(x, y, z) = 0$ passes through the origin.
2. Find the point of intersection of the tangent planes at (1, 1, 1) ; (5, 5, 7) and (7, 11, 13) on the conicoid $x^2 + y^2 - z^2 = 1$.
3. Find the equations of the tangent planes of the conicoid $7x^2 - 3y^2 - z^2 + 21 = 0$ which passes through the line $7x - 6y + 9 = 3$; $z = 3$.
4. Show that a common tangent plane of the ellipsoids $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; $\frac{x^2}{b^2} + \frac{y^2}{c^2} + \frac{z^2}{a^2} = 1$; $\frac{x^2}{c^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ is also a tangent plane of the sphere $x^2 + y^2 + z^2 = \frac{1}{3}(a^2 + b^2 + c^2)$. Also prove that the points of contact of a common tangent plane and the ellipsoids lie on the sphere $x^2 + y^2 + z^2 = (a^4 + b^4 + c^4) / (a^2 + b^2 + c^2)$.

5. If the tangent planes at P and Q on the paraboloid $ax^2 + by^2 = 2z$ meet in a line L, prove that the plane determined by L and the middle point of PQ is parallel to the axis of the paraboloid.

6. If the normal at $P(\alpha, \beta, \gamma)$ to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ cut it again in Q, then show that $PQ^2 = 4\left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right)\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right)^{-2}$

7. If the polar line of L w.r.t a conicoid is coplanar with the line M, is coplanar with L (such lines are called conjugate lines).

8. If the two lines $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$ are mutually polar lines w.r.t $ax^2 + by^2 + cz^2 = 1$, then prove that

$$a\,\alpha\alpha' + b\,\beta\beta' + c\,\gamma\gamma' = 1$$

$$a\,\alpha l' + b\,\beta m' + c\,\gamma n' = a\,\alpha' l + b\,\beta' m + c\,\gamma' n$$

$$= all' + bmm' + cnn'.$$

9. Show that the polar line of $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ w.r.t the conicoid $x^2 - 2y^2 + 3z^2 - 4 = 0$ is $\frac{x+6}{3} = \frac{y-2}{3} = \frac{z-2}{1}$

10. If a line L is normal to the conicoid $\frac{x^2}{pa+q} + \frac{y^2}{pb+q} + \frac{z^2}{pc+q} = 1$, prove that the polar line of L w.r.t. the conicoid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$ is normal to L.

11. Prove that the locus of poles of the planes $lx + my + nz = p$ w.r.t the system of conicoids $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$ where λ is a parameter, is a stright line perpendicular to the given plane.

12. Find the locus of the straight lines drawn through a fixed point (f, g, h) whose polar lines w.r.t the conicoids $ax^2 + by^2 + cz^2 = 1$ and $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ are coplanar.

13. Show that the lines $\frac{x}{1} = \frac{y}{4} = \frac{z}{3}$; $\frac{x}{4} = \frac{y}{1} = \frac{z}{-9}$; $\frac{x}{26} = \frac{y}{-28} = \frac{z}{45}$ are three mutually conjugate diameters of the ellipsoid $\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

14. Find the equation of the diameter in the plane $x + y + z = 0$ conjugate to $x = -\frac{1}{2}y = \frac{1}{3}z$ w.r.t the conicoid $3x^2 + y^2 - 2z^2 = 1$. What are the equations of the third conjugate diameter?

15. Show that for the ellipsoid $x^2 + 4y^2 + 5z^2 = 1$, two diameters $\frac{1}{3}x = -\frac{1}{2}y = \frac{1}{4}z$ and $x = 0, 2y = 5z$ are conjugate. Obtain the equations of the third conjugate diameter.

16. From a fixed point H, perpendiculars HA, HB, HC are drawn to the conjugate diameters OP, OQ, OR respectively. Show that $OP^2 . HA^2 + OQ^2 . HB^2 + OR^2 . HC^2$ = constant.

17. Show that if the plane $lx + my + nz = p$ passes through the extremities of conjugate semi diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then $a^2l^2 + b^2m^2 + c^2n^2 = 3p^2$.

18. If the lengths of chords through a given point on the ellipsoid parallel to OP, OQ, OR (conjugate semi diameters) are $2\xi, 2\eta, 2\zeta$ then prove that $\frac{\xi^2}{OP^2} + \frac{\eta^2}{OQ^2} + \frac{\zeta^2}{OR^2} = 1$.

19. Prove that the foot of the perpendicular from the origin on the plane PQR (where OP, OQ, OR are conjugate semi diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$) lies on the surface

$$a^2x^2 + b^2y^2 + c^2z^2 = 3(x^2 + y^2 + z^2)^2$$

II. Find the centre of the following conicoids

1. $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 14y + 6z + 5 = 0$
2. $18x^2 + 2y^2 + 20z^2 - 12zx + 12yz + 2x - 22y - 6z + 7 = 0$
3. $3x^2 - 7yz + x - 8y + 2z + 13 = 0$

Answers

I. 2. $\left(\frac{3}{2}, \frac{3}{2}, 2\right)$

3. $7x - 6y - 4z + 21 = 0,\ 7x - 6y - \frac{1}{2}z + \frac{21}{2} = 0$

12. $\Sigma \frac{(\alpha - a)(b\gamma - c\beta) f}{(x - f)} = 0$

14. $\frac{x}{4} = \frac{y}{-9} = \frac{z}{5};\ \frac{x}{34} = \frac{y}{42} = \frac{z}{3}$

15. $\frac{x}{16} = y = -\frac{z}{2}$

II. 1. $\left(-\frac{16}{19}, -\frac{50}{19}, -\frac{66}{19}\right)$

2. There is no centre

3. $\left(-\frac{1}{6}, \frac{2}{7}, -\frac{8}{7}\right)$

UNIT–23 : RULED SUFRACES – CONE AND CYLINDER

23.0 Contents

23.1 Aims and objectives

After going through this unit, you will be able to :

(i) identity a ruled surface as the union of straight lines on it (generating lines) and

(ii) verify the properties of cone and cylinder interms of generators.

23.2 Ruled surface

If there is a generating line (generator) through every point of a given surface, the surface is called a ruled surface. A ruled surface can be regarded as a union of straight lines which are its generating lines.

A plane, pair of planes, cone and cylinder are examples of ruled surfaces. In this unit, the properties of ruled conicoids and in particular a cone and a cylinder are studied.

Theorem. 1 : If a conicoid has a generator, it is a ruled surface.

Proof : Suppose L is a generator of a conicoid α. Take a point P on α.

If $P \varepsilon L$, then L is a generator through P.

If $P \notin L$, let Π be the plane defined by P and line L. The section of the conicoid α by the plane Π contains line L and point P. Since α is a conicoid, this section is an empty set or the plane Π or a curve of not more than second degree (theorem 4 of unit-21).

$P \varepsilon \alpha \cap \Pi \Rightarrow \alpha \cap \Pi$ is not empty.

$P \varepsilon \alpha \cap \Pi, L \varepsilon \alpha \cap \Pi, P \notin L \Rightarrow \alpha \cap \Pi$ is not a straight line (first degree).

$\therefore$ $\alpha \cap \Pi$ is either the plane Π or a conic.

If $\alpha \cap \Pi = \Pi$, then every line through P in Π is in α. Hence there are generators of α through P.

If $\alpha \cap \Pi$ is a conic, then $L \varepsilon \alpha \cap \Pi$ is a pair of lines. The straight line other than L in this pair passes through P.

Hence there is a generator of α through P. This proves the theorem.

Note : If a Conicoid has one generator, it is enough to conclude that the surface is a rule

For example, the line $z = 0$, $x = y$ is a generator of the hyperbolic paraboloid x^2
it is a ruled surface.

Theorem. 2 : If a conicoid has three concurrent generators, it is a cone or a pair of planes.

Proof : Let the three given generators to the conicoid α through a point p on it be L_1, L_2 and L_3. Then

(i) If P is a singular point of α then α is a cone (theorem 1 of Unit - 22).

(ii) If P is a non-singular point, then all the generators at P are in the tangent plane Π at the point $P \varepsilon \alpha$. Hence L_1, L_2 and L_3 are in $\Pi \cap \alpha$.

$\therefore \Pi \cap \alpha = \Pi$ (theorem 4 of Unit - 21, note iii)

$\therefore \Pi \in \alpha$

Since α is a second degree surface and $\Pi \subset \alpha$, α is a pair of planes.

Theorem. 3 : If a conicoid has three mutually parallel generators, it is a cylinder.

Proof : Let L_1, L_2 and L_3 be three mutually parallel generators of conicoid α. Then

(i) If L_1, L_2 and L_3 determine a plane Π, then $\alpha \cap \Pi$ has three straight lines in it. Hence α is a pair of planes (theorem 4 of Unit - 21, Note iii). The pair of plane is also a cylinder.

(ii) Suppose L_1, L_2 and L_3 are not in a plane. Choose L_1 as the z-axis and a common normal to L_1 and L_2 as the x-axis. Let L_2 cut the xy – plane in a point $(k, 0, 0)$ on the x – axis. Let the line L_3 meet the xy – plane in the point $(\lambda, \mu, 0)$.

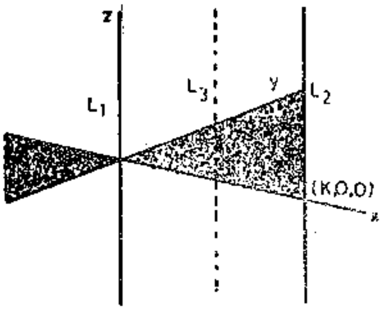


Fig. 1

Since L_1, L_2, L_3 are non coplanar, $k \neq 0$, $\mu \neq 0$. Let the given conicoid α be

$$S(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uv + 2vy + 2wz + d = 0$$

$$L_1 \subset \alpha \Rightarrow S(0, 0, z) = 0 \text{ for any real } z$$

$$\Rightarrow cz^2 + 2wz + d = 0 \text{ for any real } z$$

$$\Rightarrow c = w = d = 0.$$

$$L_2 \subset \alpha \Rightarrow S(k, 0, z) = 0 \text{ for any real } z.$$

$\therefore$ Eqn. (1) and $L_2 \subset \alpha \Rightarrow ak^2 + 2gkz + 2uk = 0$ for any real z.

$$\Rightarrow g = 0 \qquad \dots (2)$$

$$L_2 \subset \alpha \Rightarrow S(\lambda, \mu, z) = 0 \text{ for any real } z$$

$$(1), (2) \text{ and } L_3 \subset \alpha \Rightarrow 2f\mu z + S(\lambda, \mu, 0) = 0 \text{ for any real } z.$$

$$\Rightarrow f = 0$$

(1), (2) and (3) $\Rightarrow$ The equation of α, namely $S(x, y, z) = 0$ is independent of z.

$\Rightarrow$ α is a cylinder with z – axis as the axis.

Theorem. 4 : A cylinder with z – axis as axis is represented by an equation independent of z. (Converse of theorem 3 of Unit – 22).

Proof : Let z – axis be the axis of the cylinder α given by $f(x, y, z) = 0$. If the xy plane $(z = 0)$ is denoted by Π and $f(x, y, 0)$ by $F(x, y)$, then the equation of $\alpha \cap \Pi$ is $F(x, y) = 0, z = 0$.

$$\therefore (x, y, z) \in \alpha \Leftrightarrow (x, y, 0) \in \alpha \; (\because z \text{ axis is axis of the cylinder})$$

$$\Leftrightarrow (x, y, 0) \in \alpha \cap \Pi$$

$$\Leftrightarrow F(x, y) = 0$$

$\therefore$ α is represented by the equation $F(x, y) = 0$.

Theorem. 5 : The equation of a second degree cone with origin as vertex is homogeneous and of second degree in x, y, z.

(Converse of theorem 2 of Unit 20 in the case of conicoid).

Proof : Let $S(x, y, z) \equiv H(x, y, z) + 2ux + 2vy + 2wz + d = 0$ be the given cone α with origin O as vertex. By the definition of a cone

$$P \neq 0, P \in \alpha \Rightarrow \overleftrightarrow{OP} \subset \alpha.$$

$\therefore (x, y, z) \in \alpha \Rightarrow (tx, ty, tz) \in \alpha$ for any real t.

In particular $(-x, -y, -z) \in \alpha$ and hence

$$S(x, y, z) = 0 = S(-x, -y, -z).$$

$$\therefore S(x, y, z) - S(-x, -y, -z) = 0$$

$$\therefore ux + vy + wz = 0 \text{ for every } (x, y, z) \in \alpha. \qquad \ldots (1)$$

$$\text{Also } (0, 0, 0) \in \alpha \Rightarrow S(0, 0, 0) = 0$$

$$\Rightarrow d = 0 \qquad \ldots (2)$$

Let one of u, v, w be non-zero. For if all of u, v, w are zero, the theorem is proved. Since the cone is second degree (hypothesis), $H(x, y, z) \not\equiv 0$

$\therefore (ux + vy + wz)\, H(x, y, z) \not\equiv 0$

Hence there exist real numbers x_0, y_0, z_0 such that

$$(ux_0 + vy_0 + wz_0)\, H(x_0, y_0, z_0) \neq 0$$

If $H(x_0, y_0, z_0) = \lambda$ and $L(x_0, y_0, z_0) = ux_0 + vy_0 + wz_0 = \mu$, then

$$H(tx_0, ty_0, tz_0) = t^2 H(x_0, y_0, z_0) = t^2 \lambda$$

($\because H$ is homogeneous second degree polynomial)

$$\text{and } L(tx_0, ty_0, tz_0) = t \cdot L(x_0, y_0, z_0) = t\mu$$

($\because L$ is homogeneous first degree polynomial)

$$\text{Now } S(tx_0, ty_0, tz_0) = H(tx_0, ty_0, tz_0) + 2L(tx_0, ty_0, tz_0)$$

$$= t^2 \lambda + 2t\mu = t(t\lambda + 2\mu).$$

Choosing $t = -2\mu\lambda$, $S(tx_0, ty_0, tz_0) = 0$ and hence $(tx_0, ty_0, tz_0) \in \alpha$ for this value of t. But then

$$L(tx_0, ty_0, tz_0) = t\mu = -2\mu^2/\lambda \neq 0, \text{ for a point } (tx_0, ty_0, tz_0) \in \alpha.$$

This contradicts (1) and hence $u = v = w = 0$. ... (3)

$$\therefore S(x, y, z) \equiv H(x, y, z) = 0$$

is the equation of the required cone.

Definition : (i) Straight lines and pairs of planes are called degenerate conicoids. Others are called non-degenerate conicoids.

(ii) If a conicoid is not a cone or a cylinder, it is called a proper conicoid.

Note : (i) A single point set (point sphere, point cone) is a non-degenerate conicoid.

(ii) All proper conicoids are non-degenerate.

(iii) Ellipsoids, Praboloids and Hyperboloids are known conicoids. In Unit - 24 it is shown that they are the only proper conicoids.

Definition : If a plane not passing through the vertex of a non-degenerate cone cuts all its generators, then the curve of intersection is called a base curve of the cone by the plane.

Note : The set of all straight lines obtained by joining vertex to the points of a base curve is the given cone.

23.3 Equation of a cone when vertex and base are given

Let $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ z = 0$ be the given base conic. The equation of any straight line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n};\ n \neq 0 \qquad \dots (1)$$

This line meets the plane $z = 0$ in the point

$\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$. This will lie on the conic if

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2$$
$$+ 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0 \qquad \dots (2)$$

Writing $\frac{l}{n} = \frac{x-\alpha}{z-\gamma}$; $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$ from (1) and using them in (2), the following equation is obtained.

$$a\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)^2 + 2h\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)$$
$$+ b\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2 + 2g\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right) + 2f\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + c = 0$$

or $a(\alpha z - x\gamma)^2 + 2h(\alpha z - x\gamma)(\beta z - y\gamma) + b(\beta z - y\gamma)^2$

$+ 2g(\alpha z - x\gamma)(z - \gamma) + 2f(\beta z - y\gamma)(z - \gamma) + c(z - \gamma)^2 = 0$ is the required cone.

23.4 Plane sections of a cone

Consider the circular cone

$$x^2 + y^2 = z^2 \qquad \dots (1)$$

This is a surface of revolution of the pair of intersecting straight lines $z = \pm x$ about the z-axis. These straight lines make an angle $\frac{\pi}{4}$ with the z – axis.

(a) *Circle* : The section of cone (1) by a plane $z = a$ is a circle $x^2 + y^2 = a^2$, of radius a. Since a is arbitrary, a circle of required radius can be obtained (see figure–2).

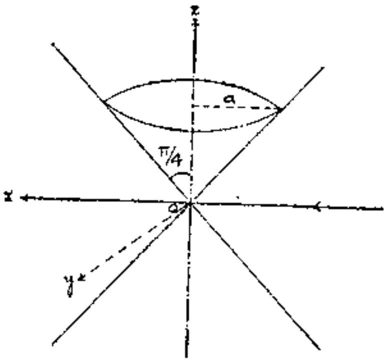


Fig. 2

(b) *Pairs of intersecting lines* : Define a new set of axes $OXYZ$ by the transformations

$$x = X$$
$$y = Y\cos\theta + Z\sin\theta$$
$$z = -Y\sin\theta + Z\cos\theta$$

(rotation about the x – axis by an angle θ from the z – axis towards y – axis)

The equation of the cone in the new system is

$$X^2 + \left(Y^2 - Z^2\right)\cos 2\theta + 2YZ\sin 2\theta = 0 \quad \ldots (2)$$

The section of this cone by the XY plane is

$$X^2 + Y^2\cos 2\theta = 0,\ Z = 0$$

For $\frac{\pi}{4} < \theta \le \frac{\pi}{2}$, $\cos 2\theta < 0$ and the section is

$$X^2 - \alpha^2 Y^2 = 0,\ Z = 0\ \left(-\alpha^2 = \cos 2\theta\right)$$

∴ The section is the pair of intersecting lines $X = \pm\alpha Y$. By choosing θ properly, the straigt lines with required angle of intersection are obtained.

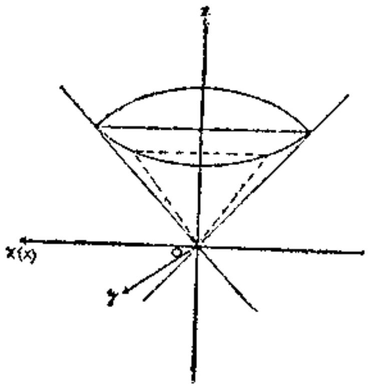


Fig. 3

(dotted lines show the XY plane)

(c) *Ellipses* : The section of the cone (2) by the plane

$Z = c$ is $X^2 + Y^2\cos 2\theta + 2cY\sin 2\theta = c^2\cos 2\theta;\ Z = c.$

$$\text{or, } \frac{X^2}{c^2\sec 2\theta} + \frac{(Y + c\tan 2\theta)^2}{c^2\sec^2 2\theta} = 1,\ Z = c. \quad \ldots (3)$$

If $0 \le \theta < \frac{\pi}{4}$, $\sec 2\theta$ is greater than zero and hence this section (3) is an ellipse. For any given a and b such that $0 < a \le b$, choosing $c = \frac{a^2}{b}$, $\theta = \frac{1}{2}\cos^{-1}\frac{a^2}{b^2}$, (3) represents an ellipse with a and b as principal semi axes.

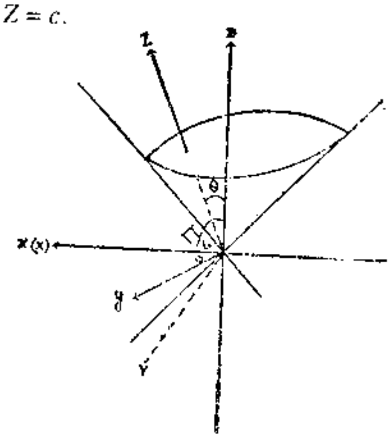


Fig. 4

(d) *parabolas* : Choosing $\theta = \frac{\pi}{4}$ equation (2) reduces to $X^2 + 2YZ = 0$. The Y and Z axes are now generators of the cone. The section of the cone by the plane $Z = c$ is

$$X^2 + 2cY = 0, Z = c.$$

Choosing c properly, *a* parabola of required latus rectum is obtained.

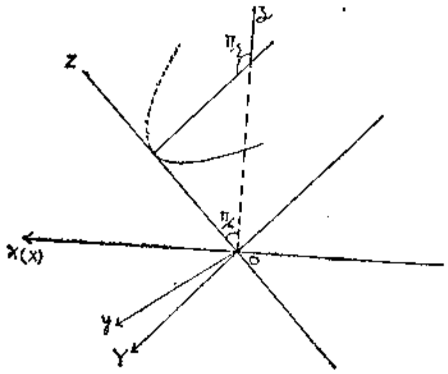


Fig. 5

(e) *Hyperbolas* : The equation $\frac{x^2 + y^2}{b^2} = \frac{z^2}{c^2}$ represents a circular cone. The section by the plane $x = b$ is

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1.$$

This is a hyperbola with b, c as principal semi axes.

The following results on cones of second degree are noteworthy.

1. If a cone α has two vertices A and B, then

(i) $\alpha = \overleftrightarrow{AB}$ or α is a pair of planes intersecting in $\overleftrightarrow{AB}$.

(ii) Every point on $\overleftrightarrow{AB}$ is a vertex.

2. The cone $H(x, y, z) = 0$ is a non-degenerate cone $\Leftrightarrow$ Origin is the only vertex

$$\Leftrightarrow D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

(f) If a plane cuts a cone in a pair of lines, to find the angle between the lines

Notation :

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$A = bc - f^2, B = ca - g^2, C = ab - h^2$

$F = gh - af, G = hf - bg, H = fg - ch.$

It can be verified that

$BC - F^2 = aD, CA - G^2 = bD, AB - H^2 = cD$

$GH - AF = fD,\ HF - BG = gD,\ FG - CH = hD.$

$$K = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

$$= -\left(Au^2 + Bv^2 + Cw^2 + 2Fuv + 2Gwu + 2Huv\right).$$

Suppose the plane $ux + vy + wz = 0$ cuts the given cone.

$H(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ in a pair of intersecting lines.

If the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies in the plane, then

$$ul + vm + wn = 0. \qquad \dots (1)$$

If the lines lies on the cone,

$$H(l, m, n) = 0. \qquad \dots (2)$$

Eliminating n between (1) and (2),

$$l^2\left(cu^2 + aw^2 - 2gwu\right) + 2lm\left(hw^2 + cuv - fuw - gvw\right)$$
$$+ m^2\left(cv^2 + bw^2 - 2fvw\right) = 0 \qquad \dots (3)$$

The direction cosines (l, m, n) of the two lines of section satisfy (1) and (2) and therefore they satisfy (3). If the direction cosines of these two lines are l_1, m_1, n_1 and l_2, m_2, n_2 then $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of,

$$\left(\frac{l^2}{m^2}\right)\beta + 2\left(\frac{l}{m}\right)\delta + \alpha = 0$$

where

$$\alpha = bw^2 + cv^2 - 2fvw$$
$$\beta = cu^2 + aw^2 - 2gwu$$
$$\gamma = av^2 + bu^2 - 2huv,$$
$$\delta = hw^2 + cuv - fuw - gvw.$$

Then,

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{\alpha}{\beta} \qquad \dots (4)$$

$$\frac{l_1}{m_1} + \frac{l_2}{m_2} = \frac{l_1 m_2 + l_2 m_1}{m_1 m_2} \qquad \dots (5)$$

From (4) and (5)

$$\frac{l_1 l_2}{\alpha} = \frac{m_1 m_2}{\beta} = \frac{l_1 m_2 + l_2 m_1}{-2\delta} = \lambda \text{ (say)} \qquad \dots (6)$$

$$\therefore \left(l_1 m_2 - l_2 m_1\right)^2 = \left(l_1 m_2 + l_2 m_1\right)^2 - 4 l_1 l_2 m_1 m_2$$

$$= (-2\delta\lambda)^2 - 4\lambda\alpha . \lambda\beta$$

$$= 4\lambda^2\left(\delta^2 - \alpha\beta\right) = 4\lambda^2 w^2 k^2$$

$$\therefore \frac{l_1 m_2 - l_2 m_1}{\pm 2wk} = \lambda \qquad \ldots (7)$$

From (6) and (7)

$$\frac{l_1 l_2}{\alpha} = \frac{m_1 m_2}{\beta} = \frac{l_1 m_2 - l_2 m_1}{\pm 2wk} \qquad \ldots (8)$$

From symmetry, each of the expressions in (8) is equal to,

$$\frac{n_1 n_2}{\gamma} = \frac{m_1 n_2 - m_2 n_1}{\pm 2uk} = \frac{n_1 l_2 - n_2 l_1}{\pm 2vk}$$

If θ is the angle between the lines, then

$$\frac{\cos\theta}{l_1 l_2 + m_1 m_2 + n_1 n_2} = \frac{\sin\theta}{\left\{\Sigma\left(m_1 n_2 - m_2 n_1\right)^2\right\}^{\frac{1}{2}}}$$

$$\therefore \frac{\cos\theta}{(a+b+c)\left(u^2+v^2+w^2\right) - H(u, v, w)} = \frac{\sin\theta}{\pm 2\left(u^2+v^2+w^2\right)^{\frac{1}{2}} k}$$

Theorem 6 : The necessary and sufficient condition that the cone $H(x, y, z) = 0$ has three mutually perpendicular generators is $a + b + c = 0$.

Necessity : Suppose the cone has three mutually perpendicular generators. Let the plane through two of them be, $ux + vy + wz = 0$. Then this plane cuts the cone in perpendicular generators.

$$\therefore \cos\theta = 0$$

$$\therefore (a+b+c)\left(u^2+v^2+w^2\right) = H(u, v, w) \qquad \ldots (1)$$

The normal to the plane at the origin also lies on the cone

$$\therefore H(u, v, w) = 0 \qquad \ldots (2)$$

From (1) and (2), $\quad a + b + c = 0$

Sufficiency : If $a + b + c = 0$, consider any plane $ux + vy + wz = 0$ whose normal at the origin is a generator of the cone.

$$\text{Then} \quad (u, v, w) = 0.$$

$$\therefore (a+b+c)\left(u^2+v^2+w^2\right) = (u, v, w)$$

Hence $\cos\theta = 0$ or $\theta = \frac{\pi}{2}$.

$\therefore$ The plane through the origin which is normal to a generator cuts the cone in perpendicular lines. Hence there are two generators of the cone at right angles to each other and at right angles to any given generator.

There are an infinite number of such sets of three mutually perpendicular generators.

23.5 Tangent planes

For the cone $\quad H(x, y, z) = 0,$

$$u = v = w = d = 0$$

$$\therefore \delta_0 = 0 \text{ for any point } \left(x_0, y_0, z_0\right).$$

The equation of the tangent plane at $\left(x_0, y_0, z_0\right)$ on the cone is

$$\xi_0 x + \eta_0 y + \zeta_0 z = 0$$

or, $a x_0 x + b y_0 y + c z_0 z + f\left(y_0 z + z_0 y\right) + g\left(z_0 x + x_0 z\right) + h\left(x_0 y + y_0 x\right) = 0.$

Theorem. 7 : All the tangent planes of a cone pass through any vertex of the cone.

Proof : Choosing the given vertex as origin, the equation of the cone is $H(x, y, z) = 0$. The tangent plane at $\left(x_0, y_0, z_0\right)$ different from the vertex is $x\xi_0 + y\eta_0 + z\zeta_0 = 0$. This plane passes through the vertex (0, 0, 0). Since this is true for all vertices and all tangent planes, the theorem follows.

Note : The polar plane of a point $\left(x_0, y_0, z_0\right)$ different from vertex (centre) w.r.t. the cone $H(x, y, z) = 0$ is

$$x\xi_0 + y\eta_0 + z\zeta_0 = 0.$$

All polar planes pass through every vertex of the cone.

Theorem. 8: The tangent planes of a cylinder are parallel to any axis of the cylinder.

Proof : Choosing any axes of the cylinder as z-axis, the equation of the cylinder is of the form $ax^2 + 2hxy + by^2 + 2ux + 2vy + d = 0$. At a point $\left(x_0, y_0, z_0\right)$ on the cylinder, $\zeta_0 = gx_0 + fy_0 + cz_0 + w = 0$ because g, f, c, w are all zero. Hence the tangent plane at $\left(x_0, y_0, z_0\right)$, namely $x\xi_0 + y\eta_0 + z\zeta_0 + \delta_0 = 0$ reduces to $x\xi_0 + y\eta_0 + \delta_0 = 0$. This is parallel to the z-axis. Since any given axis of the cylinder can be chosen as the z-axis, the theorem is proved.

Note : The polar plane of any point different from the centre, w.r.t. a cylinder is parallel to any axis of the cylinder.

Note : The section of a non-degenerate cone or cylinder by a tangent plane is a generator.

Theorem. 9 : The necessary and sufficient condition for the plane $lx + my + nz = 0$ to be a tangent plane of the non-degenerate cone $H(x, y, z) = 0$ is

$$k = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0 \qquad \ldots (1)$$

First we note that a system of equations in n variables $x_1, x_2, \ldots x_n$ namely

$$a_{11} x_1 + \ldots + a_{1n} x_n = 0$$

$$a_{21} x_1 + \ldots + a_{2n} x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1} x_1 + \ldots + a_{nn} x_n = 0$$

has a non-trival solution $\left(x_1, x_2, \ldots x_n\right)$ if and only if the determinant of the matrix $\left|\left(a_{ij}\right)\right| = 0$.

Proof :

Necessity : The given plane is $lx + my + nz = 0$ where l, m, n are not all zero. ... (2)

If this plane is a tangent plane at (x_0, y_0, z_0), its equation is

$$\xi_0 x + \eta_0 y + \zeta_0 z = 0$$

This is same as $lx + my + nz = 0$

$\therefore \xi_0 = -\lambda l, \eta_0 = -\lambda m, \zeta_0 = -\lambda n$ ... (3)

for some non-zero real number λ. Therefore, the equations

$$\left.\begin{aligned} ax + hy + gz + lt &= 0 \\ hx + by + fz + mt &= 0 \\ gx + fy + cz + nt &= 0 \\ lx + my + nz &= 0 \end{aligned}\right\} \quad \dots (4)$$

have a non-zero solution (x_0, y_0, z_0, λ).

$$\therefore k = 0.$$

Sufficiency : If $k = 0$, then the system of equations (4) have a non-zero solution (x_0, y_0, z_0, λ). [by (1)].

$$\therefore \text{if } \lambda = 0, \text{ then } (x_0, y_0, z_0) \neq (0, 0, 0).$$

Also, $\xi_0 = \eta_0 = \zeta_0 = 0$ [from (3)]

$H(x_0, y_0, z_0) = \xi_0 x_0 + \eta_0 y_0 + \zeta_0 z_0 = 0$. Hence (x_0, y_0, z_0) is on the cone.

$\therefore (x_0, y_0, z_0) \neq (0, 0, 0)$ is another vertex of the cone.

This contradicts the hypothesis that the cone is non degenerate. Hence $\lambda \neq 0$.

Also if $\lambda \neq 0$,

$$x_0 = y_0 = z_0 = 0 \Rightarrow \xi_0 = 0, \eta_0 = 0, \zeta_0 = 0$$

$$\Rightarrow l = m = n = 0$$

This contradicts (2). Hence $\lambda \neq 0$, $(x_0, y_0, z_0) \neq (0, 0, 0)$

For the point (x_0, y_0, z_0) determined by the solution (x_0, y_0, z_0, λ),

$\xi_0 = -\lambda l, \eta_0 = -\lambda m, \zeta_0 = -\lambda n$

and $lx_0 + my_0 + nz_0 = 0$

$\therefore H(x_0, y_0, z_0) = \xi_0 x_0 + \eta_0 y_0 + \zeta_0 z_0 = -\lambda (lx_0 + my_0 + nz_0) = 0$ and hence the point (x_0, y_0, z_0) is on the cone. The equation of the tangent plane at this point is

$$\xi_0 x + \eta_0 y + \zeta_0 z = 0$$

$$\text{i.e., } -\lambda (lx + my + nz) = 0$$

$$\text{i.e., } lx + my + nz = 0.$$

23.6 Reciprocal Cones

If $k = 0$, then

$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0$ ... (1)

Then $\sin \theta = 0$ and hence $\theta = 0$. The two lines of section coincide. The plane $ux + vy + wz = 0$ touches the given non-degenerate cone. Equation (1) shows that the line $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$ i.e., the normal to the plane through the origin, is a generator of the cone.

$$H^* (x, y, z) = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \dots (2)$$

Using equations $BC - F^2 = aD$ etc., given in 23.5, it follows that a normal through the origin to a tangent plane of the cone (2) is a generator of the given cone

$$H (x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \dots (3)$$

The cones (2) and (3) are such that they have the same vertex (origin) and the locus of the normals drawn through the origin to the tangent planes of any one of them is the other cone.

Definition : Two non-degenerate cones with the same vertex such that the generators of any one of them are normals to the tangent planes of the other cone are called reciprocal cones.

Theorem. 10 : The necessary and sufficient condition for the non-degenerate cone. $H (x, y, z) = 0$ to have three mutually perpendicular tangent planes is $A + B + C = 0$.

Proof: (i) Suppose the cone $H (x, y, z) = 0$ has three mutually perpendicular tangent planes. Then its reciprocal cone $H^* (x, y, z) = 0$ has three mutually perpendicular generators. Hence by theorem 9, $A + B + C = 0$.

(ii) Suppose $A + B + C = 0$. Then by theorem 9, the cone $H^* (x, y, z) = 0$ has three mutually perpendicular generators. Hence its reciprocal cone $H (x, y, z) = 0$ has three mutually perpendicular tangent planes.

Note : For degenerate cone, there cannot be three distinct tangent planes at all.

Examples

1. Prove that the equation $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$ represents a cone whose vertex is $(-1, -2, -3)$.

If the equation can be reduced to a homogeneous equation by a suitable choice of coordinate axes, then it is a cone. Using the transformations

$$x = X + \alpha$$
$$y = Y + \beta$$
$$z = Z + \gamma$$

the equaion transforms to

$$4 (X + \alpha)^2 - (Y + \beta)^2 + 2 (Z + \gamma)^2 + 2 (X + \alpha) (Y + \beta)$$
$$- 3 (Y + \beta) (Z + \gamma) + 12 (X + \alpha) - 11 (Y + \beta) + 6 (Z + \gamma) + 4 = 0 \quad \dots (1)$$

If this is a cone, then the coefficients of X, Y and Z and the independent term are all zero

i.e.
$$8\alpha + 2\beta + 12 = 0 \quad \dots (2)$$
$$2\alpha - 2\beta - 3\gamma = 11 \quad \dots (3)$$
$$- 3\beta + 4\gamma = -6 \quad \dots (4)$$
$$4\alpha^2 - \beta^2 + 2\gamma^2 + 2\alpha\beta - 3\beta\gamma + 12\alpha - 11\beta + 6\gamma + 4 = 0 \quad \dots (5)$$

 Solving (2), (3) and (4), $\alpha = -1, \beta = -2, \gamma = -3$

These values satsify (5). Hence the equation (1) reduces to the form

$$aX^2 + bY^2 + cZ^2 + 2fYZ + 2gZX + 2hXY = 0$$

which is a cone with vertex (0, 0, 0) in the new system. In the old coordinate system, the vertex is given by (α, β, γ) or $(-1, -2, -3)$.

2. Find the equations of the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be the equation of any one of the lines in which the given plane meets the given cone.

Then

$$2l + m - n = 0$$
$$4l^2 - m^2 + 3n^2 = 0$$

These two equations are to be solved for l, m, n. Eliminating n, the following equation is obtained.

$$4l^2 - m^2 + 3(2l + m)^2 = 0$$

$$\text{i.e. } 8l^2 + 6lm + m^2 = 0$$

$$\therefore \frac{l}{m} = \frac{-6 \pm \sqrt{36 - 32}}{16} = -\frac{1}{4} \text{ or } -\frac{1}{2}$$

Also, $$2\frac{l}{m} + 1 - \frac{n}{m} = 0$$

(i) $\frac{l}{m} = -\frac{1}{4}$ gives $\frac{n}{m} = \frac{1}{2}$

$\therefore \frac{l}{-1} = \frac{m}{4} = \frac{n}{2}$

(ii) $\frac{l}{m} = -\frac{1}{2}$ gives $\frac{n}{m} = 0$ (i.e., $n = 0$)

$\therefore \frac{l}{-1} = \frac{m}{2}, n = 0$

The two required lines are

$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{-1} = \frac{y}{2}, z = 0$

3. If the plane $lx + my + nz = 0$ cuts the cones $ax^2 + by^2 + cz^2 = 0$ and $fyz + gzx + hxy = 0$ in the same pair of lines, prove that

$$\frac{bn^2 + cm^2}{fmn} = \frac{cl^2 + an^2}{gnl} = \frac{am^2 + bl^2}{hlm}$$

The plane is $lx + my + nz = 0$ ... (1)

The cones are $ax^2 + by^2 + cz^2 = 0$ ... (2)

and $fyz + gzx + hxy = 0$ ... (3)

The lines in which (1) and (2) intersect are given by

$$ax^2 + by^2 + c\left[\frac{lx + my}{-n}\right]^2 = 0 \quad \text{... (4)}$$

The lines in which (1) and (3) intersect are given by

$$(fy + gx)\left(\frac{lx + my}{-n}\right) + hxy = 0 \quad \text{... (5)}$$

Since (4) and (5) represent the same pair of lines, simplifying (4) and (5) and comparing,

$$\frac{an^2 + cl^2}{gl} = \frac{bn^2 + cm^2}{fm} = \frac{2lmc}{fl + gm - hn} \quad \ldots (6)$$

Similarly, writing $x = \frac{my + nz}{-l}$ and eliminating x from (2), and (3),

$$\frac{am^2 + bl^2}{hm} = \frac{an^2 + cl^2}{gn} = \frac{2mna}{gm + hn - fl} \quad \ldots (7)$$

from (6) and (7),

$$\frac{bn^2 + cm^2}{fmn} = \frac{cl^2 + an^2}{gnl} = \frac{am^2 + bl^2}{hmn} \quad \ldots (8)$$

4. Prove that the locus of points from which three mutually perpendicular lines can be drawn to meet the conic $ax^2 + by^2 = 1, z = 0$ is

$$ax^2 + by^2 + (a + b) z^2 = 1$$

from a point $P (\alpha, \beta, \gamma)$ three mutually perpendicular lines can be drawn to cut the given conic $\Leftrightarrow$ The cone with the given conic as base curve and $P (\alpha, \beta, \gamma)$ as vertex has three mutually perpendicular generators

$\Leftrightarrow$ The cone $a (\alpha z - \gamma x)^2 + b (\beta z - \gamma y)^2 - (z - \gamma)^2 = 0$ has three mutually perpendicular generators.

$\Leftrightarrow a\alpha^2 + b\beta^2 + (a + b) \gamma^2 - 1 = 0$ (from theorem 7)

$\therefore$ The locus of (α, β, γ) is

$$ax^2 + by^2 + (a + b) z^2 - 1 = 0.$$

5. Tangent planes to the cone

$H (x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ are drawn through a line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ to the cone in lines L_1 and L_2. Prove that the plane through L_1 and L_2 is

$$x (al + hm + gn) + y (hl + bm + fn) + z (gl + fm + cn) = 0$$

The tangent plane at any point (α, β, γ) of the given cone is

$$x (a\alpha + h\beta + g\gamma) + y (h\alpha + b\beta + f\gamma) + z (g\alpha + f\beta + c\gamma) = 0$$

it will contain the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ if

$$l (a\alpha + h\beta + g\gamma) + m (h\alpha + b\beta + f\gamma) + n (g\alpha + f\beta + c\gamma) = 0$$

i.e. $\alpha (al + hm + gn) + \beta (hl + bm + fn) + \gamma (gl + fm + cn) = 0$

The point (α, β, γ) lies on the plane

$$x (al + hm + gn) + y (hl + bm + fn) + z (gl + fm + cn) = 0$$

6. Show that the locus of the line of the intersection of tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ which touch the cone along perpendicular generators is the cone

$$a^2 (b + c) x^2 + b^2 (c + a) y^2 + c^2 (a + b) z^2 = 0.$$

Let the tangent planes to the cone, along the two perpendicular generators meet in the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

The equation of the plane containing the two generators is (by example 5),

$$alx + bmy + cnz = 0$$

Since the plane cuts the cone in perpendicular generators,

$$(a + b + c)(a^2l^2 + b^2m^2 + c^2n^2) = a(al)^2 + b(bm)^2 + c(cn)^2$$

$$\therefore (b + c)a^2l^2 + (c + a)b^2m^2 + (a + b)c^2n^2 = 0$$

$\therefore$ The locus of the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is the cone

$$(b + c)a^2x^2 + (c + a)b^2y^2 + (a + b)c^2z^2 = 0$$

7. Show that the general equation of the cone which touches the three coordinate planes is

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

Consider a cone which touches the three coordinate planes. Its reciprocal cone has the coordinate axes as generators. The equation of this cone is of the form $fyz + gzx + hxy = 0$

[$\because H(x, y, z) = 0$ is satisfied by $y = 0 = z$. Hence $a = 0$, Similarly $b = c = 0$]

The reciprocal cone is

$$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$$

i.e. $(fx + gy - hz)^2 = 4fg\,xy$

$$\therefore fx + gy - hz = \pm 2\sqrt{fgxy}$$

$$\therefore \left(\sqrt{fx} \pm \sqrt{gy}\right)^2 = \left(\pm\sqrt{hz}\right)^2$$

$$\therefore \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

8. Find the equation of the quadric cone whose vertex is at the origin and which passes through the curve given by the equations.

$$ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p.$$

Since the required equation is homogeneous, of second degree, and satisfies the equations

$$ax^2 + by^2 + cz^2 = 1$$

and $$\frac{lx + my + nz}{p} = 1,$$

the required equation is

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2$$

which gives on simplification, the cone

$$(ap^2 - l^2)x^2 + (bp^2 - m^2)y^2 + (cp^2 - n^2)z^2$$

$$- 2lmxy - 2mnyz - 2nlzx = 0$$

23.7 Summary

If a line lies on a surface, then the surface becomes a ruled surface. If a surface has three concurrent lines on it then it will be either a cone or a pair of planes. If a surface has three generators which are parallel to each other then the conicoid is a cylinder. If a quadratic cone has its vertex as origin then its equation is homogeneous in x, y, z. Straight lines, pair of planes are degenerate conicoids. Conicoids which are not degenerate are called nondegenerate conicoids.

23.8 Sample Examination Questions

I. Answer the following Questions in detail

1. Prove that the equation $x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$ represents a cone with vertex $(1, -2, 3)$.

2. Find the equation of the cone with vertex at the origin and passes through the curve given below.
   (i) $x^2 + y^2 + z^2 + 2ax + b = 0, \ lx + my + nz = p$
   (ii) $ax^2 + by^2 = 2z, \ lx + my + nz = p$
   (iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 2z$

3. Find the equations of the lines of intersection of the following planes and cones
   (i) $x + 3y - 2z = 0; \ x^2 + 9y^2 - 4z^2 = 0$
   (ii) $3x + 4y + z = 0, \ 15x^2 - 32y^2 - 7z^2 = 0$
   (iii) $x + 7y - 5z = 0, \ 3yz + 14zx - 30xy = 0.$

4. Find the equation of the cone whose vertex is (α, β, γ) and base curve
   (i) $ax^2 + by^2 = 1, \ z = 0$
   (ii) $y^2 = 4ax, \ z = 0$

5. Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ z = 0$ is the sphere $x^2 + y^2 + z^2 = a^2 + b^2$.

6. Find the angle between the lines of intersection of the following planes and cones.
   (i) $6x - 10y - 7z = 0, \ 108x^2 - 20y^2 - 7z^2 = 0$
   (ii) $3x + y + 5z = 0, \ 6yz - 2zx + 5xy = 0.$

7. If the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines prove that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

8. Prove that the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has three mutually perpendicular tangent planes if $bc + ca + ab = f^2 + g^2 + h^2$.

9. Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal cones.

10. Prove that the cones $ayz + bzx + cxy = 0$ and $\sqrt{ax} + \sqrt{by} + \sqrt{cz} = 0$ are reciprocal cones.

Answers

2. (i) $\left(x^2 + y^2 + z^2\right)p^2 + 2apx\,(lx + my + nz) + b\,(lx + my + nz)^2 = 0$
   (ii) $\left(ax^2 + by^2\right)p = 2z\,(lx + my + nz)$
   (iii) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)4z^2 = \left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}\right)^2$

3. (i) $x = 2z, \ y = 0; \ 3y = 2z, \ x = 0.$
   (ii) $\frac{x}{-3} = \frac{y}{2} = \frac{z}{1}; \ \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$
   (iii) $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}; \ \frac{x}{3} = \frac{y}{1} = \frac{z}{-2}.$

4. (i) $z^2\left(a\alpha^2 + b\beta^2 - 1\right) - 2z\gamma\,(a\alpha x + b\beta y - 1) + \gamma^2\left(ax^2 + by^2 - 1\right) = 0$
   (ii) $z^2\left(\beta^2 - 4a\alpha\right) - 2z\gamma\,(\beta y - 2a\,(x + \alpha)) + \gamma^2\left(y^2 - 4ax\right) = 0.$

6. (i) $\cos^{-1}(16/21)$ (ii) $\cos^{-1}$ (1

UNIT-24 : REDUCTION AND CLASSIFICATION

24.0 Contents

24.1 Aims and Objectives

After going through this unit, you will be able to :

(i) reduce the equation of a conicoid to a standard form and classify the conicoids.

24.2 The general equation of second degree in x, y, z.

The general equation of second degree in x, y, z is

$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$

$$\text{Let } \Delta = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \text{ and}$$

$$D_\lambda = D(\lambda) = \begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}$$

$$D_0 = D(0) = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Expansion of the determinant D_λ gives

$$D_\lambda = D_0 - \lambda(A + B + C) + \lambda^2(a + b + c) - \lambda^3.$$

This is a cubic equation in λ. The coefficients

$$-1, a + b + c, -(A + B + C), D_0$$

are invariant under transformation of coordinate axes.

If $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic $D_\lambda = 0$, then

$$\lambda_1 + \lambda_2 + \lambda_3 = a + b + c, \quad \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = A + B + C, \quad \lambda_1\lambda_2\lambda_3 = D_0.$$

The cofactors of $a - \lambda$, $b - \lambda$, $c - \lambda$, f, g, h in D_λ are denoted by $A_\lambda, B_\lambda, C_\lambda, F_\lambda, G_\lambda, H_\lambda$, and the cofactors of a, b, c, f, g, h in D_0 by A, B, C, F, G, H.

Theorem : If the diameter plane of a stright line whose direction ratios are l, m, n is perpendicular to the straight line, then

$$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n}$$

[illegible] with direction ratios l, m, n has a diametral plane, its equation is

$$l\xi + m\eta + n\zeta = 0$$

or $(al + hm + gn)\ x + (hl + bm + fn)\ y + (gl + fm + cn)\ z + (ul + vm + wn) = 0.$

If this is perpendicular to the given line, the condition in the statement of the theorem is necessary.

Definition : If the direction cosines l, m, n of a straight line satisfy

$$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n}.$$

then the line is called a principal line w.r.t. the given conicoid $S = 0$. The vector (l, m, n) is called a principal vector.

Note : (i) There may be no chords parallel to a principal line. But if there are chords parallel to a principal line, the middle points of these chords determine the diametral plane of the given line w.r.t. the conicoid. This diametral plane is perpendicular to the principal line.

(ii) If a straight line is a principal line, all lines parallel to it are principal lines.

(iii) (l, m, n) is a principal vector if and only if there is a real number λ such at

$$\left.\begin{aligned} al + hm + gn &= \lambda l \\ hl + bm + fn &= \lambda m \\ gl + fm + cn &= \lambda n \end{aligned}\right\} \qquad \dots (1)$$

If $\lambda = 0$, there are no chords of the conicoid parallel to (l, m, n).

Example : The z – axis is a principal line of the paraboloid $ax^2 + by^2 = 2z$. There are no chords parallel to this principal line.

24.2.1 The roots of the equation $D_t = 0$ are real

Suppose k is a complex root of $D_t = 0$. Then $D_k = 0$. Therefore the equations in unknowns l, m, n, namely

$$(a - k)l + hm + gn = 0 \qquad \dots (i)$$

$$hl + (b - k)\ m + fn = 0 \qquad \dots (ii)$$

$$gl + fm + (c - k)\ n = 0 \qquad \dots (iii)$$

have a non-trivial solution (l, m, n). Let $(\bar{l}, \bar{m}, \bar{n})$ be the complex conjugates of (l, m, n). Multiplying the equations (i), (ii), (iii) respectively by $\bar{l}, \bar{m}, \bar{n}$ and adding, the following equation is obtained

$$al\bar{l} + bm\bar{m} + cn\bar{n} + f(m\bar{n} + \bar{m}n) + g(n\bar{l} + \bar{n}l)$$
$$+ h(l\bar{m} + \bar{l}m) - k(l\bar{l} + m\bar{m} + n\bar{n}) = 0.$$

The coefficients of a, b, c, f, g, h in this equation are real numbers. The coefficient of $-k$ is a positive number because l, m, n are not all zero. Hence k is a real number.

$$\therefore\ D_k = 0 \Rightarrow k \text{ is a real number.}$$

Definition : The equation $D_\lambda = 0$ is called the characteristic equation of the conicoid. The roots of $D_\lambda = 0$ are called characteristic roots.

Corresponding to a root λ, a vector (l, m, n) satisfying the system of equations (1) is called a principal vector corresponding to the characteristic ro[illegible]

The following results can be observed ...

(i) The principal directions (vectors) corresponding to two different values of λ are mutually perpendicular.

(ii) If λ is a root of $D_t = 0$, then

(a) λ is a non-repeated root $\Leftrightarrow A_\lambda + B_\lambda + C_\lambda \neq 0$.

(b) λ is a twice repeated root $\Leftrightarrow A_\lambda = B_\lambda = C_\lambda = F_\lambda = G_\lambda = H_\lambda = 0$.

(c) λ is a thrice repeated root $\Leftrightarrow a = b = c = \lambda$; $f = g = h = 0$.

Theorem 2 : For any conocoid, there exist three mutually perpendicular principal directions.

Proof : Let $\lambda_1, \lambda_2, \lambda_3$ be the characteristic roots of the conicoid (they are real).

(i) If $\lambda_1, \lambda_2, \lambda_3$ are all different, there exist principal directions corresponding to these values of λ. By (i) of above stated result, these are three mutually perpendicular directions.

(ii) Let $\lambda_1 = \lambda_2 = \lambda_3$, $\lambda_3 \neq \lambda$.

Since λ is a double root of $D_t = 0$, using (ii) b of above stated result

$$A_\lambda = B_\lambda = C_\lambda = F_\lambda = G_\lambda = H_\lambda = 0.$$

$$\therefore \quad (a-\lambda, h, g)\ ;\ (h, b-\lambda, f)\ ;\ (g, f, c-\lambda)$$

are parallel vectors. At least one of these three vectors is non-zero (if all of them are zero then $a = b = c = \lambda$ and $f = g = h = 0$ and hence λ is a thrice repeated root by (ii) (c) of the above stated result).

Let $(a-\lambda, h, g)$ be a non-zero vector. Then consider any non-zero vector (l, m, n) perpendicular to $(a-\lambda, h, g)$.

$$\therefore \quad (a-\lambda)\, l + hm + gn = 0.$$

Since $(h, b-\lambda, f)$ is parallel to $(a-\lambda, h, g)$, it is also parallel to (l, m, n).

$$\therefore \quad hl + (b-\lambda)\, m + fn = 0.$$

Similarly,

$$gl + fm + (c-\lambda)\, n = 0.$$

Therefore, any non-zero vector (l, m, n) perpendicular to $(a-\lambda, h, g)$ is a principal direction corresponding to λ. But by (i) of above stated result any principal direction corresponding to λ is perpendicular to any principal direction corresponding to λ_3. If (l_3, m_3, n_3) is a principal direction corresonding to λ_3, then $(l_3, m_3, n_3) \,||\, (a-\lambda, h, g)$ and any non - zero vector perpendicular to (l_3, m_3, n_3) is a principal direction corresponding to λ. Now choosing non - zero vectors (l_1, m_1, n_1) and (l_2, m_2, n_2) perpendicular to each other and perpendicular to (l_3, m_3, n_3), three mutually perpendicular principal directions are obtained.

(iii) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, by (ii) (c) of the above result,

$$a = b = c = \lambda\ ;\quad f = g = h = 0.$$

$\therefore$ Every non-zero vector is a principal direction corresponding to λ. Three mutually perpendicular directions from among the principal directions can be chosen.

Examples :

1. Find the characteristic roots and principal directions of the conicoid

$$2x^2 + y^2 - 4xy - 4yz + 3x - 5y + z - 7 = 0.$$

Here $a = 2,\ b = 1,\ c = 0,\ f = -2,\ g = 0,\ h = -2,$

$$D_t = \begin{vmatrix} 2-t & -2 & 0 \\ -2 & 1-t & -2 \\ 0 & -2 & 0-t \end{vmatrix} = t^3 - 3t^2 - 6t + 8 = 0.$$

The characteristic roots are 1, 4, – 2. If (l_1, m_1, n_1) is a principal direction corresponding to $\lambda = 1$, then l_1, m_1, n_1 are got by solving

$$al_1 + hm_1 + gn_1 = \lambda l_1$$
$$hl_1 + bm_1 + fn_1 = \lambda m_1$$
$$gl_1 + fm_1 + cn_1 = \lambda n_1$$

i.e.,

$$l_1 - 2m_1 = 0$$
$$-2l_1 - 2n_1 = 0$$
$$-2m_1 - n_1 = 0$$

Solving, $(l_1, m_1, n_1) \,||\, (2, 1, -2)$

Similarly if (l_2, m_2, n_2) and (l_3, m_3, n_3) are principal directions corresponding to 4, – 2 respectively, then $(l_2, m_2, n_2) \,||\, (2, -2, 1)$ and $(l_3, m_3, n_3) \,||\, (1, 2, 2)$.

∴ The three mutually perpendicular principal directions are (2, 1, – 2) ; (2, – 2, 1) and (1, 2, 2).

2. Find the characteristic roots and principal directions of $3x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + 7x - 5 = 0$.

$a = 3,\ b = 1,\ c = 1,\ f = -1,\ g = 1,\ h = -1.$

The characteristic equation is

$$D_t = \begin{vmatrix} 3-t & -1 & 1 \\ -1 & 1-t & -1 \\ 1 & -1 & 1-t \end{vmatrix} = -4t + 5t^2 - t^3 = 0$$

The characteristic roots are 0, 1, 4. If (l_1, m_1, n_1) is the principal direction corresponding to 0, then l_1, m_1, n_1 are found by solving

$$3l_1 - m_1 + n_1 = 0$$
$$-l_1 + m_1 + n_1 = 0$$
$$l_1 - m_1 + n_1 = 0$$

∴ $(l_1, m_1, n_1) \,||\, (0, 1, 1).$

Similarly $(l_2, m_2, n_2) \,||\, (1, 1, -1)$

and $(l_3, m_3, n_3) \,||\, (2, -1, 1).$

Theorem 3 : Let (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be three mutually perpendicular principal unit vectors corresponding to the characteristic roots $\lambda_1, \lambda_2, \lambda_3$. Then by the transformation of coordinate axes,

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \quad \ldots (1)$$

the equation of the conicoid reduces to the form

$$a'x'^2 + b'y'^2 + c'z'^2 + 2u'x' + 2v'y' + 2w'z' + d = 0. \quad \ldots (2)$$

Proof : In view of the transformation (1), x, y, z can be written in terms of x', y', z' as

$$x = l_1 x' + l_2 y' + l_3 z'$$
$$y = m_1 x' + m_2 y' + m_3 z'$$
$$z = n_1 x' + n_2 y' + n_3 z'$$

0	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Then

$$\begin{aligned} \lambda_1 x' &= \lambda_1 (x l_1 + y m_1 + z n_1) \\ &= x(\lambda_1 l_1) + y(\lambda_1 m_1) + z(\lambda_1 n_1) \\ &= x(a l_1 + h m_1 + g n_1) + y(h l_1 + b m_1 + f n_1) + \\ &\quad + z(g l_1 + f m_1 + c n_1) \\ &= l_1 (ax + hy + gz) + m_1 (hx + by + fz) + n_1 (gx + fy + cz) \end{aligned}$$

Similarly,

$$\lambda_2 y' = l_2 (ax + hy + gz) + m_2 (hx + by + fz) + n_2 (gx + fy + cz),$$

$$\lambda_3 z' = l_3 (ax + hy + gz) + m_3 (hx + by + fz) + n_3 (gx + fy + cz).$$

Multiplying these equations by x', y', z' respectively and adding,

$$\begin{aligned} \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 &= x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz) \\ &= H(x, y, z). \\ \therefore\ S(x, y, z) &= H(x, y, z) + 2ux + 2vy + 2wz + d \\ &= \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2u(l_1 x' + l_2 y' + l_3 z') \\ &\quad + 2v(m_1 x' + m_2 y' + m_3 z') \\ &\quad + 2w(n_1 x' + n_2 y' + n_3 z') + d \\ &= \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2(u l_1 + v m_1 + w n_1) x' \\ &\quad + 2(u l_2 + v m_2 + w n_2) y' + 2(u l_3 + v m_3 + w n_3) z' + d \end{aligned}$$

which is the required form in the statement of the theorem.

Note : a', b', c' in equation (2) are the characteristic roots $\lambda_1, \lambda_2, \lambda_3$ of the given conicoid.

Theorem 4 : Using a suitable transformation of coordinate axes, the equation $S = 0$ of the conicoid can be reduced to the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2 - \delta = 0 \text{ or } \alpha X^2 + \beta Y^2 + 2\epsilon Z = 0$$

Proof : Using theorem 3, the equation $S = 0$ reduces to the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2u'x' + 2v'y' + 2w'z' + d = 0 \qquad \text{... (1)}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the characteristic roots of $S = 0$.

(i) $\lambda_1 \lambda_2 \lambda_3 \neq 0$ (i.e., each of $\lambda_1, \lambda_2, \lambda_3$ is not zero)

changing the origin to $\left(\frac{-u}{\lambda_1}, \frac{-v}{\lambda_2}, \frac{-w}{\lambda_3}\right)$ with the coordinate axes remaining parallel, the transformations are

$$X = x + \frac{u}{\lambda_1}, \ Y = y + \frac{v}{\lambda_2}, \ Z = z + \frac{w}{\lambda_3}.$$

Using these transformations, the equation (1) reduces to

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 - \left(\frac{u'^2}{\lambda_1} + \frac{v'^2}{\lambda_2} + \frac{w'^2}{\lambda_3}\right) + d = 0$$

This is of the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2 - \delta = 0.$$

(ii) $\lambda_1 \lambda_2 \neq 0, \ \lambda_3 = w' = 0.$

Using the transformations $X = x' + \frac{u'}{\lambda_1}$, $Y = y' + \frac{v'}{\lambda_2}$, $Z = z'$, the equation (1) reduces to

$$\lambda_1 X^2 + \lambda_2 Y^2 - \left(\frac{u'^2}{\lambda_1} + \frac{v'^2}{\lambda_2}\right) + d = 0$$

This is of the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2 - \delta = 0.$$

Such reduction can also be done in the cases

$\lambda_1 \lambda_3 \neq 0, \ \lambda_2 = v' = 0$ and $\lambda_2 \lambda_3 \neq 0, \ \lambda_1 = u' = 0$

(iii) $\lambda_1 \lambda_2 \neq 0, \ \lambda_3 = 0, w' \neq 0.$

Using the transformations

$$X = x' + \frac{u}{\lambda_1}, Y = y + \frac{v}{\lambda_2}, \ Z = z + \frac{d - \frac{u'^2}{\lambda_1} + \frac{v'^2}{\lambda_2}}{2w'}$$

the equation (1) reduces to

$$\lambda_1 X^2 + \lambda_2 Y^2 + 2w'Z = 0$$

(iv) $\lambda \neq 0, \ \lambda_2 = \lambda_3 = 0.$ In this case zero is a double root of $D_t = 0$. If (l_1, m_1, n_1) is a principal unit vector corresponding to λ, then any direction perpendicular to (l_1, m_1, n_1) is a principal direction corresponding to zero. Choose one such unit vector (l_2, m_2, n_2) perpendicular to (l_1, m_1, n_1) such that

$$v' = ul_2 + vm_2 + wn_2 = 0,$$

and a unit cector (l_3, m_3, n_3) perpendicular to both (l_1, m_1, n_1) and (l_2, m_2, n_2).

These are three mutually perpendicular principal directions of the conicoid corresponding to the characteristic roots.

(a) If $w' = 0$, choose $X = x' + \frac{u'}{\lambda_1}$, $Y = y'$, $z = z'$

Then (1) reduces to $\lambda_1 X^2 - \frac{u'^2}{\lambda_1} + d = 0$.

This is of the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2 - \delta = 0.$$

(b) If $w' \neq 0$, choose $X = x' + \frac{u'}{\lambda_1}$, $Y = y'$, $Z = z' + \frac{d - \frac{u'^2}{\lambda_1}}{2w'}$

then (1) reduces to

$$\lambda_1 X^2 + 2w'Z = 0$$

This is of the form

$$\alpha X^2 + \beta Y^2 + 2\epsilon Z = 0.$$

Note : (i) 21.11 and theoem 4 of unit 24 can be combined to give the following result :

Any conicoid $S = 0$ is one of the following :

(i) a cone (may be degenerate)

(ii) a cylinder (may be degenerate)

(iii) an ellipsoid

(iv) a paraboloid

(v) a hyperboloid.

(ii) The coordinate transformations in Theorem 3 correspond to a rotation of coordinate axes Ox, Oy, Oz to a new set of three mutually perpendicular principal lines Ox', Oy', Oz' through the origin O.

If the given conicoid has a centre (for definition of centre, see unit 22), then the transformations in theorem 4, translate the origin to the centre of the conicoid. Thus, through the transformations in theorems 3 and 4, the coordinate axes Ox', Oy', Oz' are transferred to the principal axes through the centre.

Theorem 5 : If $D_0 \neq 0$, $\Delta \neq 0$, the conicoid $S = 0$ is an ellipsoid or a hyperboloid.

If $D_0 \neq 0$, $\Delta = 0$ it is a non - degenerate cone.

Proof : Since $D_0 \neq 0$, the three characteristic roots $\lambda_1, \lambda_2, \lambda_3$ of $D_0 = 0$ are non - zero. Using the transformation of coordinates in theorem 4, the equation of the conicoid becomes

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 - \delta = 0$$

where
$$\delta = \frac{u'^2}{\lambda_1} + \frac{v'^2}{\lambda_2} + \frac{w'^2}{\lambda_3} - d$$

$$= \frac{1}{\lambda_1 \lambda_2 \lambda_3}\left[\lambda_2\lambda_3 u'^2 + \lambda_3\lambda_1 v'^2 + \lambda_1\lambda_2 w'^2 - d\,\lambda_1\lambda_2\lambda_3\right]$$

$$= -\frac{1}{\lambda_1 \lambda_2 \lambda_3} \begin{vmatrix} \lambda_1 & 0 & 0 & u' \\ 0 & \lambda_2 & 0 & v' \\ 0 & 0 & \lambda_3 & w' \\ u' & v' & w' & d \end{vmatrix}$$

$= -\frac{\Delta'}{D_0'}$ ($\because$ in Δ' and D_0' obtained from therorem 4, equation (1),

$a' = \lambda_1, b' = \lambda_2, c' = \lambda_3, f' = g' = h' = 0$).

$= -\frac{\Delta}{D_0}$ ($\because$ Δ and D_0 are invariant under transformation of coordinates).

$\therefore$ The equation of the conicoid is

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + \frac{\Delta}{D_0} = 0.$$

If $\Delta \neq 0$, multiplying this equation by $\frac{D_0}{\Delta}$, the equation reduces to the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2 = 1.$$

If $\alpha < 0, \beta < 0, \gamma < 0$, the conicoid is an empty set.

If $\alpha > 0, \beta > 0, \gamma > 0$, it is an ellipsoid.

If some of α, β, γ are positive and some negative, it is a hyperboloid.

If $\Delta = 0$, the equation of the conicoid becomes

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 0.$$

Since $D = \lambda_1 \lambda_2 \lambda_3 \neq 0$, it is a non-degenerate cone.

Theorem 5: If $D_0 = 0$, $Au + Hv + Gw \neq 0$, the conicoid $S = 0$ is an paraboloid.

Proof : Since $D_0 = 0$, $\lambda = 0$ is a characteristic root.

$Au + Hv + Gw \neq 0 \Rightarrow$ one of A, H, G in non-zero.

Hence $\lambda = 0$ is a non-repeated root of $D_t = 0$. Let $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$. If (l_3, m_3, n_3) is a principal unit direction corresponding to λ_3, then

$$hl_3 + bm_3 + fn_3 = \lambda_3 m_3 = 0$$

$$gl_3 + fm_3 + cn_3 = \lambda_3 n_3 = 0$$

$$\therefore \frac{l_3}{bc - f^2} = \frac{m_3}{fg - ch} = \frac{n_3}{hf - bg}$$

$$\therefore \quad \frac{l_3}{A} = \frac{m_3}{H} = \frac{n_3}{G} = \frac{ul_3 + vm_3 + wn_3}{Au + Hv + Gw}$$

$$\therefore \quad ul_3 + vm_3 + wn_3 \neq 0$$

i.e. $w' \neq 0$. Using the transformations as in theorem (4). (iii) the equation of the conicoid reduces to $\lambda_1 X^2 + \lambda_2 Y^2 + 2w'Z = 0$. Since $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $w' \neq 0$, this represents a paraboloid.

Theorem 6: If $D_0 = 0, Au + Hv + Gw = 0, A \neq 0$, the conicoid $S = 0$ is a degenerate cone, an elliptic cylinder or a hyperbolic cylinder.

Proof : Since $A = bc - f^2 \neq 0$, the two equations in β and γ namely

$$b\beta + f\gamma + v = 0$$
$$f\beta + c\gamma + w = 0$$

have a solution (β, γ). If we take the point $(0, \beta, \gamma)$ for (x, y, z) in the following set of equations,

$$hx + by + fz + v = 0 \quad \ldots(i)$$
$$gx + fy + cz + w = 0 \quad \ldots(ii)$$
$$x = 0 \quad \ldots(iii)$$

these equations are satisfied. Hence $(0, \beta, \gamma)$ is a common point of these planes. Also noting that

$$\beta = \frac{fw - cv}{A}, \quad \gamma = \frac{fv - bw}{A}$$

and substituting $(0, \beta, \gamma)$ for (x, y, z) in $ax + hy + gz + u$, we get

$$h\beta + g\gamma + u = \frac{h(fw - cv)}{A} + \frac{g(fv - bw)}{A} + u$$
$$= [v(fg - ch) + w(hf - bg) + Au]/A.$$
$$= (Au + Hv + Gw)/A = 0.$$

$\therefore$ The point $(0, \beta, \gamma)$ is also on the plane $ax + hy + gz + u = 0$ $\quad \ldots(iv)$

Again, substituting $(0, \beta, \gamma)$ for (x_0, y_0, z_0), we get

$$\xi_0 = \eta_0 = \zeta_0 = 0 \text{ (using (i), (ii), (iv))}$$
$$\therefore S_{00} = \delta_0 = ux_0 + vy_0 + wz_0 + d$$
$$= u\beta + w\gamma + d \quad \ldots(v)$$

Now, shifting the origin to $(0, \beta, \gamma)$, the transformations

$$x = x^*, y = y^* + \beta, z = z^* + \gamma$$

reduce the equation of conicoid to the form

$$H(x^*, y^*, z^*) + 2x^*(h\beta + g\gamma + u) + 2y^*(b\beta + f\gamma + v)$$
$$+ 2z^*(f\beta + c\gamma + w) + S(0, \beta, \gamma) = 0$$

i.e. $H(x^*, y^*, z^*) + v\beta + w\gamma + d = 0$ (using (v)).

Now transforming the equation to the principal directions, it reduces to

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + \delta_0 = 0$$

Since $D_0 = 0, A \neq 0, \lambda = 0$ is non-repeated root of $D_t = 0$ choosing $\lambda_1 \lambda_2 \neq 0, \lambda_3 = 0$, the following classification can be arrived at. The conic is

(i) A straight line $X = 0 = Y$ if $\delta_0 = 0, \lambda_1 \lambda_2 > 0$

(both λ_1 and λ_2 are positive or negative).

(ii) A pair of intersecting planes if $\delta_0 = 0, \lambda_1 \lambda_2 < 0$

(one of λ_1, λ_2, is positive and other negative).

(iii) empty set if $\delta_0 \neq 0$, $\lambda_1, \lambda_2, \delta_0$ are all positive or all negative.

(iv) elliptic cylinder if $\delta_0 \neq 0, \lambda_1 > 0, \lambda_2 > 0, \delta_0 < 0$ or $\lambda_1 < 0, \lambda_2 < 0, \delta_0 > 0$.

(v) hyperbolic cylinder if $\delta_0 \neq 0, \lambda_1 \lambda_2 < 0$.

Theorem : If $A = B = C = F = G = H = 0$, the conicoid $S = 0$ is a parabolic cylinder or a pair of parallel planes.

Proof : By hypothesis $D_t = 0$ has $\lambda = 0$ as a double root. Let $\lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0$. As in the proof of Therorem 2 (ii) with $\lambda = 0$, it follows that vectors (a, h, g), (h, b, f), (g, f, c) are parallel. Since S is of second degree, at least one of these three vectors is nonzero. Without loss of generality, suppose $(a, h, g) \neq (0, 0, 0)$. As in Theorem 2 it can be shown that any vector perpendicular to (a, h, g) is a principal vector corresponding to $\lambda = 0$. Choose a non-zero unit vector (l_2, m_2, n_2) perpendicular to both (a, h, g) and (u, v, w). Choose a non-zero unit vector (l_3, m_3, n_3) perpendicular to (a, h, g) and (l_2, m_2, n_2). Choose non-zero unit vector (l_1, m_1, n_1) perpendicular to both (l_2, m_2, n_2) and (l_3, m_3, n_3). Then (l_1, m_1, n_1) is a principal vector corresponding to λ_1.

Now transforming the coordinates to a set of axes parallel to (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) through the origin, (rotation of the frame), the transformed equation of the conicoid is

$$\lambda_1 x'^2 + 2(ul_1 + vm_1 + wn_1)x' + 2(ul_2 + vm_2 + wn_2)y' + 2(ul_3 + vm_3 + wn_3)z' + d = 0$$

Since $ul_2 + vm_2 + wn_2 = 0$, this equation is of the form

$$\lambda_1 x'^2 + 2u'x' + 2w'z' + d = 0$$

If $w' = 0$, then using $X = x' + \frac{u'}{\lambda_1}, Y = y', Z = z'$, the above equation reduces to the form

$$\lambda_1 X^2 + d' = 0.$$

(i) If $\lambda_1 d' > 0$, this is an empty set

(ii) If $\lambda_1 d' < 0$, this is a pair of parallel planes

(iii) If $w' = d' = 0$, this does not represent a conicoid and it represents a plane.

Since $w' = ul_3 + vm_3 + wn_3$,

$$\begin{aligned} w' = 0 &\Leftrightarrow (u, v, w) \perp (l_3, m_3, n_3) \\ &\Leftrightarrow (u, v, w) \parallel (a, h, g) \parallel (h, b, f) \parallel (g, f, c) \\ &\Leftrightarrow fu = gv = hw \end{aligned}$$

$\therefore$ If $A = B = C = F = G = H = 0$ and $fu = gv = hw$, $S = 0$ represents an empty set or a pair of parallel planes.

Now consider the case when $A = B = C = F = G = H = 0$ and atleast two of fu, gv, hw are different. Then $w' \neq 0$. Translating the coordinate system using

$$X = x' + \frac{u'}{\lambda_1}, Y = y', Z = z' + \frac{d - \frac{u'^2}{\lambda_1}}{2w'},$$ the equation of the conic reduce to

$$\lambda_1 X^2 + 2w'Z = 0$$

This represents a parabolic cylinder.

Examples

1. Find the characteristic roots, principal vectors and reduce the the equation

$6x^2 + 6y^2 + 13z^2 - 10yz - 10zx + 4xy + 4x - 12y + 14z + 11 = 0$ to standard form. Classify the conicoid.

Sol : For the given equation,

$a = 6, b = 6, c = 13,$

$f = -5, g = -5, h = 2,$

$u = 2, v = -6, w = 7, d = 11$

The characteristic determinant is

$$D_0 = \begin{vmatrix} 6 & 2 & -5 \\ 2 & 6 & -5 \\ -5 & -5 & 13 \end{vmatrix} \neq 0$$

$$\Delta = \begin{vmatrix} 6 & 2 & -5 & 2 \\ 2 & 6 & -5 & -6 \\ -5 & -5 & 13 & 7 \\ 2 & -6 & 7 & 11 \end{vmatrix} \neq 0$$

Since $D_0 \neq 0$, $\Delta \neq 0$, the conicoid is either an ellipsoid or a hyperboloid. The characteristic equation is

$$\lambda^3 - 25\lambda^2 + 138\lambda - 216 = 0.$$

The characteristic roots are 3, 4, 18.

$\therefore \lambda_1 = 3, \lambda_2 = 4, \lambda_3 = 18$

The principal direction corresponding to $\lambda = 3$ is obtained by solving

$$6l + 2m - 5n = 3l$$

$$2l + 6m - 5n = 3m$$

$$-5l - 5m + 13n = 3n$$

These equations give $(l, m, n) = (1, 1, 1)$. If (l_1, m_1, n_1) is a principal unit vector corresponding to $\lambda = 3$, then

$$(l_1, m_1, n_1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Similarly, the principal unit vectors corresponding to $\lambda = 4$ and $\lambda = 18$ are,

$$(l_2, m_2, n_2) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{and } (l_3, m_3, n_3) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

$u' = ul_1 + vm_1 + wn_1$ where (l_1, m_1, n_1) is a principal unit vector.

$$\therefore u' = \frac{2 \times 1 + (-6 \times 1) + (7 \times 1)}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{3}{\sqrt{3}}$$

Similarly

$$v' = \frac{2 \times (-1) + (-6 \times 1) + 7 \times 0}{\sqrt{1^2 + 1^2 + 0^2}} = \frac{-8}{\sqrt{2}}$$

$$w' = \frac{2 \times 1 - 6 \times 1 - 7 \times 2}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{-18}{\sqrt{6}}$$

$$\frac{u'^2}{\lambda_1} + \frac{v'^2}{\lambda_2} + \frac{w'^2}{\lambda_3} - d = \frac{3}{3} + \frac{32}{4} + \frac{54}{18} - 11 = 1$$

Using the transformations in theorem 3 and the theorem 4 (i), the equation of the conicoid reduces to the canonical form

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 - \left[\frac{u'^2}{\lambda_1} + \frac{v'^2}{\lambda_2} + \frac{w'^2}{\lambda_3} - d\right] = 0$$

i.e. $3X^2 + 4Y^2 + 18Z^2 = 1$

This is an ellipsoid.

2. Prove that the equation $2x^2 + 20y^2 + 18z^2 - 12yz + 12xy + 22x + 6y - 2z - 2 = 0$ represents a paraboloid.

Sol : $a = 2, b = 20, c = 18, f = -6, g = 0, h = 6$

$u = 11, v = 3, w = -1, d = -2.$

$$D = \begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & -6 & 18 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 6 & 2 & -6 \\ 0 & -6 & 18 \end{vmatrix} = 0$$

$A = 324, H = -108, G = -36$

$$Au + Hv + Gw = 324 \times 11 - 108 \times 3 - 36 \times -1 \neq 0$$

$\therefore$ The conicoid is a paraboloid.

The characteristic equation $t^3 - 40t^2 + 364t = 0$

The characteristic roots are 0, 14, 26.

$\lambda_1 = 14, \lambda_2 = 26, \lambda_3 = 0.$

The principal direction corresponding to zero is given by

$$2l + 6m = 0$$

$$6l + 20m - 6n = 0$$

$$-6m + 18n = 0.$$

From the first two, $\frac{l}{-36} = \frac{m}{12} = \frac{n}{4}$

$$\therefore (l, m, n) \parallel (-9, 3, 1)$$

Coorresponding to $\lambda = 0$, the principal unit vector is

$$\left(\frac{-9}{\sqrt{91}}, \frac{3}{\sqrt{91}}, \frac{1}{\sqrt{91}}\right) = (l_3, m_3, n_3)$$

$$\therefore w' = ul_3 + vm_3 + wn_3 = \frac{11 \times -9 + 3 \times 3 - 1 \times 1}{\sqrt{91}}$$

$$= \frac{-91}{\sqrt{91}} = -\sqrt{91}$$

The equation of the conicoid reduces to $\lambda_1 X^2 + \lambda_2 Y^2 + 2w'Z = 0$, using the transformations of theorem 3 and theorem 4 (i)

i.e., $$14X^2 + 26Y^2 = 2\sqrt{91}\, Z$$

This is an elliptic paraboloid.

3. Reduce and classify the conicoid

$2y^2 - 2yz - 2zx + 2xy - 4x - 3y + 2z - 1 = 0$

Sol : Here $a = 0, b = 2, c = 0, f = -1, g = -1, h = 1,$

$u = -2, v = -3/2, w = 1, d = -1.$

$$D_\lambda = \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & 2-\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 - 3\lambda = 0$$

is the characteristic equation

The characteristic roots are 3, –1, 0

$\lambda_1 = 3, \lambda_2 = -1, \lambda_3 = 0$

$$D_0 = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 0 \end{vmatrix} = 0$$

$A = -1, H = 1, G = 1$

$$Au + Hv + Gw = -1 \times -2 + 1 \times \frac{-3}{2} + 1 \times 1 = 2 - \frac{3}{2} + 1 = \frac{3}{2} \neq 0$$

If (l, m, n) is a principal direction corresponding to $\lambda = 0$,

$$m - n = 0$$
$$l + 2m - n = 0$$
$$-l - m = 0.$$
$$\therefore (l, m, n) \parallel (1, -1, -1)$$

The principal unit vector corresponding to $\lambda = 0$ is $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$$w' = ul_3 + vm_3 + wn_3$$

$$= (-2)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{-3}{2}\right)\left(\frac{-1}{\sqrt{3}}\right) + 1 \cdot \left(\frac{-1}{\sqrt{3}}\right)$$

$$= \frac{-4 + 3 - 2}{2\sqrt{3}} = \frac{-3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

Using the transformations in theorem 3 and 4 (iii), the equation of the conicoid reduces to

$$3X^2 - Y^2 = \sqrt{3}\, Z$$

This is a hyperbolic parabolid.

4. Reduce the following equation to standard form and classify the conicoid:

$$9x^2 + y^2 + z^2 + 2yz - 6zx - 6xy - 6x + 2y + 2z = 0.$$

Sol : Here $a = 9, b = 1, c = 1, f = 1, g = -3, h = -3,$

$u = -3, v = 1, w = 1, d = 0.$

$$D = \begin{vmatrix} 9 & -3 & -3 \\ -3 & 1 & 1 \\ -3 & 1 & 1 \end{vmatrix} = 0$$

$A = B = C = F = G = H = 0.$

The characteristic equation is

$$D_\lambda = \begin{vmatrix} 9-\lambda & -3 & -3 \\ -3 & 1-\lambda & 1 \\ -3 & 1 & 1-\lambda \end{vmatrix} = \lambda^2 (11 - \lambda) = 0$$

The characteristic roots are $\lambda = 11, 0, 0$

The principal vector corresponding to $\lambda = 11$ is

$$9l - 3m - 3n = 11l$$
$$-3l + m + n = 11m$$
$$-3l + m + n = 11n$$

i.e.,

$$-2l - 3m - 3n = 0$$
$$-3l - 10m + n = 0$$
$$-3l + m - 10n = 0$$

$$\therefore \frac{l}{-33} = \frac{m}{11} = \frac{n}{11}$$

$\therefore (l, m, n) \parallel (-3, 1, 1)$. A unit vector in this direction

is $(l_1, m_1, n_1) = \left(\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right) \parallel (a, h, g)$

The principal vectors will be chosen now as in theorem 7.

A vector perpendicular to (a, h, g) and (u, v, w) ;

i.e. perpendicular to $(-3, 1, 1)$

is $(1, 3, 0)$. $\quad \therefore (l_2, m_2, n_2) = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0\right)$

A vector perpendicular to $(-3, 1, 1)$ and $(1, 3, 0)$ is $(-3, 1, -10)$

$$\therefore (l_3, m_3, n_3) = \left(\frac{-3}{\sqrt{110}}, \frac{1}{\sqrt{110}}, \frac{-10}{\sqrt{110}}\right)$$

$$\text{Now } w' = ul_3 + vm_3 + wn_3$$

$$= (-3)\left(\frac{-3}{\sqrt{110}}\right) + 1\left(\frac{1}{\sqrt{110}}\right) + 1\left(\frac{-10}{\sqrt{110}}\right) = 0$$

Using theorem 7 and the transformations in theorems 3 and 4 (iv) a, the equation of the given conicoid reduces to $\lambda_1 X^2 - \frac{u^2}{\lambda_1} + d = 0$

$$u' = ul_1 + vm_1 + wn_1 = (-3)\frac{(-3)}{\sqrt{11}} + 1\cdot\frac{1}{\sqrt{11}} + 1\cdot\frac{1}{\sqrt{11}} = \sqrt{11}$$

The equation reduces to

$$11X^2 - 1 = 0$$

This is a pair of parallel planes.

5. Reduce the following equation to standard form and classify the conicoid

$2x^2 + 4y^2 + 2z^2 - 4yz - 4xy + 10x - 8y - 2z + 12 = 0.$

Sol : $a = 2, b = 4, c = 2, f = -2, g = 0, h = -2$

$u = 5, v = -4, w = -1, d = 12$

$$D = \begin{vmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{vmatrix} = 0$$

$A = 4, (\neq 0), H = 4, G = 4$

$$Au + Hv + Gw = 4 \times 5 + 4 \times -4 + 4 \times -1 = 0$$

The characteristic equation is

$$D_\lambda = \begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 4-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = 0$$

i.e., $\lambda(\lambda^2 - 8\lambda + 12) = 0$

The characteristic roots are $\lambda_1 = 2, \lambda_2 = 6, \lambda_3 = 0.$

Following theorem 6, solving the equations

$b\beta + f\gamma + v = 0$, i.e., $4\beta - 2\gamma - 4 = 0,$

and $f\beta + c\gamma + w = 0$, i.e., $-2\beta + 2\gamma - 1 = 0,$

$$\beta = \frac{5}{2}, \gamma = 3.$$

$$\therefore v\beta + w\gamma + d = -4 \times \frac{5}{2} - 1 \times 3 + 12 = -1$$

This equation reduces to

$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + (v\beta + w\gamma + d) = 0$

$$\text{i.e. } 2X^2 + 6Y^2 - 1 = 0$$

$$\text{or } 2X^2 + 6Y^2 = 1$$

This is an elliptic cylinder.

24.3 Summary

The general second degree equation in three variables, x, y, z is

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

The equation $D_\lambda = \begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$ is called

the characteristic equation of the conicoid $S \equiv 0$. The roots of $D_\lambda = 0$ are called the characteristic roots. For any conicoid, there exist three mutually perpendicular principal directions. These are determined by the three characteristic roots $\lambda_1, \lambda_2, \lambda_3$ of $D_\lambda = 0$. Using the suitable transformation of coordinate axes, the equation $S = 0$ of the conicoid can be reduced to the form $\alpha X^2 + \beta Y^2 + \gamma Z^2 - \delta_0$ or $\alpha X^2 + \beta Y^2 + 2\,\varepsilon\, Z = 0$. If $D_0 \neq 0$,

$$\Delta = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \neq 0 \text{ then the}$$

conicoid $S = 0$ is an ellipsoid or a hyperboloid.

If $D_0 \neq 0$, $\Delta = 0$ it is a non-degenerate cone. If $D_0 = 0$, $Au + Hv + Gw \neq 0$ where A, H, G are cofactors of a, h, g respectively in D_0; then S is a paraboloid.

If $D_0 = 0$, $Au + Hv + Gw = 0$, $A \neq 0$, the conicoid $S = 0$ is a degenerate cone (or) an elliptic cylinder or a hyperbolic cylinder.

If $A = B = C = F = G = H = 0$, the conicoid S = 0 is a parabolic cylinder or a pair of parallel planes.

24.4 Sample Examination Questions

I. *Answer the following in detail.*

(1) Find the characteristic roots and principal direction of

(i) $x^2 + 2yz - 4x + 6y + 2z = 0.$

(ii) $x^2 + 3y^2 + 5z^2 - 8yz - 8xy = 0$

(2) Find (a) characteristic roots (b) principal vectors (c) standard form and classify the conicoids represented by the following equations.

(i) $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y - 4z + 1 = 0$

(ii) $16x^2 + 25y^2 + 40xy + 46x + 62y - 2z + 41 = 0$

(iii) $7x^2 - 8y^2 - 8z^2 - 2yz - 8zx + 8xy - 16x + 14y - 14z - 5 = 0$

(iv) $60x^2 + 40y^2 - 24z^2 + 44yz - 4zx + 124xy + 47x + 41y + 5z + 7 = 0$

(v) $5x^2 + 10y^2 + 26z^2 - 28yz - 18zx + 14xy + 28x + 42y - 70z + 49 = 0$

Answers

(1) (i) The characteristic roots are 1, 1, –1.

The principal directions are (0, 1, 1) ... 2, –1, 1)

(ii) The characteristic roots are –3, 3, 9.

The principal directions are (2, 2, 1) ; (–2, 1, 2) ; (1, –2, 2).

(2) (i) (a) –3, 4, 12 (b) (1, –1, 1) ; (1, 0, –1) ; (1, 2, 1)

(c) $3X^2 - 4Y^2 - 12Z^2 = 1$, hyperboloid of two sheets.

(ii) (a) 41, 0, 0 ; (b) (4, 5, 0) ; and any non - zero vector perpendicular to it.

(c) $41x^2 - \frac{244}{\sqrt{5002}} z = 0$, parabolic cylinder

(iii) (a) 9, –9, 9; (b) (4, 1, 1); (0, 1, 1); (1, –2,2)

(c) $X^2 + Y^2 - Z^2 = 1$; hyperboloid of one sheet

(iv) (a) 114, –38, 0; (b) (8, 7, 1); (2, –3, 5); (–1, 1, 1)

(c) $114X^2 - 38Y^2 = \frac{1}{\sqrt{3}} Z$, hyperbolic paraboloid.

(v) (a) 38, 3, 0 (b) (2, 3, –5); (1, 1, 1); (–8, 7, 1)

(c) $38X^2 + 3Y^2 = 0$, straight line.

BLOCK – 6 : THEORY OF EQUATIONS

Introduction

You are all familiar with solving the linear equations, (that is, equations of the first degree, which are of the form $ax + b = 0$, a and b being real numbers, $a \neq 0$) and quadratic equations, (that is, equations of the second degree, which are of the form $ax^2 + bx + c = 0$, a, b and c being real numbers, $a \neq 0$). The only one root of the equation $ax + b = 0$, $a \neq 0$ is given by $x = -\frac{b}{a}$ while the two roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In this block we are interested in developing some methods for solving equations of degree greater than 2, preferably of degree 3 and 4. For better understanding of the subject matter let us familiarize ourselves with the basic concepts. First let us understand what is meant by an equaton and a root of an equation.

UNIT–25 : ROOTS, RELATION BETWEEN ROOTS AND COEFFICIENTS OF AN EQUATION.

25.0 Contents

25.1 Aims and objectives

After going through this unit, you will be able to :

(i) find the roots of some of the polynomial equations

(ii) obtain the relation between the roots and coefficients of an equation

(iii) obtain the HCF of two polynomials and find $f(x + h)$ for a given polynomial $f(x)$ using Horner's Process.

25.2 Polynomials and polynomial equations

An expression $f(x)$ of the form $a_0x^n + a_1x^{n-1} + \ldots + a_n$, where $a_0, a_1, \ldots a_n$ are real numbers, not all zero, is called a non-zero *polynomial* over the real numbers, or reals. x is called a variable and a_i's are called the coefficients. If all the a_i's are zero then the *polynomial* is called a zero *polynomial.*

Two *polynomials* $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, and $g(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m$, are said to be *equal* if (i) $m = n$ and (ii) $a_0 = b_0$; $a_1 = b_1$; $a_2 = b_2$ ..., $a_n = b_n$.

Suppose $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial. If α is a number, real, or complex, then the expression $a_0\alpha^n + a_1\alpha^{n-1} + \dots + a_n$, is called the value of $f(x)$ at $x = \alpha$. A number α for which $f(\alpha) = 0$ is called a zero of $f(x)$. A polynomial may have more than one zero. For example, if $f(x) = x^3 - 6x^2 + 11x - 6$, then $f(1) = 1 - 6 + 11 - 6 = 0$, and $f(2) = 8 - 24 + 22 - 6 = 0$, so that 1 and 2 are zeros of $x^3 - 6x^2 + 11x - 6$.

Consider a non-zero polynomial $f(x)$ over the reals. Then the expression $f(x) = 0$ (that is, the polynomial $f(x)$ equated to zero) is called a polynomial equation or an equation over the real numbers. Thus if the polynomial $f(x)$ is given by $a_0x^n + a_1x^{n-1} + \dots + a_n$, then the corresponding polynomial equation is $a_0x^n + a_1x^{n-1} + \dots + a_n = 0$. The degree of the equation $f(x) = 0$ is the degree of the polynomial $f(x)$. The roots of the equation $f(x) = 0$ are the zeros of the polynomial $f(x)$.

Equation of degree one, two, three and four are respectively called the *linear, quadratic, cubic and biquadratic* (or *quatric*) equations.

25.3 Factor Theorem

An imporant question that may arise is does every equation have a root?. The answer to this question is 'yes', although the proof of this is beyond the scope of the course. The result that every equation has a root is known as the *Fundamental Theorem of Algebra.* Proof of this theorem can be found in any book on complex Analysis.

Our first aim is to prove that every equation of degree n has at most n roots (real or complex). For this we need some results and the first in this direction is the division algorithm which we state below without proof:

Division algorithm : Let $f(x)$ and $g(x)$ be two polynomials and $g(x) \neq 0$. Then there exist polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)\, q(x) + r(x)$, where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

Theorem-1 : (Factor Theorem) Let $f(x)$ be a polynomial. A number α is a zero of $f(x)$ if and only if $f(x) = (x - \alpha)\, g(x)$ for some polynomial $g(x)$.

Proof : Let $f(x) = (x - \alpha)\, g(x)$. Then $f(\alpha) = (\alpha - \alpha)\, g(\alpha) = 0$ so that α is a zero of $f(x)$.

Conversely, let α be a zero of $f(x)$ so that $f(\alpha) = 0$. By division algorithm (applied to $f(x)$ and $(x - \alpha)$) we can find polynomials $g(x)$ and $r(x)$ such that $f(x) = (x - \alpha)\, g(x) + r(x)$. But the degree of $r(x)$ is less than the degree of $(x - \alpha)$ implies that $r(x)$ is a constant, say β. Thus $f(x) = (x - \alpha)\, g(x) + \beta$. This, together with $f(\alpha) = 0$ gives that $0 = (\alpha - \alpha)\, g(\alpha) + \beta$ or $\beta = 0$. Thus $f(x) = (x - \alpha)\, g(x)$. This proves the theorem.

Suppose that α is a root of the equation $f(x) = 0$. Then $f(x) = (x - \alpha)\, g(x)$ for some polynomial $g(x)$. It may happen that α is a root of $g(x) = 0$ also, so that $g(x) = (x - \alpha)\, h(x)$ for some $h(x)$. Thus $f(x) = (x - \alpha)^2\, g(x)$. In general, if α is a root of $f(x) = 0$, we may have $f(x) = (x - \alpha)^m\, q(x)$ for some polynomial $q(x)$ and some positive integer m. If $m \geq 2$, we say that α is a repeated root or a multiple root of $f(x) = 0$. A non repeated root is called a simple root. If α is a repeated root of $f(x) = 0$ such that $(x - \alpha)^m$ divides $f(x)$ but $(x - \alpha)^{m+1}$ does not divide $f(x)$, then α is called a root of $f(x) = 0$ of multiplicity m. For instance consider the equation $x^3 - 2x^2 + x = 0$. It is easy to see that 0, 1, 1 are the roots. So 1 is a repeated root of multiplicity 2, while 0 is a simple root. A repeated root of multiplicity m is treated as m roots.

Theorem-2 : An equation of degree $n \geq 1$ has at most n roots.

Proof : Let $f(x) = 0$ be an equation of degree n. We prove the therorem by using the principle of mathematical induction.

An equation of degree one is of the form $ax + b = 0$, $a \neq 0$ and $x = -\frac{b}{a}$ is its unique solution.

So assume that every equation of degre $k < n$, has at most k roots. Using this we shall show that $f(x) = 0$ has at most n roots.

By the Foundamental Theorem of Algebra, $f(x) = 0$ has a root, say α. Suppose that the multiplicity of α is m. Then for some polynomial $g(x)$,

$$f(x) = (x-\alpha)^m g(x)$$

Thus degree of $f(x)$ = degree of $(x-\alpha)^m$ + degree of $g(x)$. That is $n = m$ + degree of $g(x)$. So degree of $g(x) = n - m < n$.

Hence by induction hypothesis, $g(x) = 0$ has at most $n - m$ roots. But the roots of $f(x) = 0$ are precisely α (which is treated as m roots because of its multiplicity m) and the roots of $g(x) = 0$. Thus $f(x) = 0$ at most has $m + n - m$ or n roots.

25.4 Formation of an equation with given roots

Suppose that $\alpha_1, \alpha_2 \ldots \alpha_n$ are any given numbers, real or complex. We are interested in finding an equation for which $\alpha_1, \alpha_2 \ldots \alpha_n$ are the roots. By factor theorem, the number a is a root of the equation $f(x) = 0$ if and only if $x - a$ divides $f(x)$. So the polynomial associated with the required equation must be divisible by $(x-\alpha_1), (x-\alpha_2), \ldots (x-\alpha_n)$. The simplest form of the equation which has this property is $k(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = 0$ where k is some non-zero real constant. Simplifying the product on the left hand side and arranging it in the descending powers of x, we get the required equation. In practice we take $k = 1$.

Example. 1 : Form the equation whose roots are $\frac{2}{3}, -1, -1$.

Sol : The desired equation is

$$\left(x - \frac{2}{3}\right)(x+1)(x+1) = 0$$

$$\text{or } \left(\frac{3x-2}{3}\right)\left(x^2 + 2x + 1\right) = 0$$

$$\text{or } (3x-2)\left(x^2 + 2x + 1\right) = 0$$

$$\text{or } 3x^3 + 4x^2 - x - 2 = 0.$$

25.5 Complex roots and surd roots

It is well known that a complex number is expressed in the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$ and the complex number $a - ib$ is called the conjugate of $a + ib$.

Let u, v, w be rational numbers such that $\sqrt{w}$ is not a rational number and $v, w \neq 0$. Then $u + v\sqrt{w}$ is called a *surd*. Further $u - v\sqrt{w}$ is also a surd. Each of $u + v\sqrt{w}$ and $u - v\sqrt{w}$ is called the conjugate of the other.

We shall show that if a complex or a surd is a root of an equation then its conjugate is also a root.

Theorem-3 : Complex roots of an equation over real numbers occur in pairs.

Proof : Let $a + ib$, $b \neq 0$ be a root of the equation $f(x) = 0$, degree $f(x) \geq 2$. We shall show that $a - ib$ is also a root of $f(x) = 0$. Let $g(x)$ and $r(x)$ be the quotient and remainder got by dividing the polynomial $f(x)$ by $(x-a)^2 + b^2$. Since the degree of $(x-a)^2 + b^2$ is 2, $r(x) = 0$ or $r(x) = cx + d$ for some real numbers c and d. Thus, let

$$f(x) = \left[(x-a)^2 + b^2\right] g(x) + (cx + d)$$

$$= [\{x - (a+ib)\}\{x - (a-ib)\}]\, g(x) + (cx+d) \qquad \ldots (1)$$

Since $a + ib$ is a root of $f(x) = 0$, we have $f(a + ib) = 0$ so that (1) gives that

$$0 = 0 . g(a + ib) + [c(a + ib) + d]$$

That is, $(ca + d) + icb = 0$, or $ca + d = 0$ and $cb = 0$. As $b \neq 0$, $cb = 0$ implies that $c = 0$ and this together with $ca + d = 0$ gives $d = 0$.

Thus $$f(x) = [\{x - (a + ib)\} \{x - (a - ib)\}] g(x).$$

This shows that $x - (a - ib)$ divides $f(x)$ so that $a - ib$ is a root of $f(x) = 0$.

Theorem-4 : For an equation with rational coefficients surd roots occur in pairs.

Proof : Let $f(x) = 0$ be an equation with rational coefficients of degree ≥ 2 and $u + v\sqrt{w}$, a root of $f(x) = 0$. Let $f(x) = \left[(x - u)^2 - v^2 w\right] g(x) + r(x)$, where $r(x) = 0$, or degree $r(x) <$ degree$\left[(x - u)^2 - v^2 w\right]$, so that $r(x) = ax + b$ for some rational a and b. Thus

$$f(x) = \left[(x - u)^2 - v^2 w\right] g(x) + (ax + b)$$

$$= \left[\left\{x - (u + v\sqrt{w})\right\}\left\{x - (u - v\sqrt{w})\right\}\right] g(x) + (ax + b)$$

Since $u + v\sqrt{w}$ is a root of $f(x) = 0$, we have $f\left(u + v\sqrt{w}\right) = 0$

so that $$0 = 0 . g\left(u + v\sqrt{w}\right) + \left[a\left(u + v\sqrt{w}\right) + b\right].$$

This implies that $au + b = 0$ and $av = 0$. Since $v \neq 0$, these imply that $a = 0$ and $b = 0$. So $r(x) = 0$. Thus

$$f(x) = \left[\left\{x - (u + v\sqrt{w})\right\}\left\{x - (u - v\sqrt{w})\right\}\right] g(x)$$

so that $x - \left(u - v\sqrt{w}\right)$ is a factor of $f(x)$, or, $u - v\sqrt{w}$ is a root of $f(x) = 0$.

Example. 2 : Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $\sqrt{3}$ and another $1 - 2i$.

Sol : Here $\sqrt{3}$ is a surd root and $1 - 2i$ is a complex root. So their conjugates, namely, $-\sqrt{3}$ and $1 + 2i$ are also rootes of the given equation. Thus $x - \sqrt{3}$, $x + \sqrt{3}$, $x - (1 - 2i)$ and $x - (1 + 2i)$ are factors of $f(x) = x^5 - x^4 + 8x^2 - 9x - 15$. Equivalently $(x - \sqrt{3})\ (x + \sqrt{3})\ [x - (1 - 2i)]\ [x - (1 + 2i)]$ is a factor of $f(x)$. Therefore, $\left(x^2 - 3\right)\left(x^2 - 2x + 5\right)$, or $x^4 - 2x^3 + 2x^2 + 6x - 15$ divides $f(x)$. By actual division we get

$$x^5 - x^4 + 8x^2 - 9x - 15 = \left(x^4 - 2x^3 + 2x^2 + 6x - 15\right)(x + 1)$$

Thus $(x + 1)$ is a factor of $f(x)$, or $x = -1$ is a root of $f(x) = 0$. So the roots of the given equation are $-\sqrt{3}, \sqrt{3}, 1 + 2i, 1 - 2i, -1$.

Example. 3 : Solve the equation $3x^3 - 4x^2 + x + 88 = 0$, one root being $2 + \sqrt{-7}$.

Sol: Since $i = \sqrt{-1}$, the given root is $2 + i\sqrt{7}$ and this is a complex root. So $2 - i\sqrt{7}$ is also a root of $f(x) = 3x^3 - 4x^2 + x + 88 = 0$. So $(x - (2 + i\sqrt{7})(x - (2 - i\sqrt{7})$, or $\left(x^2 - 2\right)^2 + 7$ is a factor of $f(x)$. Dividing $f(x)$ by this factor, namely $x^2 - 4x + 11$ we get

$$3x^3 - 4x^2 + x + 88 = \left(x^2 - 4x + 11\right)(3x + 8).$$

Thus $3x + 8$ is also a factor of $f(x)$, so that $x = -\frac{8}{3}$ is another root of the given equation. So the roots of the given equation are

$2 \pm \sqrt{-7}, \frac{-8}{3}$.

25.6 Synthetic Division

Below we give a process of obtaining the quotient and remainder when a polynomial is divided by a binomial. This process will also help us in removing a known factor of the given equation.

Theorem-5 : When a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ is divided by a binomial $x - h$, the quotient and remainder are respectively given by $q(x) = b_0x^{n-1} + b_1x^{n-2} + \ldots + b_{n-1}$ and R where $b_0 = a_0, b_1 = a_1 + b_0h, \ldots, b_{n-1} = a_{n-1} + b_{n-2}h$ and $R = a_n + b_{n-1}h$.

Proof : Suppose that

$$a_0x^n + a_1x^{n-1} + \ldots + a_n \equiv (x-h)\left(b_0x^{n-1} + b_1x^{n-2} + \ldots + b_{n-1}\right) + R.$$

Equating the coefficients of like powers or x on both sides of this identity, we have

$$\begin{array}{lll} a_0 = b_0, & \text{or} & b_0 = a_0 \\ a_1 = b_1 - b_0h, & \text{or} & b_1 = a_1 + b_0h \\ a_2 = b_2 - b_1h, & \text{or} & b_2 = a_2 + b_1h \\ \ldots\ldots\ldots & & \ldots\ldots\ldots \\ \ldots\ldots\ldots & & \ldots\ldots\ldots \\ a_{n-1} = b_{n-1} - b_{n-2}h & \text{or} & b_{n-1} = a_{n-1} + b_{n-2}h \\ a_n = R - b_{n-1}h, & \text{or} & R = a_n + b_{n-1}h. \end{array}$$

These equations give the coefficients $b_0, b_1, \ldots, b_{n-1}$ of $q(x)$ and remainder R. This process is exhibited in the following schematic form with detached coefficients.

h	a_0	a_1	a_2	a_r	..... a_{n-1}	a_n
		b_0h	b_1h	$b_{r-1}h$	.....$b_{n-2}h$	$b_{n-1}h$
	$b_0\,(=a_0)$	b_1	b_2	b_r	b_{n-1}	R

$$\begin{array}{lll} b_r & = & \text{the sum of elements in the } (r+1)\text{ th column} \\ & = & a_r + b_{r-1}h. \end{array}$$

Corollary 1 : If h is a root of $f(x) = 0$, then $f(x) = (x-h)\,q(x)$, where $q(x)$ is given as in the theorem.

Proof : Since h is a root of $f(x) = 0$, $(x-h)$ divides $f(x)$ so that $R = 0$ and $f(x) = (x-h)\,q(x)$.

Corollary 2 : If R is the remainder when $f(x)$ is divided by $(x-h)$ then $R = f(h)$.

Proof : If $q(x)$ and R are respectively the quotient and remainder when $f(x)$ is divided by $(x-h)$, then

$$f(x) = (x-h)\,q(x) + R$$

$$\text{So } f(h) = 0\,.\,q(h) + R, \text{ or, } R = f(h).$$

Example 4 : Obtain the quotient and remainder when

$$x^6 - 6x^4 + 3x^3 - 2x^2 + 3 \text{ is divided by } x + 1.$$

Solution : The process is arranged as follows :

-1	1	0	-6	$+3$	-2	0	$+3$
		-1	$+1$	$+5$	-8	$+10$	-10
	1	-1	-5	$+8$	-10	$+10$	$-7 = R$

So the quotient is $x^5 - x^4 - 5x^3 + 8x^2 - 10x + 10$ and the remainder is -7.

Example 5 : If 4 is a root of $x^3 - 12x^2 + 39x - 28 = 0$, obtain the equation by removing the factor $x - 4$ from the given equation.

Solution : The process (with detached coefficients) is exhibited below :

	1	– 12	+ 39	– 28
– 4		+ 4	– 32	+ 28
	1	– 8	+ 7	0 = R

So the required equation is $x^2 - 8x + 7 = 0$.

Note : When $f(x)$ is divided by $(x - h)$ the remainder is $f(h)$. So, to find $f(h)$, calculate the remainder when $f(x)$ is divided by $(x - h)$.

Example 6 : Find $f(-3)$, where $f(x) = 4x^4 - 3x^3 - 2x^2 - 3$.

	4	– 3	– 2	0	– 3
– 3		– 12	+ 45	– 129	+ 387
	4	– 15	+ 43	–129	+ 384

Thus the required value is 384

Example : Find the value of $x^5 - 5x^4 - 10x^3 - 10x^2 + 5x - 1$ when $x = 2, 1, 0, -1, -2$.

(*Ans :* 1, 0, – 1, – 32, – 243)

25.7 Horner's Process

Let $f(x) \equiv a_0x^n + a_1x^{n-1} + ... + a_n$. ... (1)

For any number h, let $f(x + h) \equiv q_0x^n + q_1x^{n-1} + ... + q_n$ ... (2)

where q's are to be determined.

Replacing x by $(x - h)$ in (2) we get

$$f(x) \equiv q_0(x-h)^n + q_1(x-h)^{n-1} + ... + q_{n-1}(x-h) + q_n$$

$$\equiv (x-h)\,g(x) + q_n,$$

where $g(x) = q_0(x-h)^{n-1} + q_1(x-h)^{n-2} + ... + q_{n-1}$ which is a polynomial of $(n-1)^{th}$ degree.

Thus q_n is the remainder when $f(x)$ is divided by the binomial $(x - h)$.

Now the quotient resulting from this division is

$$q_0(x-h)^{n-1} + q_1(x-h)^{n-2} + ... + q_{n-1}.$$

Thus q_{n-1} is the remainder obtained on dividing this quotient by $(x - h)$. The other q's are similarly obtained by dividing the quotients successively by $x - h$. Finally we have $q_0 = a_0$.

Example 7 : If $f(x) = 2x^4 - 13x^2 + 10x - 19$, find $f(x + 3)$.

Solution : Here we have to divide $f(x)$ successively by $x - 3$. Thus

	2	0	−13	+10	−19
3		+6	+18	+15	+75
	2	+6	+5	+25	$+56 = q_4$
3		+6	+36	+123	
	2	+12	+41	$+148 = q_3$	
3		+6	+54		
	2	+18	$+95 = q_2$		
3		+6			
	$2 = q_0$	$+24 = q_1$			

Hence $f(x + 3) = 2x^4 + 24x^3 + 95x^2 + 148x + 56$.

Example 8 : If $f(x) = x^4 + 10x^3 + 39x^2 + 76x + 65$ find $f(x - 4)$.

Solution : We have to divide $f(x)$ successively by $x + 4$. So,

	1	+10	+39	+76	+65
−4		−4	−24	−60	−64
	1	+6	+15	+16	$+1 = q_4$
−4		−4	−8	−28	
		+2	+7	$-12 = q_3$	
−4		−4	+8		
	1	−2	$+15 = q_2$		
−4		−4			
	$1 = q_0$	$-6 = q_1$			

Thus $f(x - 4) = x^4 - 6x^3 + 15x^2 - 12x + 1$.

25.8 The highest common factor (*H. C. F.*) of two given polynomials

Suppose $f(x)$ and $g(x)$ are two polynomials. Then $d(x)$ is called the *H. C. F.* of $f(x)$ and $g(x)$ if

(i) $d(x)$ is a common factor of $f(x)$ and $g(x)$

(ii) any common factor of $f(x)$ and $g(x)$ and $d(x)$ is also a factor of $d(x)$.

The method of finding the *H. C. F.* of two polynomials is outlined below without going deep into the details.

Let the two given polynomials be f and f_1 degree f > degree f_1. Dividing f by f_1, let q_1 be the quotient and f_2, deg of f_2 < deg of f_1, be the remainder so that

$$f = f_1 q_1 + f_2$$

If f_2 is not identically zero, we may further divide f_1 by f_2 obtaining the quotient q_2 and the remainder f_3 so that

$$f_1 = f_2 q_2 + f_3.$$

Again if f_3 does not vanish identically, the division of f_2 by f_3 leads to another identity

$$f_2 = f_3 q_3 + f_4.$$

In this process the degrees of the polynomials f_1, f_2, f_3, ... diminish successively. So, if we continue this process we arrive at two posibilities viz.,

(i) there is a remainder f_r (deg > 0) such that

$$f_{r-1} = f_r q_r$$

(ii) The process continues till we get a remainder of degree zero

In (i) We say that f_r is the $H.C.F.$

and (ii) 1 is the $H.C.F.$

Now f_r is the highest common factor of f and f_1.

Example 9 : Find the $H.C.F.$ of

$8x^4 + 4x^3 - 18x^2 + 11x - 2$ and $32x^3 + 12x^2 - 36x + 11$

Solution : First let us exhibit the process in its fullness.

$$8x^4 + 4x^3 - 18x^2 + 11x - 2$$
$$\underline{\quad 4 \quad}$$
$$32x^3 + 12x^2 - 36x + 11)\ 32x^4 + 16x^3 - 72x^2 + 44x - 8\ (x$$
$$\underline{32x^4 + 12x^3 - 36x^2 + 11x}$$
$$4x^3 - 36x^2 + 33x - 8$$
$$\underline{\quad 8 \quad}$$
$$32x^3 - 288x^2 + 264x - 64\ (1$$
$$\underline{32x^3 + 12x^2 - 36x + 11}$$
$$-75)\ \underline{-300x^2 + 300x - 75}$$
$$4x^2 - 4x + 1$$

$$4x^2 - 4x + 1)\ 32x^3 + 12x^2 - 36x + 11\ (8x + 11$$
$$\underline{32x^3 - 32x^2 + 8x}$$
$$+44x^2 - 44x + 11$$
$$\underline{44x^2 - 44x + 11}$$
$$0$$

So the required $H.C.F.$ is $4x^2 - 4x + 1$.

In practice the successive divisions are performed with detached coefficients in the following way.

f	8	+4	−18	+11	−2	32	+12	−36	+11	f_1
	4					32	−32	8		8
1	32	+16	−72	+44	−8		+44	−44	+11	11
	32	+12	−36	+11			44	−44	+11	
		+4	−36	+33	−8		0	0	0	
		8								
1		+32	−288	+264	−64					
		+32	+12	−36	+11					
		−75)	−300	+300	−75					
f_2			4	−4	1					

So the required *H. C. F.* is $4x^2 - 4x + 1$.

Example 10 :. Find the *H. C. F.* of $3x^3 - 7x^2 - 18x - 8$ and $2x^3 - 3x^2 - 17x - 12$.

Let us perform the successive divisions with detached coefficients in the following way :

f	3	−7	−18	−8	2	−3	−17	−12	f_1
	2				2	−6	−8		2
3	6	−14	−36	−16		3	−9	−12	3
	6	−9	−51	−36		3	−9	−12	
	−5)	−5	+15	+20		0	0	0	
f_2		1	−3	−4					

Thus the required *H. C. F.* $x^2 - 3x - 4$.

25.9 Relation between roots and coefficients

Let $f(x) = 0$ be an equation of degree n and $\alpha_1, \alpha_2, \ldots, \alpha_n$ its roots. We adopt the following notation. S_1 denotes the sum of $\alpha_1, \alpha_2 \ldots, \alpha_n$, namely, $\alpha_1 + \alpha_2 + \ldots + \alpha_n$. S_2 denotes the sum of the products of $\alpha_1, \alpha_2, \ldots, \alpha_n$ taken two at a time, namely, $S_2 = \Sigma\alpha_1 \alpha_2$. In general S_r denotes the sum of the products of $\alpha_1, \alpha_2, \ldots, \alpha_n$ taken r at a time, namely, $S_r = \Sigma\alpha_1 \alpha_2 \ldots \alpha_r$

In particular $S_n = \Sigma\alpha_1 \alpha_2 \ldots \alpha_n$, the product of $\alpha_1, \alpha_2, \ldots \alpha_n$. For example, if $n = 2$, $S_1 = \alpha_1 + \alpha_2$ and $S_2 = \alpha_1 \alpha_2$. If $n = 3$, $S_1 = \alpha_1 + \alpha_2 + \alpha_3$, $S_2 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1$ and $S_3 = \alpha_1 \alpha_2 \alpha_3$. If $n = 4$, $S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $S_2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_1 \alpha_4 + \alpha_2 \alpha_3 + \alpha_2 \alpha_4 + \alpha_3 \alpha_4$, $S_3 = \alpha_1 \alpha_2 \alpha_3 + \alpha_2 \alpha_3 \alpha_4 + \alpha_3 \alpha_4 \alpha_1 + \alpha_4 \alpha_1 \alpha_2$ and $S_4 = \alpha_1 \alpha_2 \alpha_3 \alpha_4$.

Theorem 6 : If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$, then $S_r = (-1)^r \dfrac{a_r}{a_0}$, for $r = 1, 2, 3, \ldots, n$.

Proof : Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0,$$

we have

$$\left(x - \alpha_1\right)\left(x - \alpha_2\right) \cdots \left(x - \alpha_n\right) \equiv R\left(a_0 x^n + a_1 x^{n-1} + \ldots + a_n\right),$$ for some constant R.

That is

$$x^n - \left(\Sigma \alpha_1\right) x^{n-1} + \left(\Sigma \alpha_1 \alpha_2\right) x^{n-2} + \ldots + (-1)^r \left(\Sigma \alpha_1 \alpha_2 \ldots \alpha_r\right) x^{n-r} + \ldots$$

$$+ (-1)^n \alpha_1 \alpha_2 \ldots \alpha_n \equiv Ra_0 x^n + Ra_1 x^{n-1} + \ldots + Ra_n = 0.$$

That is,

$$x^n - S_1 x^{n-1} + S_2 x^{n-2} + \ldots + (-1)^r S_r x^{n-r} + \ldots + (-1)^n S_n$$

$$\equiv Ra_0 x^n + Ra_1 x^{n-1} + \ldots + Ra_n .$$

Comparing the coefficients of the like terms on both sides of this identity we have $Ra_0 = 1$ and $S_r = (-1)^r Ra_r, r = 1, 2, \ldots, n$.

Thus we get

$$S_r = (-1)^r \frac{a_r}{a_0}, r = 1, 2, \ldots, n.$$

Example 12 : Solve the equation $x^3 - 5x^2 - 16x + 80 = 0$, the sum of two of its roots being zero

Solution : Here $a_0 = 1, a_1 = -5, a_2 = -16, a_3 = 80$. The given equation being a cubic equation it has three roots. Let the roots be $\alpha, -\alpha, \beta$.

Then

$$S_1 = \alpha - \alpha + \beta = -\frac{a_1}{a_0} = \frac{-(-5)}{1}, \text{ or, } \beta = 5$$

$$S_2 = -\alpha\alpha + \alpha\beta - \alpha\beta = \frac{a_2}{a_0} = \frac{-16}{1} = -16$$

$$S_3 = -\alpha^2 \beta = -\frac{a_3}{a_0} = \frac{-80}{1}, \text{ or } \alpha^2 = \frac{80}{5} = 16.$$

That is $\alpha = 4$ or -4, so that the roots are $-4, 4, 5$.

Example 13 : Solve the equation $x^3 - 12x + 16 = 0$, two of its roots being equal.

Solution : Here $a_0 = 1, a_1 = 0, a_2 = -12, a_3 = 16$.

So, if α, α, β are the roots, then

$$S_1 = \alpha + \alpha + \beta = -\frac{0}{1} = 0, \text{ or } \beta = -2\alpha,$$

$$S_2 = \alpha^2 + \alpha\beta + \alpha\beta = -\frac{-12}{1} = -12 \quad \ldots \text{(i)}$$

$$S_3 = \alpha^2 \beta = \frac{-16}{1} = -16 \quad \ldots \text{(ii)}$$

From (ii), we have $\alpha^2(-2\alpha) = -16$, or $\alpha^3 = 8$. So $\alpha = 2$ and the roots are $2, 2, -4$.

25.10 Note on numbers in arithmetic progression (A. P.)

A sequence of numbers $a_1, a_2, \ldots, a_n \ldots$ is said to be in arithmetic progression ($A.P$), if the difference between every pair of consecutive numbers is the same constant that is,

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_{r+1} - a_r = \text{constant}.$$

Three numbers which are in $A.P$ are represented by $a - d, a, a + d$.

Four numbers which are in *A. P* are represented by $a-3d, a-d, a+d, a+3d$. If these are respectively denoted by $\alpha, \beta, \gamma, \delta$ then $\alpha+\delta = \beta+\gamma$.

Example 14 : Solve the equation $x^3 - 12x^2 + 39x - 28 = 0$, given that the roots are in *A. P.*

Solution : Let the roots be $\alpha - d, \alpha, \alpha + d$. Then

$$\alpha - d + \alpha + \alpha + d = -\frac{(-12)}{1} = 12.$$

So $3\alpha = 12$, or, $\alpha = 4$.

Taking the product of the roots, we get

$$(\alpha - d)\,\alpha\,(\alpha + d) = -\frac{(-28)}{1} = 28.$$

That is,

$$\alpha\left(\alpha^2 - d^2\right) = 28, \text{ or, } \alpha^2 - d^2 = 7.$$

Thus

$$d^2 = 9, \text{ or, } d = \pm\ 3.$$

If $d = 3$, the roots are 1, 4, 7 and if $d = -3$ the roots are 7, 4, 1.

Example 15 : Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, given that the roots are in *A. P.*

Solution : This is an equation of degree 4. So it has four roots and these are given to be in *A. P.* So if the roots are $\alpha, \beta, \gamma, \delta$ then

$$\alpha + \delta = \beta + \gamma.$$

But $\quad S_1 = \alpha + \beta + \gamma + \delta = -\frac{(-2)}{1} = 2.$

So $\alpha + \delta = \beta + \gamma = 1.$

Now

$$x^4 - 2x^3 - 21x^2 + 22x + 40 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$= \left[x^2 - x(\alpha + \delta) + \alpha\delta\right]\left[x^2 - x(\beta + \gamma) + \beta\gamma\right].$$

$$= \left(x^2 - x + p\right)\left(x^2 - x + q\right), \qquad \text{... (1)}$$

where $p = \alpha\delta$ and $q = \beta\gamma$.

Comparing the coefficients of x and constant terms on both sides of (1) we get, $-p-q = 22$ and $pq = 40$.

So $(p-q)^2 = (p+q)^2 - 4pq = (-22)^2 - 4 \times 40 = 324$

$$\therefore \quad p - q = \pm\ 18.$$

Taking $p - q = 18$, we get $p = -2$ and $q = -20$. (If we take $p - q = -18$, we get $p = -20$ and $q = -2$). Putting these values in (1) we have

$$x^4 - 2x^3 - 21x^2 + 22x + 40 = \left(x^2 - x - 2\right)\left(x^2 - x - 20\right).$$

Thus the roots of the given equation are nothing but the roots of $x^2 - x - 2 = 0$ and $x^2 - x - 20 = 0$.

From $x^2 - x - 2 = 0$, we get $(x-2)(x+1) = 0$ or $x = -1, 2$. Similarly $x^2 - x - 20 = 0$ gives $x = -4, 5$. So the roots of the given equation are $-4, -1, 2, 5$.

25.11 Note on numbers which are in Geometric Progression

A sequence of numbers $a_1, a_2, \dots, a_r \dots$ are said to be in geometric progression (G. P)

$$\text{if } \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = \frac{a_{r+1}}{a_r} = \dots = a \text{ constant.}$$

Three numbers which are in G. P. are represented by $\frac{a}{r}, a, ar$

Four numbers which are in G. P. are represented by $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$. If these are respectively denoted by $\alpha, \beta, \gamma, \delta$ then $\alpha\delta = \beta\gamma$.

Example 16 : If the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in G. P, find the relation between p, q, r.

Solution : Let the roots be $\frac{a}{t}, a, at$. Then, the product of the roots

$$= \frac{a}{t} \, a \, at = -\frac{r}{1}, \text{ or, } a^3 = -r. \qquad \dots \text{(i)}$$

Since a is a root of the given equation $a^3 + 3pa^2 + 3qa + r = 0$ ... (ii)

From (i) and (ii) we have

$-r + 3pa^2 + 3qa + r = 0$, or $a(3pa + 3q) = 0$.

Since $a \neq 0$, this gives $3qa + 3q = 0$, or $a = \frac{-q}{p}$.

Substituting this value of a in (i), we get

$$\left(\frac{-q}{p}\right)^3 = -r, \text{ or } q^3 = p^3 r$$

Example 17 : Solve $3x^3 - 26x^2 + 52x - 24 = 0$, given that the roots are in G. P. (with real terms).

Sol : Let the roots be $\frac{a}{r}, a, ar$.

Then $\frac{a}{r} \, a \, ar = -\frac{(-24)}{3} = 8$. That is

$$a^3 = 8 \text{ or } a = 2.$$

Taking sum of the roots, we have

$$\frac{a}{r} + a + ar = -\frac{(-26)}{3} = \frac{26}{3}.$$

This is $a\left(\frac{1}{r} + 1 + r\right) = \frac{26}{3}$, or,

$$\frac{r^2 + r + 1}{r} = \frac{26}{2 \times 3} = \frac{13}{3}.$$

From this we get, $3r^2 - 10r + 3 = 0$. This gives $r = 3$ or $\frac{1}{3}$.

If $r = 3$, the roots are $\frac{2}{3}, 2, 6$, and if $r = \frac{1}{3}$, the roots are $6, 2, \frac{2}{3}$, which are the same in different order.

Example 18 : Solve $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$, whose roots are in G. P.

Solution : Let α, β, γ be the roots of the given equation which are in G. P. Then

$$\alpha\delta = \beta\gamma$$

Taking product of the roots, we get

$$\alpha\beta\gamma\delta = \frac{64}{1} = 64.$$

That is $\qquad (\alpha\,\delta)^2 = (\beta\,\gamma)^2 = 64$ (Since $\alpha\,\delta = \beta\,\gamma$)

$$\therefore\ \alpha\,\delta = \beta\,\gamma = \pm 8.$$

Let us take $\qquad \alpha\,\delta = \beta\,\gamma = 8.$

Now

$$x^4 + 15x^3 + 70x^2 + 120x + 64 = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$$

$$= \left[x^2 - x(\alpha + \delta) + \alpha\,\delta\right]\left[x^2 - x(\beta + \gamma) + \beta\,\gamma\right].$$

$$= \left(x^2 - px + 8\right)\left(x^2 - qx + 8\right) \qquad \text{... (1)}$$

where $p = \alpha + \delta$ and $q = \beta + \gamma$.

Comparing the coefficients of x and x^2 respectively on both sides of (1) we have

$$-8(p + q) = 120, \text{ or, } p + q = -15, \text{ and}$$

$$pq + 16 = 70 \text{ or } pq = 54.$$

Thus $\qquad (p - q)^2 = (p + q)^2 - 4pq = 9$

So $\qquad p - q = 3 \text{ or } -3.$

If $p - q = 3$, then from $p + q = -15$, we get $p = -6$ and $q = -9$.

Then

$$x^4 + 15x^3 + 70x^2 + 120x + 64 = \left(x^2 + 6x + 8\right)\left(x^2 + 9x + 8\right)$$

and the roots of the given equation are the same as the roots of the equations $x^2 + 6x + 8 = 0$ and $x^2 + 9x + 8 = 0$.

Since $x^2 + 6x + 8 = (x + 2)(x + 4)$, the roots of $x^2 + 6x + 8 = 0$ are the $-4, -2$.

Again $x^2 + 9x + 8 = (x + 1)(x + 8)$ implies that -1 and -8 are the roots of $x^2 + 9x + 9 = 0$.

Thus the roots of the given equation are $-8, -4, -2, -1$.

If we take $p - q = -3$, we get $p = -9$ and $q = -6$ and it is easy to see that we get the same roots.

Remarks : After solving the equations whose roots are in A. P. and G. P, one may be interested in solving the equations whose roots are in harmonic progression. We shall come to this aspect in the next unit.

25.12 Summary

Division algorithm states that, is $f(x)$, $g(x)$ are two polynomials and if $g(x) \neq 0$, then there exist two polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)\,q(x) + r(x)$, where $r(x) = 0$ or deg. of $r(x) <$ deg of $g(x)$.

An nth degree equation $f(x) = 0$ has at most n roots ($n \geq 1$). If $\alpha_1, \alpha_2 \ldots \alpha_n$ are n roots of the equation, then $f(x) = k\left(x - \alpha_1\right)\left(x - \alpha_2\right) \cdots \left(x - \alpha_n\right)$, where k is a constant generally taken as one. When the coefficients of the equations are real the complex roots occur in pairs. If the coefficients are rational numbers, then the roots of the form $\alpha + \sqrt{\beta}$, where $\sqrt{\beta}$ is irrational and α is rational occur in pairs (i.e., surds occur in pairs). Synthetic division and Horner's process help us in reducing the degree of the polynomial or in factorizing the equation.

When the roots are in *A. P.* it is convenient to take the roots as $a, a + d, a + 2d, \ldots$ and if the roots are in *G. P*, they may be taken as $a, ar, ar^2, \ldots$

25.13 Sample Examination Questions

I. Answer the following Questions in detail

1. Find the quotient and remainder when $2x^5 - 3x^3 + 2x^2 - 5x - 46$ is divided by $(x - 2)$.

   (*Ans :* $2x^4 + 4x^3 + 5x^2 + 12x + 19$; -8)

2. Find the quotient and remainder when $x^7 - 6x^4 + 3x^3 - 2x^2 + 3$ is divided by $(x - 1)$.

   (*Ans :* $x^6 + x^5 + x^4 - 5x^3 + 2x^2 - 4x - 4$; -1)

3. If $f(x) = 2x^4 - 13x^2 + 10x - 19$, find the value of (i) $f(x + 1)$, (ii) $f(x - 2)$.

4. If $f(x) = x^4 + 10x^3 + 39x^2 + 76x + 65$, find (i) $f(x + 1)$ and (ii) $(x - 1)$.

5. Find the *H. C. F* of

   (i) $x^5 - x^4 + 4x^2 - 3x + 2$ and $5x^4 - 3x^2 + 8x - 3$

   (*Ans :* $x^2 - x + 1$)

   (ii) $x^5 + x^4 - 4x + 2$ and $x^3 - 2x + 1$

   (*Ans :* $x - 1$).

   (iii) $4x^4 - 16x^3 + 108$ and $6x^5 - 14x^3 - 40x^2 + 36$

   $\left(Ans : \ 2\left(x^2 + 2x + 3\right)\right)$

   (iv) $4x^4 + 11x^3 + 27x^2 + 17x + 5$ and $6x^4 + 14x^3 + 26x^2 + 4x$

   (*Ans :* $x^2 + 2x + 5$).

6. Solve the equation $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, the product of two of its roots being 2.

   $\left(Ans : \ \frac{3}{2}, \frac{4}{3}, 1 \pm \sqrt{2}\right)$

7. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, if two of its roots are equal in magnitude but opposite in sign.

   $\left(Ans : \ \sqrt{3}, -\sqrt{3}, 1 \pm \sqrt{6}\right)$

8. Solving the equation $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$ given that one of its roots is $1 - \sqrt{5}$.

   $\left(Ans : \ 1 \pm \sqrt{5}, 1, 2\right)$

9. Solve the equation $x^4 - 6x^3 + 11x^2 - 10x + 2 = 0$ given that $2 + \sqrt{3}$ is a root of the equation.

   $\left(Ans : \ 2 \pm \sqrt{3}, 1 \pm i\right)$

10. Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $-\sqrt{3}$ and another $1 + 2\sqrt{-1}$. (*Hint :* the root $1 + 2\sqrt{-1}$ is equal to $1 + 2i$).

    $\left(Ans : -\sqrt{3}, +\sqrt{3}, 1 + 2i, 1 - 2i, 1\right)$

11. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two roots are in the ratio $3 : 2$.

    (*Ans :* 6, 4, -1)

12. Solve the following equations whose roots are in $A.P.$

(i) $9x^3 - 54x^2 + 107x - 70 = 0$

$\left(Ans: \frac{5}{3}, 2, \frac{7}{3}\right)$

(ii) $8x^3 - 12x^2 - 2x + 3 = 0$

$\left(Ans: -\frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right)$

(iii) $x^3 - 15x^2 + 71x - 105 = 0$

$(Ans:\ 3, 5, 7)$

(iv) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$

$(Ans:\ -5, -2, 1, 4)$

(v) $16x^4 - 64x^3 + 54x^2 + 16x - 15 = 0$

$\left(Ans: -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)$

(vi) $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$

$(Ans:\ -1, 1, 3, 5)$

13. Solve the following equations whose roots are in $G.P.$

(i) $54x^3 - 39x^2 - 26x + 16 = 0.$

$\left(Ans: \frac{1}{2}, -\frac{2}{3}, \frac{8}{9}\right)$

(ii) $x^3 - 7x^2 + 14x - 8 = 0$

$(Ans:\ 1, 2, 4)$

(iii) $8x^3 - 14x^2 + 7x - 1 = 0$

$\left(Ans: 1, \frac{1}{2}, \frac{1}{4}\right)$

(iv) $x^3 - 13x^2 + 39x - 27 = 0$

$(Ans:\ 1, 3, 9)$

(v) $3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$

$\left(Ans: \frac{1}{3}, 1, 3, 9\right)$

(vi) $3x^4 + 20x^3 - 70x^2 - 60x + 27 = 0$

$\left(Ans: \frac{1}{3}, -1, 3, 9\right).$

UNIT–26 : TRANSFORMATION OF EQUATIONS, AND RECIPROCAL EQUATIONS

26.0 Contents

26.1 Aims and Objectives

After going through this unit you will be able to :

i) obtain transformed equations whose roots have some assigned relations with the roots of a given equation, and reciprocal equations.

26.2 Introduction

Given an equation, one can transform it into another equation whose roots will be required to have some assigned relations with the roots of the given equation. The transformed equation may be easy to solve and help us in getting the required information about the original equation. In this unit we shall present some elementary transformations.

26.3 Transformation of Equations

26.3.1 Equation whose roots are equal in magnitude but opposite in sign to those of a given equation

Theorem-1 : If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $f(x) = 0$, then $f(-x) = 0$ is the equation whose roots are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$.

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$. Then $a_0x^n + a_1x^{n-1} + \ldots + a_n \equiv a_0\left(x - \alpha_1\right)\left(x - \alpha_2\right) \ldots \left(x - \alpha_n\right)$. Replacing x by $-x$ in this identity we get

$(-1)^n a_0x^n + (-1)^{n-1} a_1x^{n-1} + \ldots + a_n \equiv (-1)^n a_0\left(x + \alpha_1\right)\left(x + \alpha_2\right) \ldots \left(x + \alpha_n\right)$.

The right side of this identity vanishes at $x = -\alpha_1, -\alpha_2, \ldots, -\alpha_n$ so that $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ are the roots of the equation.

$$a_0x^{n-1} - a_1x^{n-1} + \ldots + (-1)^n a_n = 0, \text{ or, } f(-x) = 0.$$

Aliter : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the n^{th} degree equation $f(x) = 0$. Consider $f(-x) = 0$. This is also an equation of degree n, so that it has n roots. If α is a root of $f(x) = 0, f(\alpha) = 0$.

This is, $f[-(-\alpha)] = 0$. This shows that $-\alpha$ is a root of $f(-x) = 0$.

Thus $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ are the roots of the equation $f(-x) = 0$.

Ex. 1 : Obtain the equation whose roots are equal in magnitude but opposite in sign to those of $x^5 + 4x^4 - 3x^3 - 2x^2 - x + 5 = 0$.

Sol : The required equation is got by changing x to $-x$ in the given equation. Thus

$$-x^5 + 4x^4 + 3x^3 - 2x^2 + x + 5 = 0,$$

or $$x^5 - 4x^4 - 3x^3 + 2x^2 - x - 5 = 0$$

is the required equation.

26.3.2 Equation whose roots are *m* times the roots of a given equation

Theorem-2 : If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$, then the equation whose roots are $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$, $m \neq 0$ is $a_0x^n + a_1mx^{n-1} + a_2m^2x^{n-2} + \ldots + m^na_n = 0$.

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$. Then $a_0x^n + a_1x^{n-1} + \ldots + a_n \equiv a_0(x - \alpha_1), (x - \alpha_2) \ldots (x - \alpha_n)$.

For $m \neq 0$, substituting $\frac{x}{m}$ in place of x, $a_0\left(\frac{x}{m}\right)^n + a_0\left(\frac{x}{m}\right)^{n-1} + \ldots + a_n \equiv$

$$a_0\left(\frac{x}{m} - \alpha_1\right)\left(\frac{x}{m} - \alpha_2\right) \ldots \left(\frac{x}{m} - \alpha_n\right).$$

or $$a_0x^n + a_1mx^{n-1} + a_2m^2x^{n-2} + \ldots + m^na_n \equiv a_0\left(x - m\alpha_1\right)\left(x - m\alpha_2\right) \ldots \left(x - m\alpha_n\right).$$

Right hand side of this identity vanishes at $x = m\alpha_1, m\alpha_2, \ldots, m\alpha_n$. Thus $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$ are the roots of the equation.

$$a_0x^n + a_1mx^{n-1} + a_2m^2x^{n-2} + \ldots + m^na_n = 0.$$

Note : This result can be proved in a similar way to that given in the aliter to Theorem 1.

Ex. 2 : Form an equation whose roots are –2 times the roots of the equation

$$3x^4 - 5x^3 + x^2 - x + 1 = 0.$$

Sol : The required equation is $3x^4 - 5(-2)x^3 + (-2)^2x^2 - (-2)^3x + (-2)^4 . 1 = 0$.

That is, $$3x^4 + 10x^3 + 4x^2 + 8x + 16 = 0.$$

26.3.3 Equation whose roots are the reciprocals of the roots of a given equation

Theorem-3 : If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$, then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the roots of the equation

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0.$$

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$.

Then $$a_0x^n + a_1x^{n-1} + \ldots + a_n \equiv a_0\left(x - \alpha_1\right)\left(x - \alpha_2\right) \ldots \left(x - \alpha_n\right).$$

In this identity replacing x by $\frac{1}{x}$ and multiplying both sides by x^n we get

$$a_0 + a_1x + \ldots + a_nx^n = a_0\left(1 - x\alpha_1\right)\left(1 - x\alpha_2\right) \ldots \left(1 - x\alpha_n\right).$$

But the right hand side of this identity vanishes when $x = \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$. So $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the roots of the equation

$$a_0 + a_1x + \ldots + a_nx^n = 0.$$

Note : This result can be proved in a similar way to that given in the aliter to Theorem 1.

Ex. 3 : Obtain the equation whose roots are the reciprocals of the roots of the equation

$$4x^5 - 3x^4 + 2x^3 - 2x^2 + 3x - 4 = 0.$$

Sol. : The required equation is given by

$$4 - 3x + 2x^2 - 2x^3 + 3x^4 - 4x^5 = 0$$

or $$4x^5 - 3x^4 + 2x^3 - 2x^2 + 3x - 4 = 0.$$

Ex. 4 : Solve the equation $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$ given that the roots are in H.P.

Sol. : Since the roots of

$$40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0 \qquad \text{... (i)}$$

are in H.P., the roots of the equation

$$\frac{40}{x^4} - \frac{22}{x^3} - \frac{21}{x^2} + \frac{2}{x} + 1 = 0$$

or $$x^4 + 2x^3 - 21x^2 - 22x + 40 = 0 \qquad \text{... (ii)}$$

are in A.P.

If $\alpha, \beta, \gamma, \delta$ are the roots of the equation (ii), then

$$\alpha + \delta = \beta + \gamma$$

But $$\alpha + \beta + \gamma + \delta = -2$$

So $$\alpha + \delta = \beta + \gamma = -1$$

Now $$x^4 + 2x^3 - 21x^2 - 22x + 40 \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$\equiv \left[x^2 - x(\alpha + \delta) + \alpha\delta\right] \times \left[x^2 - x(\beta + \gamma) + \beta\gamma\right]$$

$$\equiv \left(x^2 + x + p\right)\left(x^2 + x + q\right)$$

where $p = \alpha\delta$ and $q = \beta\gamma$. Comparing the coefficients of x and constant terms on both sides we get

$$p + q = -22 \text{ and } pq = 40$$

So $$(p-q)^2 = (p+q)^2 - 4pq = 484 - 160 = 324$$

or $$p - q = \pm 18$$

Taking $$p - q = 18, \text{ we get } p = -2, q = -20.$$

So $$x^4 - 2x^3 - 21x^2 - 22x + 40 = \left(x^2 + x - 2\right)\left(x^2 + x - 20\right)$$

and the roots of the equation (ii) are the same as the roots of the equations $x^2 + x - 2 = 0$ and $x^2 + x - 20 = 0$ put together.

$x = -2, 1$ are the roots of $x^2 + x - 2 = 0$ and $x = -5, 4$ are the roots of $x^2 + x - 20 = 0$.

Thus the roots of the equation (ii) are $-5, -2, 1, 4$. Taking $p - q = -18$, it can be seen that $p = -20, q = -2$ and the roots of (ii) are the same as $-5, -2, 1, 4$.

Thus the roots of the given equation are $\frac{-1}{5}, \frac{-1}{2}, 1, \frac{1}{4}$.

Ex. 5 : Solve the equation $6x^3 - 11x^2 + 6x - 1 = 0$ given that the roots are in H.P.

Sol. : The roots of the equation

$$6x^3 - 11x^2 + 6x - 1 = 0 \qquad \text{... (i)}$$

are the reciprocals of the roots of the equation

$$\frac{6}{x^3} - \frac{11}{x^2} + \frac{6}{x} - 1 = 0, \text{ or } x^3 - 6x^2 + 11x - 6 = 0 \qquad \text{... (ii)}$$

So if the roots of the given equation are in H.P., then the roots of the equation (ii) are in A.P. Let $a - d$, a, $a + d$ be the roots of (ii), so that

$$a - d + a + a + d = 6, \text{ or } 3a = 6, \text{ or } a = 2.$$

Thus one root is 2. Let us remove the factor $(x-2)$ from (ii).

2	1	-6	$+11$	-6
		$+2$	-8	$+6$
	1	-4	$+3$	0

So $\quad x^3 - 6x^2 + 11x - 6 = (x-2)(x^2 - 4x + 3)$

But $\quad x^2 - 4x + 3 = (x-3)(x-1)$, so that $x = 1, 3$ are the other roots of (ii). Thus the roots of the (ii) are 1, 2, 3, so that the roots of (iii) are $1, \frac{1}{2}, \frac{1}{3}$.

26.3.4 Equation whose roots are the squares of the roots of a given equation

Theorem-4 : The equation whose roots are squares of the equation $f(x) = 0$, is got by replacing x^2 by x in $f(x) f(-x) = 0$.

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$,

so that $\quad a_0 x^n + a_1 x^{n-1} + \ldots + a_n \equiv a_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \quad \ldots (1)$

Changing x to $-x$ in this identity we get

$$a_0 (-1)^n x^n + a_1 (-1)^{n-1} x^{n-1} + \ldots + a_n \equiv a_0 (-x - \alpha_1)(-x - \alpha_2) \ldots (-x - \alpha_n)$$

$$\equiv (-1)^n a_0 (x + \alpha_1)(x + \alpha_2) \ldots (x + \alpha_n).$$

Multiplying throughout this identity by $(-1)^n$ we get

$$a_0 x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n \equiv a_0 (x + \alpha_1)(x + \alpha_2) \ldots (x + \alpha_n) \quad \ldots (2)$$

Multiplying (1) and (2) we obtain

$$\left(a_0 x^n + a_2 x^{n-2} + \ldots\right)^2 - \left(a_1 x^{n-1} + a_3 x^{n-3} + \ldots\right)^2$$

$$\equiv a_0^2 (x^2 - \alpha_1^2)(x^2 - \alpha_2^2) \ldots (x^2 - \alpha_n^2) \quad \ldots (3)$$

Since R.H.S. is a function of x^2, L.H.S. is also a function of x^2. So replacing x^2 by x in (3), we finally obtain on reduction, the form

$$a_0^2 x^n - (a_1^2 - 2a_0 a_2) x^{n-1} + \ldots + (-1)^n a_n^2 \equiv a_0^2 (x - \alpha_1^2)(x - \alpha_2^2) \ldots (x - \alpha_n^2) \quad \ldots (4)$$

The R.H.S. of (4) vanishes at $x = \alpha_1^2, \alpha_2^2, \ldots \alpha_n^2$.

So $\alpha_1^2, \alpha_2^2, \ldots \alpha_n^2$ are the roots of the equation

$$a_0^2 x^n - (a_1^2 - 2a_0 a_2) x^{n-1} + \ldots + (-1)^n a_n = 0.$$

Note : The fact that the left hand side of (3) is a function of x^2 can be seen directly as follows.

When n is even say $2k$, then left hand side of (3) becomes

$$\left(a_0 x^{2k} + a_2 x^{2k-2} + \ldots\right)^2 - x^2 \left(a_1 x^{2k-2} + a_3 x^{2k-4} + \ldots\right)^2$$

which is evidently a function of x^2

When n is odd, say $2k + 1$, then left hand side of (3) becomes

$$x^2 \left(a_0 x^{2k} + a_2 x^{2k-2} + \ldots\right)^2 - \left(a_1 x^{2k} + a_3 x^{2k-2} + \ldots\right)^2$$

which is again a function of x^2

Ex. 6 : Find the equation whose roots are the squares of the roots of the equation

$$x^5 + x^3 + x^2 + 2x + 3 = 0$$

Sol. : Let $f(x) = x^5 + x^3 + x^2 + 2x + 3$

Then $f(-x) = -x^5 - x^3 + x^2 - 2x + 3$ so that

$$f(x) \cdot f(-x) = -\left[\left(x^5 + x^3 + 2x\right) + \left(x^2 + 3\right)\right] \times \left[\left(x^5 + x^3 + 2x\right) - \left(x^2 + 3\right)\right]$$

$$= -\left[\left(x^5 + x^3 + 2x\right)^2 - \left(x^2 + 3\right)^2\right]$$

$$= -x^2\left[\left(x^4 + x^2 + 2\right)^2 - \left(x^2 + 3\right)^2\right]$$

Thus the required equation is got by replacing x^2 by x in $x^2\left(x^4 + x^2 + 2\right)^2 - \left(x^2 + 3\right)^2 = 0$. So the required equation is

$$x\left(x^2 + x + 2\right)^2 - (x + 3)^2 = 0$$

i.e., $x^5 + 2x^4 + 5x^3 + 3x^2 - 2x - 9 = 0.$

26.3.5 To increase or decrease the roots of a given equation by an assigned quantity

Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ be the given equation with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let us first find an equation whose roots are $\alpha_1 - h, \alpha_2 - h, \ldots \alpha_n - h$ for some real number h.

Since $\alpha_1, \alpha_2, \ldots \alpha_n$ are roots of $f(x) = 0$,

$$f(x) \equiv a_0\left(x - \alpha_1\right)\left(x - \alpha_2\right) \cdots \left(x - \alpha_n\right).$$

In this identity, let us change x to $x + h$. Then

$f(x + h) \equiv a_0\left(x + h - \alpha_1\right)\left(x + h - \alpha_2\right) \cdots \left(x + h - \alpha_n\right)$ and R.H.S. vanishes when $x = \alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$. So $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$ are the roots of $f(x + h) = 0$.

Thus $f(x + h) = 0$ is the equation whose roots are less by h than the roots of $f(x) = 0$.

If we want to increase the roots of $f(x) = 0$ by h it is enough if we decrease the roots of $f(x) = 0$ by $-h$.

Thus $f(x - h) = 0$ is the equation whose roots are greater by h than the roots of $f(x) = 0$.

To obtain $f(x + h)$, we make use of **Horners process.**

Ex. 7 : Diminish by 2 the roots of the equation

$$2x^5 + x^4 - 3x^2 - x + 2 = 0$$

Sol : We have to obtain $f(x + 2)$ where $f(x) \equiv 2x^5 + x^4 - 3x^2 - x + 2$. We do this by employing Horner's process.

2	2	+ 1	0	− 3	− 1	2
		+ 4	+ 10	+ 20	+ 34	+66
2	2	+ 5	+ 10	+ 17	+ 33	+68
		+ 4	+ 18	+ 56	+146	
2	2	+ 9	+ 28	+ 73	+179	
		+ 4	+ 26	+108		
2	2	+ 13	+ 54	+181		
		+ 4	+ 34			
2	2	+ 17	+ 88			
		+ 4				
	2	+ 21				

Thus the required equation is

$$2x^5 + 21x^4 + 88x^3 + 181x^2 + 179x + 68 = 0$$

Ex. 8 : Increase by 1 the roots of the equation $2x^5 - 2x^4 + 3x^3 - 2x^2 - x - 5 = 0$.

Sol. : We have to find $f(x-1)$ where $f(x) \equiv 2x^5 - 2x^4 + 3x^3 - 2x^2 - x - 5$.

Again we employ Horner's Process.

–1	2	– 2	3	– 2	– 1	– 5
		– 2	+ 4	– 7	+ 9	– 8
–1	2	– 4	+ 7	– 9	+ 8	– 13
		– 2	+ 6	– 13	+ 22	
–1	2	– 6	+ 13	– 22	+ 30	
		– 2	+ 8	– 21		
–1	2	– 8	+ 21	– 43		
		– 2	+ 10			
–1	2	– 10	+ 31			
		– 2				
	2	– 12				

Thus the required equation is

$$2x^5 - 12x^4 + 31x^3 - 43x^2 + 30x - 13 = 0$$

26.4 Reciprocal equations

The equation that remains unaltered when the variable x is changed to its reciprocal $\frac{1}{x}$, is called a *reciprocal equation.*

Consider the equation $x^5 - 5x^4 + 6x^3 - 6x^2 + 5x - 1 = 0$.

Replacing x by $\frac{1}{x}$ (and multiplying throughout by x^5) we get

$$1 - 5x + 6x^2 - 6x^3 + 5x^4 - x^5 = 0.$$

Changing the sign throughout we have

$$x^5 - 5x^4 + 6x^3 - 6x^2 + 5x - 1 = 0$$

which is the original equation. Thus it is a reciprocal equation.

Similarly one can see that the equation

$$x^4 + 4x^3 + 8x^2 + 4x + 1 = 0,$$ is a reciprocal equation.

26.4.1 Criterion for the general equation of *n*th degree to be a reciprocal equation

Let $$a_0x^n + a_1x^{n-1} + \dots + a_n = 0 \qquad \dots (1)$$

be a reciprocal equation. Then changing x to $\frac{1}{x}$ we get the equation

$$a_0 + a_1x + a_2x^2 + \dots + a_n x^n = 0 \qquad \dots (2)$$

By the definition of a reciprocal equation, (1) and (2) must be equal. The coefficients of like powers of x in (1) and (2) must be proportional. That is

$$\frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \dots = \frac{a_r}{a_{n-r}} = \dots = \frac{a_{n-r}}{a_r} = \dots = \frac{a_n}{a_0} = k \text{ say.}$$

Then $\frac{a_0}{a_n} \cdot \frac{a_n}{a_0} = kk = k^2$ or $k^2 = 1$ or $k = \pm 1$.

If $k = 1$, then $a_n = a_0, a_{n-1} = a_1 \dots a_{n-r} = a_r \dots$

This means that the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and of the same sign. Such equations are called *reciprocal equations of the first type*.

Ex. : $x^4 + 4x^3 + 8x^2 + 4x + 1$.

If $k = -1$, then $a_n = -a_0, a_{n-1} = -a_1, \ldots, a_{n-r} = -a_r \;\ldots$

This means that the coefficients of the terms equidistant from the beginning and the end are of the same magnitude but of opposite sign. These equations are called *reciprocal equations of the second type*.

Ex. : $x^6 + x^5 + x^4 - x^2 - x - 1 = 0$

Depending on the degree of the equation we have the following types of reciprocal equations.

(i) Reciprocal equations of even degree and first type

(ii) Reciprocal equations of even degree and second type

(iii) Reciprocal equations of odd degree and first type

(iv) Reciprocal equations of odd degree and second type

Let us discuss these one by one.

26.4.2 Standard Reciprocal Equation

A reciprocal equation of the even degree and first type is called a *standard reciprocal equation*.

Theorem 5 : A standard reciprocal equation can be reduced to an equation of half of its original degree.

Proof : Let $a_0x^{2m} + a_1x^{2m-1} + \ldots + a_{m-1}x^{m+1} + a_mx^m + a_{m+1}x^{m-1} + \ldots + a_1x + a_0 = 0$ ... (1)

be the given standard reciprocal equation.

Dividing throughout by x^m we get

$$a_0\left(x^m + \frac{1}{x^m}\right) + a_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots + a_{m-1}\left(x + \frac{1}{x}\right) + a_m = 0 \qquad \ldots (2)$$

Put $x + \dfrac{1}{x} = z$.

Using the identity

$$x^{k+1} + \frac{1}{x^{k+1}} = \left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right)$$

we obtain,
$$x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z\left(z^2 - 2\right) - z = z^3 - 3z$$

$$x^4 + \frac{1}{x^4} = z\left(z^3 - 3z\right) - \left(z^2 - 2\right) = z^4 + 4z^2 + 2$$

and so on. Thus we get $x^m + \dfrac{1}{x^m}$ as a polynomial of degree m in z. Substituting these values in (2) we obtain an equation of degree m in z.

Remark : After reducing a standard reciprocal equation to an equation of half of its degree, the transformed equation may be solved by the known methods. This facilitates obtaining the roots of the given equation.

Ex. 9 : Solve $x^4 + x^3 + x + 1 = 0$.

Sol.: This is a standard reciprocal equation. Dividing throughout by x^2 and rearranging the terms we obtain

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 0$$

Putting $x + \frac{1}{x} = z$, we have $x^2 + \frac{1}{x^2} = z^2 - 2$.

So equation (1) reduces to $z^2 - 2 + z = 0$ or $z^2 + z - 2 = 0$.

From this we obtain $z = 1, -2$.

When $z = 1$, we have $x + \frac{1}{x} = 1$, or $x^2 - x + 1 = 0$.

This gives $x = \frac{1 \pm i\sqrt{3}}{2}$

When $z = -2$, we get $x + \frac{1}{x} = -2$ or $x^2 + 2x + 1 = 0$

This gives $x = -1, -1$.

Thus the roots of the given equation are $-1, -1, \frac{1 + i\sqrt{3}}{2}$ and $\frac{1 - i\sqrt{3}}{2}$.

26.4.3 Reciprocal equation of even degree and second type

Theorem–6 : For a reciprocal equation of even degree and second type 1 and −1 are roots.

Proof : Let $f(x) = 0$ be a reciprocal equation of even degree, say $2m$ and of second type. Then it has $2m + 1$ terms and the coefficients of the terms equidistant from the beginning and the end are equal in magnitude but opposite in sign. Thus

$$a_0 = -a_{2m+1},\ a_1 = -a_{2m},\ \ldots a_{m-1} = -a_{m+1} \text{ and}$$
$$a_m = -a_m \quad \text{or } a_m = 0.$$

So the x^m terms do not appear in this type of equation and it takes the form

$$a_0\left(x^{2m} - 1\right) + a_1 x\left(x^{2m-2} - 1\right) + \ldots + a_{m-1}x^{m-1}\left(x^2 - 1\right) = 0$$

Evidently this equation has 1 and −1 as roots.

Remark : Since 1 and −1 are the roots of the reciprocal equation of even degree and second type, remove the factors $(x - 1)$ and $(x + 1)$ by Horner's Process. The resulting equation will be again a reciprocal equation of even degree. If it is in standard form, we try to solve it by the known methods. Otherwise again 1 and −1 are roots of this equation. Again remove the factors $(x - 1)$ and $(x + 1)$. We repeat this process till we obtain the standard equation.

Ex. 10 : Solve the equation $6x^6 + 5x^5 - 44x^4 + 44x^2 - 5x - 6 = 0$.

Sol. : This is a reciprocal equation of even degree and second type. So 1 and −1 are roots of this equation. Let us remove the factors $(x - 1)$ and $(x + 1)$.

−1	6	+ 5	− 44	+ 0	+ 44	− 5	− 6
		− 6	+ 1	+ 43	− 43	− 1	+ 6
1	6	− 1	− 43	+ 43	+ 1	− 6	0
		+ 6	+ 5	+ 38	+ 5	+ 6	
	6	+ 5	− 38	+ 5	+ 6	0	

Thus dividing $6x^6 + 5x^5 - 44x^4 + 44x^2 - 5x - 6 = 0$ by $x - 1$ and $x + 1$ successively we obtain

$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0.$$

This is a standard reciprocal equation and can be put in the form $\left(\text{by dividing throughout by } x^2\right)$

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$$

Putting $x + \frac{1}{x} = z$, this reduces to $6z^2 + 5z - 50 = 0$.

so that $z = \frac{-10}{3}, \frac{5}{2}$.

From $z = \frac{-10}{3}$, we have $x + \frac{1}{x} = \frac{-10}{3}$ or $3x^2 + 10x + 3 = 0$

From this we get $x = \frac{1}{3}, -3$.

When $z = \frac{5}{2}$ we get $x + \frac{1}{x} = \frac{5}{2}$ or $2x^2 - 5x + 2 = 0$.

This gives $x = \frac{1}{2}, 2$.

Thus the roots of the given equation are $1, -1, \frac{-1}{3}, -2, \frac{1}{2}, 2$.

26.4.4 Reciprocal equation of odd degree and first type

Theorem-7 : For a reciprocal equation of odd degree and first type, $x = -1$ is a root.

Proof : Since the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and sign and the equation is of odd degree, it is of the form

$$a_0x^{2m+1} + a_1x^{2m} + \ldots + a_mx^{m+1} + a_mx^m + \ldots + a_1x + a_0 = 0$$

and this can be written in the form

$$a_0\left(x^{2m+1} + 1\right) + a_1x\left(x^{2m-1} + 1\right) + \ldots + a_mx^m(x + 1) = 0.$$

This is satisfied by $x = -1$. So -1 is a root of the given equation.

Remark : If we remove the factor $x + 1$ from the given equation we obtain a reciprocal equation of the even order. This can be solved by the known methods.

Ex. 11 : Solve $7x^5 - 5x^4 - 2x^3 - 2x^2 - 5x + 7 = 0$.

Sol. : The given equation is a reciprocal equation of odd degree and first type. So -1 is a root of the given equation. Let us remove the factor $x + 1$ from the equation

−1	7	−5	−2	−2	−5	7
		−7	+12	−10	+12	−7
	7	−12	+10	−12	+7	0

The required equation is $7x^4 - 12x^3 + 10x^2 - 12x + 7 = 0$.

This is in the standard form. Dividing by x^2 and putting $x + \frac{1}{x} = z$ and simplifying we get the quadratic in z, $7z^2 - 12z - 4 = 0$.

This has roots $z = \frac{-2}{7}$ and 2.

From $z = \frac{-2}{7}$ we get $7x^2 + 2x + 7 = 0$.

So $x = \frac{-1 + i\,4\sqrt{3}}{7}$ or $\frac{-1 - i\,4\sqrt{3}}{7}$

From $z = 2$ we get $x^2 - 2x + 1 = 0$ or $x = 1, 1$

Thus the roots are $1, 1, 1, \frac{-1 + i\,4\sqrt{3}}{7}$ and $\frac{-1 - i\,4\sqrt{3}}{7}$.

26.4.5 Reciprocal equation of odd degree and second type

Theorem-8 : For a reciprocal equation of odd degree and second type $x = 1$ is a root.

Proof : This type of equation can be put in the form

$$a_0\left(x^{2m+1} - 1\right) + a_1 x\left(x^{2m-1} - 1\right) + \ldots + a_m x^m (x - 1) = 0$$

and this equation is satisfied by $x = 1$.

Thus $x - 1$ is factor of this equation. By removing this factor, it reduces to a reciprocal equation of even degree and this can be solved by the known methods.

Ex. 12 : Solve $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$.

Sol. : This is a reciprocal equation of odd degree and second type. So $x - 1$ is a factor. Let us remove this factor from the given equation.

1	6	– 1	– 43	+ 43	+ 1	– 6
		+ 6	+ 5	– 38	+ 5	+ 6
	6	+ 5	– 38	+ 5	+ 6	0

After removing the factor $x - 1$, we obtain the equation

$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$$

and this is in the standard form. Dividing throughout by x^2 and putting $x + \frac{1}{x} = z$ and simplifying, we get the quadratic in z, $6z^2 + 5z - 50 = 0$. This has roots $z = \frac{5}{2}, \frac{-10}{3}$.

From $z = \frac{5}{2}$ we get $2x^2 - 5x + 2 = 0$.

i.e., $(x - 2)(2x - 1) = 0$ or $x = 2, \frac{1}{2}$

From $z = \frac{-10}{3}$ we get $3x^2 + 10x + 3 = 0$.

i.e., $(x + 3)(3x + 1) = 0$ or $x = -3, -\frac{1}{3}$.

26.5 Summary

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of the equation $f(x) = 0$, then $f(-x) = 0$ is the equation whose roots are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$. If we replace x by $\frac{x}{m}$ in the equation $f(x) = 0$, we get an equation $g(x) = f\left(\frac{x}{m}\right) = 0$ as the equation whose roots are m times the roots of $f(x) = 0$. If you replace x by $\frac{1}{x}$ in the equation $f(x) = 0$, we get an equation $g(x) = f\left(\frac{1}{x}\right) = 0$ where the roots are reciprocals of the roots of $f(x) = 0$. The equation whose roots are the squares of the roots of the equation $f(x) = 0$, is obtained by replacing x^2 by x in the equation $f(x)\,f(-x) = 0$. If you replace x by $x + h$ in the equation $f(x) = 0$, we get an equation $g(x) = 0$, whose roots are less than the roots of $f(x) = 0$ by h. Similarly, if the roots are to be increased by h, then replace x by $x - h$ in $f(x) = 0$. Therefore, $f(x - h) = 0$ has the roots which are greater than the roots of $f(x) = 0$ by h and $f(x + h) = 0$ has the roots which are less than the roots of $f(x) = 0$ by h.

If the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and of the same sign, then such equations are called reciprocal equations of the first type. But, if the magnitudes of the terms are as said in the first type, but the sign of the terms are opposite, then it is called a reciprocal equation of the second type. A reciprocal equation of even degree and of the first type is called the standard type of reciprocal equation. It can be reduced to an equation of degree equal to the half of the degree of the original equation. The reciprocal equation of first type and of odd degree has, –1 as one of its roots.

26.13 Sample Examination Questions

I. *Answer the following questions in detail.*

1. Obtain the equations whose roots are equal in magnitude but opposite in sign to those of the equations

   a) $x^5 + 4x^4 + 3x^3 - 2x^2 - 4x - 7 = 0$

   $\left(\text{Ans. : } x^5 - 4x^4 + 3x^3 + 2x^2 - 4x + 7 = 0\right)$

   b) $x^6 - x^4 + x^2 - 1 = 0$

   $\left(\text{Ans. : } x^6 - x^4 + x^2 - 1 = 0\right)$

2. Form an equation whose roots are (i) 5 times (ii) – 6 times the roots of the equation

   $x^3 - 4x^2 + \frac{1}{2}x - \frac{1}{9} = 0$

3. Form an equation whose roots are the reciprocals of the equations

   a) $x^5 - 4x^3 + 6x^2 - 3x + 2 = 0$ $\left(\text{Ans. : } 2x^5 - 3x^4 + 6x^3 - 4x^2 + 1 = 0\right)$

   b) $x^4 - 3x^3 + x^2 - x + 5 = 0$ $\left(\text{Ans. : } 5x^4 - x^3 + x^2 - 3x + 1 = 0\right)$

4. Solve the following equations whose roots are in H.P.

   a) $6x^3 - 11x^2 - 3x + 2 = 0$ $\left(\text{Ans. : } \frac{1}{2}, 2, \frac{1}{3}\right)$

   b) $81x^3 - 18x^2 - 36x + 8 = 0$ $\left(\text{Ans. : } \frac{2}{9}, \frac{2}{3}, \frac{-2}{3}\right)$

   c) $70x^3 - 107x^2 + 54x - 9 = 0$ $\left(\text{Ans. : } \frac{5}{2}, 2, \frac{7}{3}\right)$

   d) $40x^4 + 22x^3 - 21x^2 - 2x + 1 = 0$ $\left(\text{Ans. : } -\frac{1}{4}, -1, \frac{1}{2}, \frac{1}{5}\right)$

5. Form an equation whose roots are the squares of the roots of the following equations

   a) $x^5 - 3x^4 + 2x^3 - 3x^2 - 5 = 0$ $\left(\text{Ans. : } x^5 - 5x^4 - 14x^3 - 39x^2 - 30x - 25 = 0\right)$

   b) $x^4 - x^2 + x - 3 = 0$ $\left(\text{Ans. : } x^4 - 2x^3 - 5x^2 + 5x + 9 = 0\right)$

6. Diminish by 2 the roots of the equation

   $2x^5 - 2x^4 + 3x^3 - 2x^2 - x - 5 = 0$

   $\left(\text{Ans. : } 2x^5 + 18x^4 + 67x^3 + 128x^2 + 123x + 41 = 0\right)$

7. Increase the roots of the following equation by 3

   $2x^5 + x^4 - 3x^2 - x + 2 = 0$ $\left(\text{Ans. : } 2x^5 - 29x^4 + 168x^3 - 489x^2 + 719x - 427 = 0\right)$

II. Solve the following reciprocal equations.

1. $2x^4 + x^3 - 6x^2 + x + 2 = 0.$ $\left(\text{Ans. : } 1, 1, \frac{-1}{2}, -2\right)$

2. $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ $\left(\text{Ans. : } 1, \frac{3 \pm \sqrt{5}}{2}, \frac{1 \pm i\sqrt{3}}{2}\right)$

3. $x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1 = 0$ $\left(\text{Ans. : } 1, -1, \frac{-1 \pm i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}\right)$

4. $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$ $\left(\text{Ans. : } -1, \frac{1}{3}, \frac{1}{2}, 2, 3\right)$

UNIT–27 : NATURE OF ROOTS

27.0 Contents

27.1 Aims and Objectives

After going through this unit, you will be able to obtain the maximum number of the positive and negative roots of an equation by means of Descartes' rule.

27.2 Descartes' Rule of Signs

Descartes' rule of signs gives a method of finding the maximum number of positive and negative roots of an equation. Using this one can also determine the minimum number of complex roots.

Consider a polynomial $f(x)$ and write the signs of the terms side by side in a row. If two adjucent signs are the same, they do not contribute any change of sign, otherwise they contribute one change in sign. In this way we count the number of changes in signs of $f(x)$. Similar thing can be done to $f(-x)$.

Let us explain these for $f(x) = x^7 - x^6 + 3x^4 - 2x^3 - x + 3$. Arranging the signs of the terms of $f(x)$ in a row we have

change   change   change   change

+   −   +   −   −   +

no change

thus, there are four changes of sign in $f(x)$. Next consider $f(-x) = -x^7 - x^6 + 3x^4 + 2x^3 + x + 3$. The signs of the terms of $f(-x)$ are

change

−   −   +   +   +   +

change   no change   no change   no change.

Hence, there is only one change of sign.

The number of changes of sign in terms of $f(x)$ and $f(-x)$ play very important role in determining the nature of roots of $f(x) = 0$.

Descartes' Rule explained

Let $g(x)$ be a polynomial whose signs of the terms are as follows :

+ + − − − + − + + − +

In this, there are *six* changes of sign. We shall show that if this polynomial is multiplied by a binomial of the form $x - h$, whose signs of the terms are + , −, the resulting polynomial will have at least one more change of sign than the original. Writing down only the signs of the terms in the multiplication, we have

+	+	−	−	−	+	−	+	+	−	+	
									+	−	
	−	−	+	+	+	−	+	−	−	+	−
+	+	−	−	−	+	−	+	+	−	+	
+	±	−	±	±	+	−	+	±	−	+	−

In the last row the ambiguous sign ± is placed whenever two different signs are to be added. In the product we observe

(1) an ambiguity replaces each continuation of sign in the original polynomial ;

(2) the signs before and after an ambiguity or a set of ambiguities are unlike ; and

(3) a change of sign is introduced at the end.

Let us take the most unfavourable case, in which the minimum number of changes occur in the product. This is achieved when all the ambiguities are replaced by continuations. Then the signs of the terms of the product are

$$+ \quad + \quad - \quad - \quad - \quad + \quad - \quad + \quad + \quad - \quad + \quad -$$

Here the number of changes is *seven*, one greater those of $g(x)$. Thus even in the most unfavourable case there is one more change of sign in the product than the number of changes of sign in $g(x)$. This shows that the effect of multiplying a ploynomial $g(x)$ by a binomial $(x - h)$ is to introduce at least one more change of sign than that of $g(x)$.

(i) Descartes' rule of signs for positive roots.

Suppose that $f(x) = 0$ is a given equation whose negative and complex roots make up the factor $g(x)$. Then

$$f(x) = g(x)\left(x - \alpha_1\right)\left(x - \alpha_2\right) \ldots \left(x - \alpha_k\right) \qquad \ldots (1)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the remaining positive roots of $f(x) = 0$. The above discussion makes us to infer that the effect of multiplying $g(x)$ by each of the factors $x - \alpha_1, x - \alpha_2, \ldots, x - \alpha_k$ corresponding to the positive roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ is to introduce at least one change of sign. Thus, when $f(x)$ is formed by computing the product on the right hand side of (1), $f(x)$ has at least k changes of sign that is, as many changes of sign as it has positive roots. From this we conclude that the number of positive roots of $f(x) = 0$ cannot exceed the number of changes of signs of terms of $f(x)$. This is the Descartes' rule of signs for positive roots.

(ii) Descartes' rule of signs for negative roots.

Let $f(x) = 0$ be the given equation. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x) = 0$, then $-\alpha_1, -\alpha_2, \ldots, -\alpha_k$ are the roots of $f(-x) = 0$. So the negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$ and the number of positive roots of $f(-x) = 0$ cannot exceed the number of changes of signs of terms in $f(-x)$.

The number of negative roots of $f(x) = 0$ can not exceed the number of changes of sign of the terms of $f(-x) = 0$. This is descartes' rule of negative roots.

(ii) Descartes' rule for real roots.

Combining (i) and (ii) we get an upper limit for the number of real roots of an equation. The positive roots and negative roots put together give the real roots of an equation. Thus the number of real roots of $f(x) = 0$ cannot exceed the number of changes of sign of the terms of $f(x) = 0$ and the number of changes of signs of the terms of $f(-x) = 0$.

(iv) Descrates' rule and complex roots.

Using Descartes' rule one can assertain whether an equation has complex roots or not.

Suppose $f(x) = 0$ is an equation of degree n. Let r and s be respectively the maximum number of positive and negative roots of $f(x) = 0$. Thus the maximum number of real roots of $f(x) = 0$ is $r + s$. If $r + s < n$, then $f(x) = 0$ must have at least $n - (r + s)$ complex roots.

Ex. 1 : Determine completely the nature of roots of the equation

$$x^{10} - 2x^4 + 3x^3 - x^2 + 2 = 0$$

Sol. : The series of signs of the terms of the given equation is

$+ \quad - \quad + \quad - \quad +$

So there are four changes of sign. Thus the number of positive roots cannot exceed 4.

Changing x to $-x$ in the given equation, we get

$$x^{10} - 2x^4 - 3x^3 - x^2 + 2 = 0$$

The series of signs of the terms of the equation is

$+ \quad - \quad - \quad - \quad +$

The number of changes of sign is 2, so that the given equation cannot have more than 2 negative roots.

Thus the maximum number of real roots of the given equation is 6.

Since the degree of the given equation is 10, it has 10 roots. Thus the given equation must have atleast $10 - 6 = 4$ complex roots.

Ex. 2 : Show that the equation $x^6 - 3x^2 - x + 1 = 0$ has atleast two complex roots.

Sol. : The series of signs of the terms of the given equation is

$+ \quad - \quad - \quad +$

Here there are two changes of sign. Hence there cannot be more than two positive roots.

Changing x to $-x$, the equation becomes

$$x^6 - 3x^2 + x + 1 = 0.$$

The series of signs of the terms of this equation is

$+ \quad - \quad + \quad +$

There are two changes of sign, so that there cannot be more than two negative roots.

So the given equation has at most four real roots. But the total number of roots of the equation is six. Hence there are atleast two complex roots for the given equation.

27.3 Multiple Roots

In this section we give a method of finding multiple roots of a given equation.

Theorem-1 : For any integer $m \geq 2$, $(x - \alpha)^m$ is a factor of $f(x)$ if and only if $(x - \alpha)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$.

Proof : Suppose that for $m \geq 2$, $(x - \alpha)^m$ is a factor of $f(x)$. Then for some polynomial $g(x)$

$$f(x) = (x - \alpha)^m g(x).$$

$$\therefore \quad f'(x) = (x - \alpha)^m g'(x) + m(x - \alpha)^{m-1} g(x).$$

$$= (x - \alpha)^{m-1} [(x - \alpha) g'(x) + m\, g(x)].$$

So $(x - \alpha)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$.

Conversely assume that $(x - \alpha)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$. Then $(x - \alpha)^{m-1}$ divides $f(x)$ implies that

$$f(x) = (x - \alpha)^{m-1} h(x)$$

for some polynomial $h(x)$. So

$$f'(x) = (x - \alpha)^{m-2} [(x - \alpha) h'(x) + (m - 1) h(x)] \qquad \ldots (1)$$

Since $(x-\alpha)^{m-1}$ also divides $f'(x)$, equation (1) implies that $(x-\alpha)$ divides $(x-\alpha)h'(x) + (m-1)h(x)$. That is $(x-\alpha)$ divides $h(x)$, so that $h(x) = (x-\alpha)h_1(x)$ for some polynomial $h_1(x)$. Thus $f(x) = (x-\alpha)^{m-1}h(x) = (x-\alpha)^m h_1(x)$, so that $(x-\alpha)^m$ divides $f(x)$.

Corollary 1 : If $(x-\alpha)^{m-1}$ is the *H.C.F* of $f(x)$ and $f'(x)$, then α is a repeated root of $f(x) = 0$ with multiplicity m.

Proof : Suppose that $(x-\alpha)^{m-1}$ is the *H.C.F* of $f(x)$ and $f'(x)$. Then $(x-\alpha)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$, but $(x-\alpha)^m$ is not so. Thus by the above Theorem $(x-\alpha)^m$ is a factor of $f(x)$, but $(x-\alpha)^{m+1}$ is not so. So α is a repeated root of multiplicity m of $f(x) = 0$.

The following corollary follows from corollary 1.

Corollary 2 : If $\phi(x)$ is the *H.C.F* of $f(x)$ and $f'(x)$, and $\phi(x) = (x-\alpha_1)^{m_1}(x-\alpha)^{m_2} \ldots (x-\alpha)^{m_r}$, then $\alpha_1, \alpha_2, \ldots, \alpha_r$ are repeated roots of $f(x) = 0$ of multiplicity $m_1+1, m_2+1, \ldots, m_r+1$ respectively.

Thus, to obtain the multiple roots of $f(x) = 0$, we adopt the following procedure :

1. Find $f'(x)$
2. Find the *H. C. F.*, $\phi(x)$ of $f(x)$ and $f'(x)$
3. Express $\phi(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2} \ldots (x-\alpha_r)^{m_r}$.
4. Then the repeated roots of $f(x) = 0$ are $\alpha_1, \alpha_2, \ldots, \alpha_r$ with respective multiplicity $m_1+1, m_2+1, \ldots, m_r+1$.

Example 3 : Solve the equation $4x^3 - 12x^2 - 15x - 4 = 0$, given that it has repeated roots.

Sol. : Let $f(x) = 4x^3 - 12x^2 - 15x - 4$. Then

$$f'(x) = 12x^2 - 24x - 15.$$

The *H.C.F* of $f(x)$ and $f'(x)$ is found as below :

	4	−12	−15	−4	12	−24	−15	6
	3				12	6		
1	12	−36	−45	−12		−30	−15	−15
	12	−24	−15			−30	−15	
1		−12	−30	−12		0	0	
		−12	+24	+15				
	−27		−54	−27				
			2	1				

So the *H.C.F* of $f(x)$ and $f'(x)$ is $2x+1$.

$\therefore$ $(2x+1)^2$ is a factor of $f(x)$. Dividing $f(x)$ by $(2x+1)^2$, or, $4x^2+4x+1$ we get

$$\left(4x^3 - 12x^2 - 15x - 4\right) = (2x+1)^2(x-4).$$

Thus $x = -\frac{1}{2}, -\frac{1}{2}$ and 4 are the roots of the given equation.

mple 4 : Solve $8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0$, given that it has repeated roots.

: Given that $f(x) = 8x^4 + 4x^3 - 18x^2 + 11x - 2$

n $f'(x) = 32x^3 + 12x^2 - 36x + 11$.

us find the $H.C.F$ of $f(x)$ and $f'(x)$.

	8	+4	−18	+11	−2	32	+12	−36	11	8
	4					32	−32	8		
1	32	+16	−72	+44	−8		44	−44	11	11
	32	+12	−36	+11			44	−44	11	
		4	−36	33	−8		0	0	0	
		8								
1		32	−288	264	−64					
		32	12	−36	11					
	−75)	−75	−300	300	−75					
			4	−4	1					

us the $H.C.F$ of $f(x)$ and $f'(x)$ is $4x^2 - 4x + 1$, or, $(2x-1)^2$.

So $(2x-1)^3$, or, $8x^3 - 12x^2 + 6x - 1$ is factor of $f(x)$. By actual division we have

$8x^4 + 4x^3 - 18x^2 + 11x - 2 = (2x-1)^3 (x+2)$.

Thus the roots of the given equation are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and -2.

. 5 : Solve the equation $x^5 - 7x^3 - 2x^2 + 12x + 8 = 0$, which has equal roots

l. : Let $f(x) = x^5 - 7x^3 - 2x^2 + 12x + 8$. Then

$f'(x) = 5x^4 - 21x^2 - 4x + 12$

To obtain the multiple roots of $f(x) = 0$, we have to find first, the $H.C.F$ of $f(x)$ and $f'(x)$.

	1	0	−7	−2	12	8	5	0	−21	−4	12	
	5						7					
1	5	0	−35	−10	60	40	35	0	−147	−28	84	5
	5	0	−21	−4	12		35	15	−120	−100		
		−2)	−14	−6	48	40	−3)	−15	−27	72	84	
			7	+3	−24	−20		5	9	−24	−28	5
			5					5	−5	−10		
			35	15	−120	−100			14	−14	−28	14
			35	63	−168	−196			14	−14	−28	
		−48)		−48	+48	+96			0	0	0	
				1	−1	−2						

So the $H.C.F$ of $f(x)$ and $f'(x)$ is $x^2 - x - 2$ or $(x+1)$ $(x-2)$. Thus $(x+1)^2 (x-2)^2$ is a factor $f(x)$. Dividing $f(x)$ by $(x+1)^2 (x-2)^2$ we have

−1	1	0	−7	−2	12	8
		−1	+1	+6	−4	−8
−1	1	−1	−6	+4	+8	0
		−1	+2	+4	−8	
2	1	−2	−4	+8	0	
		+2	0	−8		
2	1	0	−4	0		
		2	4			
	1	2	0			

So $x+2$ is the quotient when $f(x)$ is divided by $(x+1)^2 (x-2)^2$. Thus

$$f(x) = (x+1)^2 (x-2)^2 (x+2),$$

and the roots of $f(x) = 0$ are $-1, -1, 2, 2, -2$.

27.4 Summary

Descartes' rule of signs : The number of positive roots of $f(x) = 0$ cannot exceed the number of changes of signs of terms of $f(x)$. The number of negative roots of $f(x) = 0$ cannot exceed the number of changes in the signs of the terms of $f(-x) = 0$. Therefore, the number of real roots of $f(x) = 0$ cannot exceed the total number of changes in the sign of the terms of $f(x) = 0$ and $f(-x) = 0$. Therefore if number changes of signs in $f(x) = 0$ is r and if it is s in $f(-x) = 0$ then the number of common roots of the equation $f(x) = 0$ is $n - (r + s)$ where n is the degree the equation. If $H.C.F$ of $f(x)$ and $f'(x)$ has a factor $(x - \alpha)^{m-1}$ then $f(x) = 0$ has a multiple root with multiplicity m. Therefore to obtain multiple roots of $f(x) = 0$ find $H.C.F$ of $f(x)$ and $f'(x)$ then factorize $H.C.F$. If $H.C.F = (x - \alpha_1)(x - \alpha_2)$... then $f(x) = 0$ has a root α_1 with multiplicity $p + 1$ a root α_2 with multiplicity $q + 2$ .... etc.

27.5 Sample Examination Questions

I. *Answer the following in detail.*

1) Show that the equation $x^7 - 3x^4 + 12x^2 + 5x - 4 = 0$ has atleast two complex roots.

2) Determine completely the nature of the roots of the equation $x^8 + 10x^3 + x - 4 = 0$. (Atmost one positive root and one negative root. Atleast 6 complex roots)

3) Solve the following equations, given that each of them has repeated roots.

(i) $x^3 - 5x^2 + 3x + 9 = 0$.

(ii) $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$. (*Ans* : 3, 3, − 1)

(iii) $27x^4 - 72x^2 + 64x - 16 = 0$. (*Ans* : 1, 1, 1, 3)

(iv) $x^5 - x^4 - 4x^2 + 7x - 3 = 0$ $\left(Ans : \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -2\right)$

$\left(Ans : 1, 1, 1, -1 \pm \sqrt{2}\right)$.

Author of the Units 25 – 27 : Dr. L. Nagamuni Reddy

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(1) Higher Algebra, S. BARNARD and J. M. CHILD, Mac Millan,, 1955

(2) Theory of Equations, RAM BEHARI and HANSRAJ GUPTA, S. Chand & Co., 1959.

(3) The theory of Equations, W. S. Burnside and Awo Panton, S. Chand & Co., 1957.

ANDHRA PRADESH OPEN UNIVERSITY

(Undergraduate Programmes)

Mathematics Course–2–Syllabus

(VECTORS, THREE DIMENSIONAL GEOMETRY
AND THEORY OF EQUATIONS)

Block–1 : Vector Analysis

Unit–1 : Vectors and Scalars
Unit–2 : Scalar and Vector products
Unit–3 : Vector differentiation
Unit–4 : Divergence and Curl (Differential operators)
Unit–5 : Line, Surface and Volume integrals
Unit–6 : Applications

Block–2 : Three Dimensional Geometry: Plane and Straight Line

Unit–7 : Direction Cosines and ratios of a Line
Unit–8 : Equation of a plane
Unit–9 : Equation of a straight line
Unit–10 : Shortest line segment between two skew lines
Unit–11 : Two or more planes
Unit–12 : Change of axes

Block–3 : Sphere

Unit–13 : Equation of a Sphere
Unit–14 : Plane sections of a Sphere
Unit–15 : Tangent plane and normal at a point on the sphere
Unit–16 : Polar plane and system of spheres

Block–4 : Conics

Unit–17 : Parabola
Unit–18 : Ellipse
Unit–19 : Hyperbola

Block–5 : Conicoids

Unit–20 : Surfaces of Revolution
Unit–21 : Standard Equation of Conicoids
Unit–22 : Tangent Plantes and Normals
Unit–23 : Ruled Surfaces–Cone and Cylinder
Unit–24 : Reduction and Classification

Block–6 : Theory of Equations

Unit–25 : Roots, Relation between roots and Coefficients of an Equation
Unit–26 : Transformation of Equations, and Reciprocal Equations
Unit–27 : Nature of Roots

MATHEMATICS - COURSE - II

Assignment - I

Section A

1) Prove that

(i) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(ii) $[\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

2) State and prove Gauss divergence theorem and verify the theorem for $f = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

3) (i) A fluid motion is given by $\mathbf{v} = (y + z)\,\mathbf{i} + (z + x)\,\mathbf{j} + (x + y)\,\mathbf{k}$ show that the motion is irrotational.

(ii) Prove that $\nabla \cdot (\nabla r^m) = m\,(m + 1)\,r^{m-2}$, where $r = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r$

Section B

1) Find the unit tangent vector at any point of the curve $x = at^2$, $y = 2at$, $z = 0$.

2) If $\mathbf{F} = x^2 y\,\mathbf{i} - 2\,xz\,\mathbf{j} + 2\,yz\,\mathbf{k}$, find $\nabla \times (\nabla \times \mathbf{F})$

3) Find the value of λ such that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, where $\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\,\mathbf{j} + 3\mathbf{k}$.

MATHEMATICS - COURSE - II

Assignment - II

Section A

1) Find the length and equations of the shortest line segment between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}, \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

2) Show that $12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$ represent a pair of planes and find the angle between them.

3) (i) Prove that the locus of the poles with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, of the tangents to the auxiliary cirle $x^2 + y^2 = a^2$ is another ellipse.

(ii) Find the foci, eccentricity and latus rectum of the hyperbola $x^2 - 4y^2 = 4$.

Section B

1) Show that the lines whose, direction cosines are given by $3lm - 4ln + mn = 0, l + 2m + 3n = 0$ are perpendicular.

2) Obtain the equations of the spheres passing through the points (4, 1, 0), (2, −3, 4), (1, 0, 0) and touching the plane $2x + 2y - z = 11$.

3) Obtain the equation of a sphere which cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ and is passing through the cirle $x^2 + y^2 + z^2 - 2x + 3y - 14z + 6 = 0, 3x - 4y + 5z - 15 = 0$.

MATHEMATICS - COURSE - II

Assignment - III

Section A

1) Reduce the following equation to standard form and classify the conicoid :
$3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y - 4z + 1 = 0.$

2) Find the locus of the poles of the tangent planes to the conicoid $ax^2 + bx^2 + cx^2 = 1$ with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$.

3) Prove that "if a conicoid has three mutually parallel generators, it is a cylinder".

Section B

1) Find the surface of revolution of the parabola $y^2 = 4ax$ around the y- axis.

2) Prove that $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ is the equation of a cone of
$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$.

3) Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $\sqrt{3}$ and another $1 - 2i$.

FACULTY OF SCIENCE

B.Sc. II Year (3 Year Degree Course) Examination

Mathematics - Course II

(Vectors, Three Dimensional Geometry and Theory of Equations)

Time - 3 hours

Max. Marks - 100

Min. Marks - 35

Section A $4 \times 15 = 60$

Answer any four questions.

1. Define divergence and curl of a vector with the usual notation show that

$$\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^2}\, dS$$

2. Prove that

(i) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(ii) $[\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

3. Find the length and the equations of the shortest line segment between the lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}, \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

4. i) Obtain the necessary and sufficient condition that a homogeneous second degree equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ to represent a pair of planes.

   ii) Show that the equation $2x^2 - 2y^2 + 4z^2 + 6xz + 2yz + 3xy = 0$ represents a pair of planes and also find the angle between them.

5. i) Obtain the condition for the orthogonality of two spheres.

   ii) Find the equation of the sphere that passes through the circle

   $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0,\ 3x - 4y + 5z - 15 = 0$, and cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

6. i) Obtain the standard equation of the ellipse and state its properties.

   ii) Find the centre and eccentricity of the ellipse $9x^2 + 25y^2 - 18x - 100y - 116 = 0$.

7. Find the locus of the poles of the tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$ with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$.

8. Reduce the equation $6x^2 + 6y^2 + 13z^2 - 10yz - 10zx + 4xy + 4x - 12y + 14z + 11 = 0$ to the standard form and describe the surface.

Section B

Answer any Five of the following. $5 \times 8 = 40$

9. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

10. Find the vector equation of the line joining the points $i - 2j + k$ and $3k - 2j$.

11. Show that the lines drawn from the origin with the direction ratios (1, –1, 1), (2, –3, 0) and (1, 0, 3) are coplanar.

12. Find the distance of the point (–1, –5, –10) from the point of intersection of the line

$$\frac{1}{3}(x-2) = \frac{1}{4}(y+1) = \frac{1}{12}(z-2) \text{ and the plane } x - y + z = 5.$$

13. Find the equation of the sphere whose diameter is the line joining the origin to the point (2, –2, 4). Also find its centre and radius.

14. Show that the locus of the poles of normal chords of the parabola

$y^2 = 4ax$ is $(x + 2a)y^2 + 4a^3 = 0$

15. Obtain the equation of a tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at the point } (x_1, y_1)$$

16. Find the equation of the surface of revolution of the curve $x^2 - y^2 = 2$ in the xy–plane about the line $y + 2 = 0$.

17. Solve $x^5 - x^4 + 8x^2 - 9x - 15 = 0$. One root being $\sqrt{3}$ and another $1 - 2i$.

18. Solve the reciprocal equation $x^6 + x^5 + x^4 - x^2 - x - 1 = 0$.