MATHEMATICS

COURSE – I

Calculus, Differential Equations and Matrices

Blocks : I – IV

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Dr. B.R. AMBEDKAR OPEN UNIVERSITY
HYDERABAD
1992

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First published in 1984.

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This text forms a part of an Open University course. The complete syllabus for the course appears at the end of this text.

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Printed and Sruthi Graphics (P)Ltd,Sri Krishna Nagar,Jubilee Hills xRoad Phone:248494.

PREFACE

This book deals with the topics in Calculus, Differential Equations and Matrix theory included in the syllabus for the second year of the B.Sc. course offered by the Andhra Pradesh Open University. These topics cover the 'core area' of the subject to be studied in the Second Year of the Three Year Degree Course in Science (B.Sc.). The syllabus, for the sake of convenience, is divided into *blocks*, each of which comprises a number of units. Each block generally covers a specific area of the subject. The units are prepared by specialists in accordance with a format so designed as to enable the student read and understand them without much difficulty.

Man strives ceaselessly to discover the laws that govern nature. Scientists are trying to study the movements of the planets; the different forces that bring about changes in different objects of nature; the birth, multiplication and destruction of organisms. This book describes the changes that take place in interrelated unknown quantities, the rate at which changes take place and the manner in which they can be discussed through Calculus and Differential Equations.

The real functions – their limits and continuity, differentiation and applications of differentiation are explained in the first two blocks. Topics like Integration, evaluation of areas and volumes through integration are given in Blocks 3 and 4. How the ordinary and partial differential equations are formed, through what methods they could be solved, in what form such problems arise in geometry, Physics and Biology, these questions are discussed in Blocks 5, 6 and 7. Block 8 explains Matrix theory.

The University hopes that this course material will help the student to get acquainted with the concepts and principles of Mathematics.

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Contents

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BLOCK-1 : DIFFERENTIAL CALCULUS-I

Introduction

Very often it becomes necessary to estimate the rate of growth of increasing and decreasing physical quantities like temperature, velocity etc. For this purpose the subject of calculus was introduced. Calculus is divided into two sections. One treating Differentiation and the other treating Integration. Historically integral calculus was first invented and then differential calculus. Differential calculus was primarily concerned with the determination of rates of change of physical quantities. The integral calculus was concerned initially with the determination of areas and volumes.

Let us study the subject of calculus with the explanation of basic concepts of numbers, functions, limits etc.

Numbers

1. *Natural numbers :* The numbers used in the process of counting the members of a set are called natural numbers. These are named as one, two, three, etc. The sum $a + b$ and the product $a.b$ of any two natural numbers a and b is also a natural number as is well known. That is, the set of natural numbers satisfies the closure property with respect to the operations of addition and multiplication.

The set of natural numbers is denoted by N.

Thus $N = \{1, 2, 3, 4, ...\}$.

Natural numbers are also called positive integers.

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2. *Integers :* The system of natural numbers is quite insufficient even for elementary calculations. Given two natural numbers a and b it is not always possible to find a natural number x such that $a + x = b$. Thus we require an extended system of numbers,... ,$-3, -2, -1$ and 0 to permit solutions of equations like $a + x = b$. The set of positive, negative integers and zero is called the set of integers I.

Thus $I = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

The sum, difference and product of two integers is also an integer. Inequalities like $2 < 3$ are well known.

If n is an integer, there does not exist an integer between n and $n + 1$.

3. *Rational numbers :* The system of integers is also insufficient to make elementary calculations. There is no integer x such that $3x = 4$. There arises the necessity of extending the system to permit solutions of equations such as $ax = b$ for all integers a and b, $a \neq 0$. This leads to the operation of division or inverse of multiplication to enable us to write $x = b/a$. In effect a rational number is an ordered pair of integers. The sum, difference, product and quotient of two rational numbers is defined in the usual way and they are also rational numbers . They are denoted by the set $Q = \left\{ \frac{p}{q} : p, q \text{ are integers and } q \neq 0 \right\}$. The set of integers is a subset of rational numbers since integers are rational numbers when $q = 1$.

Let x and y be two rational numbers such that $x < y$.

$$x < y \quad \Rightarrow \quad x + x < x + y$$

$$\Rightarrow \quad 2x < x + y$$

$$\Rightarrow \quad x < \frac{x+y}{2}$$

$$\text{Also } x < y \quad \Rightarrow \quad x + y < y + y$$

$$\Rightarrow \quad x + y < 2y$$

$$\Rightarrow \quad \frac{x+y}{2} < y.$$

$$\therefore \; x < \frac{x+y}{2} < y.$$

$\therefore \; \frac{x+y}{2}$, the arithmetic mean between x and y, is a rational number which lies between x and y. If x and y are any two rational numbers then $(n-1)$ arithmetic means between x and y are

$$x + \frac{y-x}{n}, \; x + \frac{2\,(y-x)}{n}, \ldots, \; x + \frac{(n-1)\,(y-x)}{n}.$$

These numbers are all greater than x and less than y. Since n is arbitrary we conclude that between two rational numbers x and y there exist as many rationals as we please. We may say that the rational numbers are dense in the neighbourhood of any one of them.

4. *Irrational numbers :* There exists no rational number whose square is 2. i.e., $\sqrt{2}$ is not a rational number. If possible, let a/b $(b \neq 0)$ be a rational number whose square is equal to 2.

We can assume that a, b are positive integers with no common factor.

$$\therefore \; \left(\frac{a}{b}\right)^2 = 2 \Rightarrow a^2 = 2b^2.$$

Obviously, $2b^2$ is even and so a^2 is even.

$\therefore$ a is even.

Let $a = 2m$. Then $2b^2 = 4m^2$ or $b^2 = 2m^2$ and so b is also even. Hence a, b have 2 as a common factor. This contradicts the hypothesis.

Thus there exists no rational number whose square is equal to 2. In other words $\sqrt{2}$ is an irrational number. The set of all such numbers (which are not rational) constitute Irrational numbers.

5. *Real numbers :* The equation $x^2 = 2$ has no rational solution. To accommodate this deficiency we have to extend the set of rational numbers. Such an extension leading to the set of real numbers of which Q is a subset is assumed. Real numbers are the union of Rational and Irrational numbers.

The set R of real numbers with the operations of addition and multiplication satisfy the following axioms.

1. *Closure laws of addition and multiplication :*
   $a, b \in R \Rightarrow a + b \in R, ab \in R.$
2. *Associative laws of addition and multiplication :*
   For every $a, b, c \in R$,
   $a + (b + c) = (a + b) + c, a\,(bc) = (ab)\,c.$
3. *Commutative laws :*
   For every $a, b \in R, a + b = b + a, ab = ba.$

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4. *Distributive laws :*

For every $a, b, c \in R$,

$a(b+c) = ab + ac, (b+c)a = ba + ca.$

5. *Identity elements :*

The identity element w. r. t. addition is 0.

The identity element w. r. t. multiplication is 1.

That is for every real number a,

$0 + a = a + 0 = a, a.1 = a = 1.a$

6. *Inverse elements :*

Given a in R there is a number x in R such that $x + a = 0$. x is called the inverse of a w. r. t. addition and is denoted by $-a$.

For any $a \neq 0$ in R, there is a number x in R such that $a.x = 1$ then x is called the inverse of a w. r. t. multiplication and is denoted by a^{-1} or $\frac{1}{a}$.

Order relations in real numbers

There is an order relation (<, >, =) among real numbers which satisfy the following laws.

1. *Law of trichotomy :*

For each pair of real numbers a and b one and only one of the following holds.

$a > b, a = b, a < b$

2. *Tansitivity :*

$a > b$ and $b > c \Rightarrow a > c.$

3. *Monotone property for addition :*
If $a > b$ then for each real number c,

$a + c > b + c.$

Also, $a < b \Rightarrow a + c < b + c.$

4. *Monotone property for multiplication :*
If $a > b$ and $c > 0$ then $ac > bc$
Also $c > 0, a < b \Rightarrow ac < bc.$

5. $a \leq b$ means $a < b$ or $a = b$
$a \geq b$ means $b \leq a.$

6. $0 < a < b \Rightarrow \frac{1}{a} > \frac{1}{b}.$

7. $a < b, c < d \Rightarrow a + c < b + d.$

8. $ab = 0 \Leftrightarrow a = 0$ or $b = 0.$

9. If $c \neq 0$, then $ac = bc \Leftrightarrow a = b.$

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Polynomial

An expression of the form $a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ where $a_0 \neq 0, a_1, a_2 \ldots a_n$ are real constants and n is a positive integer, is called a polynomial in x of degree n.

Algebraic and transcendental numbers

A real number x which is a solution to the polynomial equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, where $a_0 \neq 0, a_1, a_2, a_3, \ldots, a_n$ are integers and n is a positive integar, is called an algebraic number.

A real number which cannot be expressed as a solution of any polynomial equation with integer coefficients is called a transcendental number.

Ex. 1: $\frac{3}{4}$ and $\sqrt{3}$ are algebraic numbers since they are solutions of $4x - 3 = 0$ and $x^2 - 3 = 0$ respectively.

Ex. 2: π and e are transcendental numbers.

Complex numbers

Since there is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$, we have to extend the set of real numbers to meet such a contingency. An ordered pair (a, b) of real numbers a and b is defined as a complex number.

By defining

$(a, b) + (c, d) = (a + c, b + d)$,

$(a, b) \times (c, d) = (ac - bd, ad + bc)$ and writing $(0, 1) = i$, we can write $(a, b) = (a, 0) + i\,(b, 0)$. Now identifying $(a, 0)$ with a we can write $(a, b) = a + ib$. Complex numbers are commutative for multiplication. Two complex numbers $a + ib$, $c + id$ are equal if and only if $a = c$ and $b = d$.

Inequalities for complex numbers are not defined. Set of real numbers is a subset of complex numbers in the sense that the numbers of the form $(a, 0)$ are complex.

The absolute value or modulus of $a + ib$ is defined as

$|a + ib| = \sqrt{a^2 + b^2}$.

$a - ib$ is called the conjugate of $a + ib$.

Mathematical Induction

The principle of mathematical induction regarding a statement involving integers is an important property of the positive integers. It is useful in proving statements involving all positive integers. The method of proof consists of the following steps.

1. Prove the statement for $n = 1$,
2. Assume the statement to be true for $n = k$,
3. From the above assumption, prove that the statement to be true for $n = k + 1$

   Then the statement is true for all n.

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Absolute value or modulus of a real number

Definition : The absolute value or modulus of a real number x denoted by $|x|$ is defined as

$$|x| = x \quad \text{if } x > 0$$
$$= -x \quad \text{if } x < 0$$
$$= 0 \quad \text{if } x = 0$$

Thus $|5| = 5$, $|0| = 0$, $|-5| = 5$

For all real numbers x, y the following results are easily verified.

(i) $|x| = +\sqrt{x^2}$

(ii) $|xy| = |x|\,|y|$, $|x/y| = |x| / |y|$, $y \neq 0$

(iii) $|x + y| \leq |x| + |y|$

(iv) $||x| - |y|| \leq |x - y|$

(v) $|x| < \delta \Rightarrow -\delta < x < \delta$, $\delta > 0$

(vi) $|x - a| < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon$

(vii) If x, y and a are real numbers such that

$|x - a| < \epsilon$ and $|y - a| < \epsilon$ then $|y - x| < 2\epsilon$.

(viii) For every $x \in R$, $|x| = \max.\{-x, x\}$.

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Intervals

Let a and b be two real numbers such that $a < b$.

The set $\{x : x \in R \text{ and } a \leq x \leq b\}$ is called a *closed interval.* It is denoted by $[a, b]$. a and b are called the end points of the interval.

The set $\{x : x \in R, a < x < b\}$ is called an *open interval.* It is denoted by (a, b) or $]a, b[$.

The set $\{x : x \in R, a \leq x < b\}$ is called a *left closed and right open interval.* It is denoted by $[a, b[$ or $[a, b)$, left end point belongs to the interval. The right end point does not belong to the interval.

Right closed and left open interval is defined as above and denoted by $(a, b]$ or $]a, b]$.

Length of an interval

For each interval with end points a and b $(a < b)$, $(b - a)$ is called the length of the interval. Each of the intervals $[a, b],]a, b[, [a, b[,]a, b]$ has the same length $b - a$.

Note: Every interval is an infinite set of reals.

Neighbourhood of a point

Let $\delta > 0$. The open interval $]a - \delta, a + \delta[$ is called a neighbourhood of a.

If x lies in a neighbourhood of a then

$a - \delta < x < a + \delta$ or $|x - a| < \delta$ for some $\delta > 0$.

Deleted neighbourhood of *a*

From a neighbourhood of a point, if the point itself is excluded, we get a deleted neighbourhood of that point.

If $\delta > 0$, then $\{x: a - \delta < x < a + \delta, x \neq a\}$ is called a deleted neighbourhood of a.

Function

Let X and Y be two non-empty sets. If to each element $x \varepsilon X$ there is associated an element $y \varepsilon Y$ then we say we have a function $f: X \rightarrow Y$, and read "f is a function from X to Y". The element y is denoted by $f(x)$. y is called the f image of x or value of the function f at x. X is called the domain of the function f, Y is called the codomain of f. Any arbitrary $x \varepsilon X$ is called the independent variable. The variable $y \varepsilon Y$ corresponding to $x \varepsilon X$ is called the dependent variable. The set of correspondents $f(x)$ is called the range of the function.

Ex. 1 : Consider $y = \sqrt{4 - x^2}$

Since x^2 should be less than or equal to 4, in order that y is real, it follows that $-2 \leq x \leq 2$.

Domain of the real function = $[-2, 2]$

Range of the function $= [0, 2]$.

Ex. 2 : Let $f(x) = \frac{1}{2x - 3}$, x being real.

f is a real valued well defined function if $2x - 3 \neq 0$ i.e. $x \neq 3/2$.

Domain of $f = R - \{3/2\}$.

Constant function

If $f: R \rightarrow \{c\}$ defined by $f(x) = c$ for every $x \varepsilon R$, then f is said to be a constant function on R. It is usually denoted by c itself.

If $f(x) = 2$,

Domain of f is $\{x/x \varepsilon R\}$. Range of f is $\{2\}$.

Composite function

Let $f: A \rightarrow B$ and $g: f(A) \rightarrow C$ be two functions gof called the composite function of f and g is defined by $(gof)(x) = g(f(x))$.

Ex. : If $f: R \rightarrow R$ is defined by $f(x) = 2x + 3$ and $g: R \rightarrow R$ is defined by $g(x) = x^2 + 1$ then $(gof)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 + 1 = 4x^2 + 12x + 10$.

$(fog)(2) = f(g(2)) = f(2^2 + 1) = f(5) = 2(5) + 3 = 13$.

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Unit-1 : Limits and Continuity

1.0 Contents

1.1 Aims and Objectives

At the end of this lesson (i) you must be able to test whether the given function at a given point has a limit or not, and evaluate the limit if it exists,

(ii) You will be able to test whether the given function at a given point is continuous or discountinuous.

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1.2 Introduction

The concept of the limit of a function is fundamental to the study of calculus. Though Newton (1642 – 1727) and Leibnitz (1646 – 1716) have introduced the concept of Limit, it is only Cauchy (1789 – 1857), who gave it a sound mathematical basis.

An important use of limits is the study of continuous functions. Many of the theorems of the calculus are simply not true unless we are dealing with a function that may be considered as continuous. Therefore, we must be able to readily identify a continuous function.

1.3 Limit of a function

In many problems we will be interested to study the behaviour of a function in the vicinity (neighbourhood) of a point. For example, consider the function given by $f(x) = 2x + 1$. Let us calculate the values of $f(x)$ for some values of x, that are very close to but not equal 3.

Table - 1

x	$f(x)$
2.96	6.92
2.98	6.96
2.99	6.98
3.01	7.02
3.02	7.04
3.04	7.08

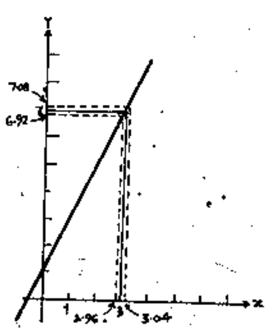

Fig. 1

From table 1 and also fig.1, you will observe that if x is very close to 3, then $2x+1$ is very close to 7.

We represent this by saying :

If x is very close to 3 in value then $2x+1$ is very close to 7 in value.

Or Limit of $2x+1$ as x approaches 3 is 7.

Or $\lim_{x \to 3} (2x+1) = 7$.

This can be stated as follows :

Given a function f and a point $x = a$, we say that $\lim_{x \to 3} f(x) = l$ if and only if the numbers $f(x)$ remain arbitrarily close to the real number l, whenever x is very near to 'a' in value:

This fact can be defined in mathematical terms as follows :

Def : Let f be a function defined over a deleted neighbourhood of 'a' and l be a real number. f is said to tend to l as x tends to 'a' if to every positive number $\in$ there exists a corresponding number $\delta > 0$ (depending on $\in$) such that $|f(x) - l| < \in$ for all values of x such that $|x - a| < \delta$. This is written as $\lim_{x \to a} f(x) = l$.

Left hand limit

f is said to tend to the limit l as x tends to a through values less than a, if to each given $\in > 0, \exists\, \delta > 0 \ni |f(x) - l| < \in$ whenever $a - \delta < x < a$. The limit is called the left limit and is denoted by $f(a-0)$.

$$f(a-0) = \lim_{x \to a-0} f(x)$$

Right hand limit

$f(x)$ is said to tend to a limit l as x tends to 'a' through values greater than 'a', if to each given $\in > 0, \exists\, \delta > 0 \ni |f(x) - l| < \in$ whenever $a < x < a + \delta$.

The limit in this case is called the right hand limit and is denoted by $f(a+0)$.

$$f(a+0) = \lim_{x \to a+0} f(x).$$

Note : $\lim_{x \to a} f(x)$ and $f(a)$ are not necessarily equal.

Limits at infinity and infinite limits

1. A function f is said to tend to l as $x \to \infty$ if given $\in > 0$ there exists a positive number N such that $|f(x) - l| < \in$, for all $x > N$. We write $\lim_{x \to \infty} f(x) = l$.

2. A function f is said to tend to l as $x \to -\infty$ if given $\in > 0$, however small, there exists a positive N such that $|f(x) - l| < \in$ for $x \le -N$.
We write $\lim_{x \to -\infty} f(x) = l$.

3. A function f is said to tend to ∞ as x tends to a, if given $N > 0$, there exists a positive number δ such that $f(x) > N$ for $0 < |x - a| < \delta$.
We write $\lim_{x \to a} f(x) = \infty$.

4. A function f is said to tend to $-\infty$ as x tends to a, if given $N > 0$, there exists a positive number δ such that $f(x) > -N$ for $0 < |x - a| < \delta$.
We write $\lim_{x \to a} f(x) = -\infty$.

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5. A function f is said to tend to ∞ as x tends to ∞, if given $N > 0$, there exists a number $M > 0$ such that $f(x) > N$ for $x \geq M$.

We write $\lim\limits_{x \to \infty} f(x) = \infty$

We can also define by $\lim\limits_{x \to \infty} f(x) = -\infty$, $\lim\limits_{x \to -\infty} f(x) = \infty$ $\lim\limits_{x \to -\infty} f(x) = -\infty$

Self Assessment Questions

SAQ 1 : Find $\lim\limits_{x \to 5} (3x - 2)$ and give an intutive meaning to your answer.

SAQ 2 : Evaluate the $\lim\limits_{x \to 3} \dfrac{x^2 - 5x + 6}{x - 3}$. Give an intutive meaning.

SAQ 3 : Find the $\lim\limits_{x \to \infty} \dfrac{-3x^2 + 4x + 2}{x^3 + 2x^2 + x + 1}$

1.4 Graphical representation of limit

The statement $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$ means that given any $\epsilon > 0$, we can find $\delta > 0$ such that the graph of $y = f(x)$

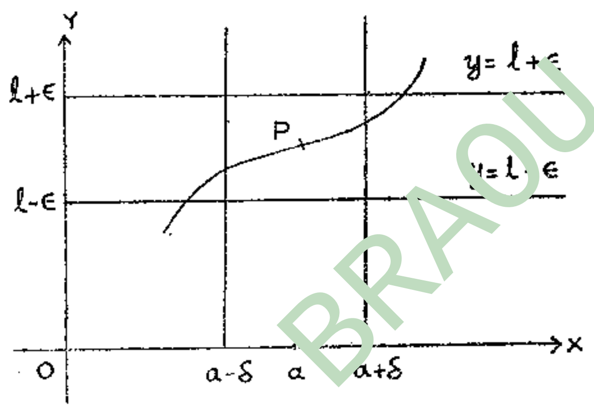

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Fig 2

lies within the parallel lines $y = l - \epsilon$ and $y = l + \epsilon$ as long as x lies within the parallel lines $x = a - \delta$ and $x = a + \delta$.

1.5 Theorems on limits

1. The limit of a function at a point, when it exists, is unique.

Proof : Suppose $\lim\limits_{x \to a} f(x)$ exists and is not unique.

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} f(x) = l'$ where $l \neq l'$

$l \neq l' \Rightarrow |l - l'| > 0$. Take $\epsilon = \dfrac{|l - l'|}{3} > 0$

Then $\lim\limits_{x \to a} f(x) = l \Rightarrow \exists\, \delta_1 > 0 \ni |f(x) - l| < \epsilon$

when $0 < |x - a| < \delta_1$. ....(i)

Again $\lim\limits_{x \to a} f(x) = l' \Rightarrow \exists\, \delta_2 > 0 \ni |f(x) - l'| < \epsilon$ when

$0 < |x - a| < \delta_2$. .... (ii)

Let $\delta = \min. \{\delta_1, \delta_2\}$ From (i) and (ii) we have

$|f(x) - l| < \epsilon,\ |f(x) - l'| < \epsilon$ when $0 < |x - a| < \delta$.

$|l - l'| = |l - f(x) + f(x) - l'|$

$\leq |l - f(x)| + |f(x) - l'|$

$< \dfrac{|l - l'|}{3} + \dfrac{|l - l'|}{3} = \dfrac{2}{3}\,|l - l'|$ when $0 < |x - a| < \delta$.

i.e. $|l - l'| < \frac{2}{3}\,|l - l'|$

This is absurd.

$\therefore$ The supposition $l \neq l'$ is wrong. Hence $l = l'$.

Algebra of limits

Let f and g be two functions and 'a' be a point of their common domain.
If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$ then

(i) $\lim\limits_{x \to a} [f(x) + g(x)] = l + m$

(ii) $\lim\limits_{x \to a} [f(x) - g(x)] = l - m$

(iii) $\lim\limits_{x \to a} [f(x)\, g(x)] = l\,m$

(iv) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$ $(m \neq 0)$

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The results are well-known. Hence we give the proof for (iv) only as an illustration.

(iv) Let us first prove that

$\lim\limits_{x \to a} g(x) = m\,(\neq 0) \Rightarrow \lim\limits_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$

$\lim\limits_{x \to a} g(x) = m \Rightarrow$ given $\epsilon > 0\ \exists\, \delta > 0 \ni |g(x) - m| < \epsilon$ for $0 < |x - a| < \delta$

$$\begin{aligned} \therefore |m| &= |m - g(x) + g(x)| \\ &\leq |m - g(x)| + |g(x)| \\ &= |g(x) - m| + |g(x)| \\ &< \epsilon + |g(x)| \text{ for } 0 < |x - a| < \delta \end{aligned}$$

$\therefore |g(x)| > |m| - \epsilon$ for $0 < |x - a| < \delta$.

Taking $\epsilon = \dfrac{|m|}{2}$, we get $|g(x)| > \dfrac{|m|}{2}$ for $0 < |x - a| < \delta$

$\therefore \dfrac{1}{|g(x)|} < \dfrac{2}{|m|}$ for $0 < |x - a| < \delta$ ... (1)

Again $\lim\limits_{x \to a} g(x) = m \Rightarrow$ given $\epsilon_1 = \frac{|m|^2}{2} \epsilon > 0, \exists \delta > 0$

$\ni |g(x) - m| < \epsilon_1$ for $0 < |x - a| < \delta$ ... (2)

$$\therefore \left|\frac{1}{g(x)} - \frac{1}{m}\right| = \left|\frac{m - g(x)}{m\, g(x)}\right|$$

$$< \frac{\epsilon_1}{|m|} \cdot \frac{2}{|m|} \text{ \{using (1) and (2)\} for } 0 < |x - a| < \delta$$

$$= \frac{2}{|m|^2} \cdot \frac{|m|^2}{2} . \epsilon = \epsilon.$$

i.e. $\left|\frac{1}{g(x)} - \frac{1}{m}\right| < \epsilon$ for $0 < |x - a| < \delta$

$$\therefore \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}.$$

$$\text{Now } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \frac{1}{g(x)}$$

$$= \lim_{x \to a} f(x) . \lim_{x \to a} \frac{1}{g(x)}$$

$$= l . 1/m = l/m$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}.$$

Corollary 1 : If $\lim\limits_{x \to a} f_1(x) = l_1$, $\lim\limits_{x \to a} f_2(x) = l_2$, ...

$\lim\limits_{x \to a} f_n(x) = l_n$ then we can prove by induction that

1. $\lim\limits_{x \to a} (f_1 + f_2 + ... + f_n)(x) = l_1 + l_2 + ... + l_n.$

2. $\lim\limits_{x \to a} (f_1 . f_2 ... f_n)(x) = l_1 . l_2 ... l_n.$

Corollary 2 : If n is a positive integer and $f(x) = x^n$ then we can prove that $\lim\limits_{x \to a} f(x) = a^n$

3. If n is a positive integer and $f(x) = \sqrt[n]{x}$ for every positive real number,

$$\lim_{x \to a} f(x) = \sqrt[n]{a} \quad (a \geq 0)$$

Proof : We have to prove that there exists $\delta > 0$ corresponding to every given $\epsilon > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$

$$|f(x) - \sqrt[n]{a}| < \epsilon \Leftrightarrow |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon$$

$$\Leftrightarrow \sqrt[n]{a} - \epsilon < \sqrt[n]{x} < \sqrt[n]{a} + \epsilon$$

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$$\Leftrightarrow (\sqrt[n]{a} - \epsilon)^n < x < (\sqrt[n]{a} + \epsilon)^n$$

$$\Leftrightarrow [(\sqrt[n]{a} - \epsilon)^n - a)] < (x - a) < [(\sqrt[n]{a} + \epsilon)^n - a)] \qquad (x \neq a)$$

(By taking δ as the smaller of the two $[a - (\sqrt[n]{a} - \epsilon)^n]$ and $[(\sqrt[n]{a} + \epsilon)^n - a]$)

$$\Leftrightarrow -\delta < (x - a) < \delta$$

$$\Leftrightarrow 0 < |x - a| < \delta.$$

$$\therefore \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

Corollary : If n is a rational number and $f(x) = x^n$, for every postive real number a,

$$\lim_{x \to a} f(x) = a^n.$$

1.6 Standard limits

The student is familiar with the following standard limits –

1. If n $(\neq 0)$ is a rational number $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = n \, . \, a^{n-1}$

2. $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$

3. $\lim_{\theta \to 0} \dfrac{\tan \theta}{\theta} = 1$

4. Prove that $\lim_{n \to \infty} (1 + 1/n)^n = e.$

   Let $u_n = \left(1 + \dfrac{1}{n}\right)^n$ be a sequence

(A sequence is a function whose domain is the set of natural numbers).

By the binomial theorem,

$$u_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \, . \, \frac{1}{n} + \frac{n\,(n-1)}{1.2} \cdot \frac{1}{n^2}$$

$$+ \frac{n\,(n-1)\,(n-2)}{1.2.3} \cdot \frac{1}{n^3} + \ldots + \frac{n\,(n-1)\ldots(n-n+1)}{1.2.3\ldots n} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots..$$

$$+ \frac{1}{n!} \, . \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right)$$

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$$u_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)$$

$$+ \ldots + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \ldots \left(1 - \frac{n}{n+1}\right).$$

We have, $\frac{1}{n+1} < \frac{1}{n} \Leftrightarrow -\frac{1}{n+1} > -\frac{1}{n}$

$$\Leftrightarrow 1 - \frac{1}{n+1} > 1 - \frac{1}{n}$$

Similarly, $1 - \frac{k}{n+1} > 1 - \frac{k}{n}$ for $k = 2, 3 \ldots n-1$.

It follows that each term after the second of u_{n+1} is greater than the corresponding term of u_n.

$\therefore u_{n+1} \geq u_n \forall n$

$\therefore \{u_n\}$ is monotonic increasing.

(A sequence which is either increasing or decreasing is called a monotomic sequence).

$$\text{Also } u_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right)$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} \ldots + \frac{1}{2^{n-1}} \quad \left(\because \frac{1}{3!} = \frac{1}{6} < \frac{1}{2^2}\right)$$

$$= 1 + \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3 \forall n.$$

Clearly $u_n \geq 2$ and < 3.

Thus u_n is bounded and monotonic increasing and so has a limit which we denote by e. (From the theorem, which states that every bounded monotonic sequence has a limit point).

$$\therefore \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Corollary : $\lim_{n \to 0} (1 + n)^{1/n} = e.$

Put $n = 1/x$. Then $x = 1/n$. As $n \to 0, x \to \infty$.

$$\lim_{n \to 0} (1 + n)^{1/n} = \lim_{x \to \infty} (1 + 1/x)^x = e.$$

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5. Prove that $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a, a > 0.$

Proof : Let $a^x - 1 = y$.

$\therefore$ $a^x = 1+y$. Hence $x \log_e a = \log_e (1 + y)$ and $y\to 0$ as $x\to 0$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y \log_e a}{\log_e (1+y)}$$

$$= \lim_{y\to 0} \frac{\log_e a}{(1/y) \log_e (1+y)} = \lim_{y\to 0} \frac{\log_e a}{\log_e (1+y)^{1/y}}$$

$$= \frac{\log_e a}{\log_e e} = \log_e a$$

6. Prove that $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1.$

Put $a = e$ in (5)

then $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = \log_e e = 1.$

Examples

Ex 1 : If $f(x) = |x|/x$, as $x\to 0$, show that limit of $f(x)$ does not exist.

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Sol : Left hand limit :

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} |x|/x = \lim_{x\to 0} -x/x, \ (\because x < 0)$$

$$= \lim_{x\to 0-} (-1) = -1$$

Right hand limit:

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} |x|/x = \lim_{x\to 0+} x/x, \ (\because x > 0)$$

$$= 1$$

$$\therefore \lim_{x\to 0-} f(x) \neq \lim_{x\to 0+} f(x)$$

$\therefore$ $\lim\limits_{x\to 0} f(x)$ does not exist.

Ex 2 : Prove that $\lim\limits_{x\to 0} x \,.\, \sin 1/x = 0.$

Sol : $|x.\sin 1/x - 0| = |x \,.\, \sin 1/x| = |x| \, |\sin 1/x|$

$\leq |x|$ $\quad (\because |\sin 1/x| \leq 1)$

$\therefore$ $|x \,.\, \sin 1/x - 0| < \epsilon$ whenever $0 < |x| < \epsilon$

Choosing $\delta = \epsilon$, $|x \,.\, \sin 1/x - 0| < \epsilon$ whenever $0 < |x| < \delta$.

Hence $\lim\limits_{x\to 0} x \,.\, \sin 1/x = 0$

1.7 Continuity of a function

Definition : Let $x = a$ be an interior point of the domain of f. Let f be defined in the deleted neighbourhood of a. Then f is said to be continuous at $x = a$, if to any positive number ϵ, there corresponds a positive number δ such that

$$|f(x)-f(a)| < \epsilon \text{ when } |x-a| < \delta.$$

Note : From the above definition, it follows that f will be continuous at $x = a$, if all the following three conditions are satisfied.

1. f is defined at $x = a$ i.e., $f(a)$ exists.
2. $\lim\limits_{x\to a} f(x)$ exists i.e., $\lim\limits_{x\to a-0} f(x)$ and $\lim\limits_{x\to a+0} f(x)$ both exist and are equal.
3. $\lim\limits_{x\to a} f(x) = f(a)$.

SAQ 4 : If $f(x) = \dfrac{x^2-9}{x-3}$, is f continuous at $x = 3$?

SAQ 5 : If $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ is f continuous at $x = 3$?

1.8 Graphical representation of continuity

Consider the function $y = f(x)$. Draw its graph and consider the point $P[a, f(a)]$ on it. Draw the lines

$$y = f(a) - \epsilon \text{ and } y = f(a) + \epsilon \quad \dots (1)$$

parallel to x–axis and having the point P inside these lines.

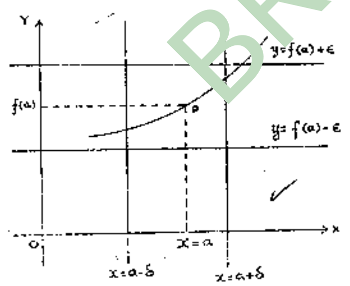

Fig.3

Here ϵ, a positive number, is a measure of how close the lines (1) from each other are.

Then, the continuity of f at $x = a$ means that we should be able to draw two lines

$$x = a - \delta \text{ and } x = a + \delta \quad \dots (2)$$

parallel to y– axis such that every point of the graph between the lines (2) must also lie between lines (1).

1.9 Continuity in an interval

Definition : A function f is said to be continuous in the closed interval $[a, b]$, if it is continuous at every point of the interval.

Note : If the function f is continuous in the closed interval, then the graph of the function in this interval is without any gap or break.

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Continuity at end point

A function f is said to be continuous at the left end point 'a' of its domain $[a, b]$, if given any positive number $\in$, there exists a positive number δ such that

$$|f(a+h)-f(a)|<\in \text{ when } 0 \leq h < \delta.$$

A function f is said to be continuous at the right end point 'b' of its domain $[a, b]$, if given $\in > 0$, there exists $\delta > 0$ such that

$$|f(b-h)-f(b)|<\in \text{ when } 0 \leq h < \delta.$$

1.10 Discontinuity

Definition : f is said to be discontinuous at a point of its domain, if f is not continuous at that point.

Kinds of discontinuities

1. f is said to have discontinuity of the first kind at a, if $\lim\limits_{x\to a+0} f(x)$ and $\lim\limits_{x\to a-0} f(x)$ exist, but are not equal.

2. f is said to have discontinuity at a of the second kind, if neither $\lim\limits_{x\to a-0} f(x)$ nor $\lim\limits_{x\to a+0} f(x)$ exists.

3. f is said to have a removable discontinuity at $x = a$, if

$$\lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x) \neq f(a)$$

Continuity of the sum, difference, product and quotient of two continuous functions can be easily established.

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Examples

Ex.1: Show that

$$f(x) = x+2 \quad \text{when } x < 2$$
$$= x^2 - 1 \quad \text{when } x \geq 2$$

is discontinuous at $x = 2$.

Sol:

Left hand limit : $f(2-0) = \lim\limits_{x\to 2-0} (x+2) = 4$

Right hand limit : $f(2+0) = \lim\limits_{x\to 2+0} (x^2-1) = 3$

Since $f(2-0) \neq f(2+0)$, the function is discontinuous at $x = 2$.

Ex.2: Let

$$f(x) = \frac{x^3 + x^2 - 16x + 20}{(x-2)^2} \quad \text{when } x \neq 2$$
$$= k \quad \text{when } x = 2$$

If $f(x)$ is continuous for all x, determine the value of k.

Sol:

We have $f(x) = \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}$ when $x \neq 2$

$$= \frac{(x+5)(x-2)^2}{(x-2)^2} = x+5$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+5) = 7$$

We have $f(2) = k$

Since f is continuous at $x = 2$

We have $\lim_{x\to 2} f(x) = f(2)$

$$\therefore \quad k = 7$$

Ex.3: Examine whether the following function is continuous at the origin.

$$f(x) = \frac{x\, e^{1/x}}{1 + e^{1/x}} \quad \text{when } x \neq 0$$

$$= 0 \qquad \text{when } x = 0$$

Sol :

$$\lim_{x\to 0-0} f(x) = \lim_{x\to 0-0} \frac{x\, e^{1/x}}{1 + e^{1/x}}$$

$$= \lim_{h\to 0} \frac{(0-h)\, e^{1/(0-h)}}{1 + e^{1/(0-h)}}$$

$$= \lim_{h\to 0} \frac{-h\, e^{-1/h}}{1 + e^{-1/h}} = 0$$

$$\lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} \frac{x\, e^{1/x}}{1 + e^{1/x}}$$

$$= \lim_{h\to 0} \frac{(0+h)\, e^{1/(0+h)}}{1 + e^{1/(0+h)}} = \lim_{h\to 0} \frac{h\, e^{1/h}}{1 + e^{1/h}}$$

$$= \lim_{h\to 0} \frac{h}{e^{-1/h}+1} = 0$$

$\therefore \lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)$, $\lim_{x\to 0} f(x)$ exists and equal to zero.

Also $\lim_{x\to 0} f(x) = 0 = f(0)$

$\therefore$ f is continuous at $x = 0$.

Ex.4: If $f(x) = \dfrac{\sin^2 ax}{x^2}$ when $x \neq 0$

$= 1$ when $x = 0$,

prove that f is discontinuous at $x = 0$ unless $a = \pm 1$.

Sol :

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^2 ax}{x^2}$$

$$= \lim \left(\frac{\sin ax}{ax} \cdot a\right)^2$$

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$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right)^2 . a^2$$

$$= 1. a^2 = a^2 \text{ since } \frac{\sin ax}{ax} \to 1 \text{ as } x \to 0.$$

This is true for all a, zero or not.

We have $f(0) = 1$.

(i) When $a^2 \neq 1$, we have $\lim_{x \to 0} f(x) \neq f(0)$ and so f is discontinuous at $x = 0$.

(ii) When $a^2 = 1$, we have $\lim_{x \to 0} f(x) = f(0)$. Then f is continuous at $x = 0$.

Ex.5: Show that the function ϕ defined by $\phi(0) = 0$, $\phi(x) = \frac{1}{2} - x$ for $0 < x < \frac{1}{2}$, $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$, $\phi(x) = \frac{3}{2} - x$ for $\frac{1}{2} < x < 1$, $\phi(1) = 1$ has three points of discontinuity and examine the types of discontinuities.

Sol: (i) We have $\phi(0) = 0$

$$\lim_{x \to 0+0} \phi(x) = \lim_{x \to 0+0} \left(\frac{1}{2} - x\right) = \lim_{h \to 0} \left[\frac{1}{2} - (0 + h)\right] = \frac{1}{2}$$

Since $\lim_{x \to 0+0} \phi(x) \neq \phi(0)$, ϕ is discontinuous at $x = 0$. The discontinuity is seen to be removable in this case.

(ii) We have $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$

$$\lim_{x \to \frac{1}{2}-0} \phi(x) = \lim_{x \to \frac{1}{2}-0} \left(\frac{1}{2} - x\right)$$

$$= \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right]$$

$$= \lim_{h \to 0} h = 0$$

$$\lim_{x \to \frac{1}{2}+0} \phi(x) = \lim_{x \to \frac{1}{2}+0} \left(\frac{3}{2} - x\right)$$

$$= \lim_{h \to 0} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right]$$

$$= \lim_{h \to 0} (1 - h) = 1.$$

Since $\lim_{x \to \frac{1}{2}-0} \phi(x) \neq \lim_{x \to \frac{1}{2}+0} \phi(x) \neq \phi\left(\frac{1}{2}\right)$

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III. Answer the following questions in about 5 lines.

(i) Define the left and right limits of a function.

(ii) Define the continuity of a function in an interval.

(iii) When do you say that a function is discontinuous ?

(iv) When do you say that a function has a jump discontinuity?

1.13 Answers to Self Check Questions

SAQ 1. $\lim_{x\to 10} (3x-2) = 3 \times 10 - 2 = 28.$

when x is very close to 10, then $3x-2$ has a value very close to 28.

SAQ 2. $\lim_{x\to 3} \dfrac{x^2-5x+6}{x-3} = \lim_{x\to 3} \dfrac{(x-3)(x-2)}{(x-3)} = \lim_{x\to 3} (x-2) = 1.$

when x is very close to 3, but $x \neq 3$, then $\dfrac{x^2-5x+6}{x-3}$ has a value very close to 1.

SAQ 3. Divide both numerator and the denominator by the highest power of x present (it is x^3 here).

$$\lim_{x\to\infty} \frac{\frac{3}{x}+\frac{4}{x^2}+\frac{2}{x^3}}{1+\frac{2}{x}+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0.$$

SAQ 4. No! $f(3)$ does not exist. Hence the $\dfrac{x^3-9}{x-3}$ is not continuous at $x = 3$.

SAQ 5. Yes! $f(3)$ is given as 6 and

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)} = 6$$

$\therefore \lim_{x\to 3} f(x) = 6 = f(3) \Rightarrow f$ is continuous.

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Unit – 2 : Differentiation

2.0 Contents

2.1 Aims and Objectives

After going through this lesson: (i) you will be able to find the derivative of a given function at a given point using the limit concept and

(ii) you will be able to apply the Leibnitz's theorem to calculate the nth derivative of a function.

2.2 Introduction

Many physical phenomena involve changing quantities – the speed of an object, the voltage of an electrical signal, the number of bacteria in culture etc. All these problems involve the rate of change, which is nothing but the derivative of a function at a point. In this unit, you will learn the notion of a derivative, which is a basic mathematical tool for studying rates of change.

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2.3 Differential coefficient or derivative

Let a and $a + h$ be two points of the domain of a function f. If h is positive, $a + h$ lies to the right of a. If h is negative, $a + h$ lies to the left of a. Now for a change h in the independent variable x, there is a corresponding change $f(a + h) - f(a)$ in the function $f(x)$.

The ratio $\dfrac{f(a+h)-f(a)}{h}$ of the two changes is the rate of change of f in the interval $(a, a + h)$.

If the limit of this ratio, as h tends to zero, exists the limit is called the derivative or differential coefficient of f at a and is denoted by $f'(a)$.

$$\text{Thus } f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Ex 1: Using the definition, obtain the differential coefficient of the function $f(x) = x^2 + 2x - 1$.

Sol : The differential coefficient of a function is given by

$$f'(x) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

for the given function $f(x + h) = (x + h)^2 + 2(x + h) - 1$

$$\therefore f'(x) = \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} = 2x + 2.$$

ϕ is discontinuous at $x = \frac{1}{2}$.

The discontinuity is of the first kind.

(iii) We have $\phi(1) = 1$

$$\lim_{x \to 1-0} \phi(x) = \lim_{x \to 1-0} \left(\frac{3}{2} - x\right)$$

$$= \lim_{h \to 0} \left(\frac{3}{2} - (1-h)\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{2} + h\right) = \frac{1}{2}$$

Since $\lim_{x \to 1-0} \phi(x) \neq \phi(1)$, ϕ is discontinuous at $x = 1$.

The discontinuity is removable.

It is obvious that ϕ has only three discontinuities at $x = 0, \frac{1}{2}, 1$.

Ex.6: Consider
$$f(x) = |x|/x \text{ if } x \neq 0$$
$$= 0 \quad \text{if } x = 0.$$

Show that the function has jump discontinuity at $x = 0$.

Sol: For $x < 0, f(x) = -x/x = -1$.

For $x > 0, f(x) = x/x = 1$.

$\therefore f(0-0) = -1$ and $f(0+0) = 1$

f is defined at $x = 0$ and its value is equal to 0.

Since $\lim_{x \to 0-0} f(x) \neq \lim_{x \to 0+0} f(x)$ and these limits exist, the function has jump discontinuity at $x = 0$.

1.11 Summary

We say that $\lim_{x \to a} f(x) = l$, iff the numbers $f(x)$ remain arbitrarily close to the number l, whenever x is very near to a in value. The limit of a function at a point exists iff the left limit and right limit at that point exist and are equal.

A function f is said to be continuous at $x = x_1$ iff $\lim_{x \to x_1} f(x) = f(x_1)$. That is, for f to be continuous at $x = x_1$, three criteria must be met :

(i) $f(x_1)$ must exist,

(ii) $\lim_{x \to x_1} f(x)$ must exist and,

(iii) $\lim_{x \to x_1} f(x) = f(x_1)$.

If any one of this criteria is not met, then f is discontinuous at $x = x_1$.

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1.12 Sample Examination Questions

I. *Answer the following questions in detail.*

(i) (a) Define the limit of a function. Prove that the limit of a function is unique if it exists.

(b) Show that $\lim\limits_{x \to 0} \dfrac{x\, e^{1/x}}{1 + e^{1/x}} = 0$

(ii) (a) Show that $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ lies between 2 and 3.

(b) Prove that $\lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

(iii) (a) Define the continuity of a function at a point and give its geometrical interpretation.

(b) If $f(x) = (1 + 3x)^{1/x}, x \neq 0$ and $f(x) = e^3, x = 0$, show that f is continuous at $x = 0$.

(iv) (a) Explain various kinds of discontinuities.

(b) Let the function f defined by

$$f(x) = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \text{ if } x \neq 0 \text{ and } f(0) = 1$$

then show that f is continuous from the right at $x = 0$ and has a discontinuity of the first kind from the left at $x = 0$.

II. *Briefly answer the following questions.*

(i) Show that $\lim\limits_{x \to 0} \dfrac{x - |x|}{x}$ does not exist.

(ii) Show that $\lim\limits_{x \to a} (x - a) \operatorname{cosec} (x - a) = 1$

(iii) Show that $\lim\limits_{x \to 1} \dfrac{x - 1}{1 + e^{1/(x-1)}} = 0$

(iv) Show that $\cos x$ is continuous for every value of x.

(v) The function f is defined as follows

$$f(x) = x \text{ if } x < 0$$
$$= x^2 \text{ if } x > 0$$
$$= 1 \text{ if } x = 0$$

Discuss the continuity of the function at $x = 0$.

(vi) Define $f(0)$, if the function $f(x) = \dfrac{\cos 3x - \cos 4x}{x \sin 2x}$ is continuous.

(vii) The function f is defined as

$$f(x) = 5x - 4 \quad \text{when } 0 < x < 1$$
$$= 4x^2 - 3x \quad \text{when } 1 < x < 2.$$

Test the continuity of the function at $x = 1$.

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Left hand derivative

If $\lim\limits_{h \to 0} \dfrac{f(a-h)-f(a)}{-h}$, $h > 0$ exists, the limit is called the left hand derivative of f at $x = a$. It is denoted by $f'(a-0)$ or $Lf'(a)$.

Right hand derivative

If $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$, $h > 0$ exists, the limit is called the right hand derivative of f at $x = a$. It is denoted by $f'(a+0)$ or $Rf'(a)$.

Existence of the derivative

The derivative of f at $x = a$ exists iff both the left hand and right hand derivatives exist and are equal at $x = a$.

Derivability in an interval

A function f is said to be derivable in an interval if it is a derivable at every point of this interval.

Differentiation

The process of finding out the derivative of a function is called differentiation.

Ex 2 : A function is defined as :

$$\begin{aligned} f(x) &= 1 + \sin x \text{ for } 0 < x < \pi/2 \\ &= 2 + \left(x - \frac{\pi}{2}\right)^2 \text{ for } x \geq \pi/2. \end{aligned}$$

Examine the derivability at $x = \pi/2$

Sol : $f(\pi/2) = 2 + (\pi/2 - \pi/2)^2 = 2$

$$Lf'(\pi/2) = \lim_{h \to 0} \frac{f(\pi/2-h) - f(\pi/2)}{-h} = \lim_{h \to 0} \frac{1+\sin(\pi/2-h) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{1+\cos h - 2}{-h} = \lim_{h \to 0} \frac{1-\cos h}{h}$$

$$= \lim_{h \to 0} \frac{2\sin^2 h/2}{h} = \lim_{h \to 2} \frac{2}{h}\left(\frac{\sin h/2}{h/2}\right)^2 \cdot \frac{h^2}{4}$$

$$= \lim_{h/2 \to 0} \frac{h}{2} \lim_{h/2 \to 0} \left(\frac{\sin h/2}{h/2}\right)^2 = 0$$

$$\therefore \; Lf'(\pi/2) = 0$$

$$Rf'(\pi/2) = \lim_{h \to 0} \frac{f(\pi/2+h) - f(\pi/2)}{h}$$

$$= \lim_{h \to 0} \frac{2 + (\pi/2+h-\pi/2)^2 - 2}{h}$$

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$$= \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0$$

$$\therefore Rf'(\pi/2) = 0$$

Since $Lf'(\pi/2) = Rf'(\pi/2) = 0, f$ is differentiable at $x = \pi/2$ and $f'(\pi/2) = 0$.

Ex. 3 : Examine the function

$$f(x) = x \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, \quad x \neq 0$$

$$= 0 \qquad , x = 0$$

with regard to the existence of its derivative at the origin.

Sol:

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(0-h)\dfrac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} = \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1}$$

$$\left[\because e^{-2/h} \to e^{-\infty} \text{ as } h \to 0 \text{ and } e^{-\infty} \to 0\right]$$

$$= \frac{0-1}{0+1} = -1$$

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$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}}$$

$$= \frac{1-0}{1+0} = 1$$

Since $Lf'(0) \neq Rf'(0)$, f is not differentiable at $x = 0$.

SAQ 2. A function is defined as follows

$$f(x) = \frac{x-1}{2x^2 - 7x + 5} \quad \text{if } x \neq 1$$

$$= -\frac{1}{3} \qquad \text{if } x = 1.$$

Check whether the function is differentiable at $x = 1$.

2.4 Theorems on differentiation

Theorem : If the function f is derivable at a point a in the domain of f then f is continuous at a.

Proof : Since f is derivable at a, we have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Now, $\lim_{h\to 0} [f(a+h)-f(a)] = \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h}.h\right]$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \lim_{h\to 0} h$$

$$= f'(a)\,0 = 0$$

$\therefore \lim_{h\to 0} f(a+h) = f(a) \Rightarrow \lim_{x\to a} f(x) = f(a)$.

Hence f is continuous at $x = a$.

Note : The converse of the above theorem is not true. i.e., a function may be continuous at a point, but it may not be derivable at that point. (see the following example).

Ex. 4 : Show that $f(x) = |x|$ is continuous but not derivable at $x = 0$.

Sol : We have $f(0) = |0| = 0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} |x| = \lim_{h\to 0} |0-h| = \lim_{h\to 0} h = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} |x| = \lim_{h\to 0} |0+h| = \lim_{h\to 0} h = 0.$$

Since $\lim_{x\to 0} f(x) = 0 = f(0)$, $f(x)$ is continuous at $x = 0$.

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$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|0-h|-0}{-h}$$

$$= \lim_{h\to 0} \frac{h}{-h} = -1.$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|0+h|-0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1.$$

Since $Lf'(0) \neq Rf'(0)$, $f(x)$ is not derivable at $x = 0$

Rules of differentiation

(1) If u and v are derivable functions of x, then $u \pm v$ are also derivable functions of x and

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \qquad \text{(sum or difference rule)}$$

(2) If u and v are derivable functions of x then uv is also a derivable function of x and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{(product rule)}$$

(3) If u and v are derivable functions of x, and $v \neq 0$ then u/v is also a derivable function of x and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \text{(Quotient rule)}$$

(4) If $y = f(u)$ is a derivable function of u and $u = g(x)$ is a derivable function of x, then $y = f\{g(x)\}$ is a derivable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{(function of a function rule)}$$

(5) If the inverse of $y = f(x)$ exists and $\frac{dx}{dy} \neq 0$ then

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \qquad \text{(inverse function rule)}$$

(6) Let $x = f(\theta), y = g(\theta)$ be two derivable functions of θ.

Let $x = f(\theta)$ possess an inverse function $\theta = u(x)$. Let $\frac{dx}{d\theta} \neq 0$.

Then, $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$. But $\frac{d\theta}{dx} = \frac{1}{\left(\frac{dx}{d\theta}\right)}$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

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Ex. 5 : Find $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)\right]$

Sol : $\frac{d}{dx}\left[\tan^{-1}\frac{\left(\sqrt{1+x} - \sqrt{1-x}\right)^2}{\left(\sqrt{1+x} + \sqrt{1-x}\right)\left(\sqrt{1+x} - \sqrt{1-x}\right)}\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{1 - \sqrt{1-x^2}}{x}\right)\right] = \frac{d}{dx}\left[\tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right)\right]$ (putting $x = \sin\theta$)

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2 \,.\, \cos\theta/2}\right)\right] = \frac{d}{dx}\left[\tan^{-1}(\tan\theta/2)\right]$

$= \frac{d}{dx}(\theta/2) = \frac{1}{2}\frac{d}{dx}(\sin^{-1}x) = \frac{1}{2\sqrt{1-x^2}}$

Ex. 6: Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Sol: Let $y = \tan^{-1}\frac{2x}{1-x^2}, u = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$. Put $x = \tan\theta$.

Then $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$ and

$$u = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta$$

$$\therefore \quad \frac{dy}{du} = \frac{\frac{dy}{d\theta}}{\frac{du}{d\theta}} = \frac{2}{2} = 1$$

Ex. 7: If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, prove that $\frac{dy}{dx}$ is 1 when $\theta = \frac{\pi}{4}$.

Sol:

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \tan\frac{\pi}{4} = 1.$$

Note: The method of finding the derivative of a function after taking logarithms is called *logarithmic differentiation*. Logarithmic differentiation is useful when the function is of the form $\{f\}^g$.

Ex.8: Find $\frac{d}{dx}\left[(\tan x)^{\log x} + x^x\right]$

Sol:

$$\text{Let } y = (\tan x)^{\log x} + x^x, \; u = (\tan x)^{\log x}, \; v = x^x$$

$$\text{Then } y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\log u = \log x \, . \log\tan x$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \, . \frac{d}{dx}\log\tan x + \log\tan x \, . \frac{d}{dx}\log x$$

$$= \log x \, . \frac{1}{\tan x} . \sec^2 x + \log(\tan x) \, . \frac{1}{x}$$

$$\therefore \frac{du}{dx} = (\tan x)^{\log x}\left[\frac{\sec^2 x}{\tan x}\log x + \frac{1}{x}\log(\tan x)\right]$$

$$\log v = x\log x$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = 1 + \log x$$

$$\therefore \frac{dv}{dx} = x^x(1 + \log x)$$

$$\therefore \frac{dy}{dx} = (\tan x)^{\log x}\left[\frac{\sec^2 x}{\tan x}\log x + \frac{1}{x}\log(\tan x)\right] + x^x(1 + \log x)$$

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Ex.9: Find $\frac{dy}{dx}$ if $\sin y = x \sin(a+y)$.

Sol: The equation is an implicit equation. In such cases, we have to differentiate each term to find $\frac{dy}{dx}$.

$$\cos y \frac{dy}{dx} = x \cos(a+y) \frac{dy}{dx} + \sin(a+y)$$

$$\frac{dy}{dx}[\cos y - x\cos(a+y)] = \sin(a+y)$$

$$\text{But } x = \frac{\sin y}{\sin(a+y)}$$

$$\therefore \frac{dy}{dx}\left[\cos y - \frac{\sin y}{\sin(a+y)}\cos(a+y)\right] = \sin(a+y)$$

$$\frac{dy}{dx}\sin(a+y-y) = \sin^2(a+y)$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

2.5 Successive Differentiation

If the function y has a derivative $\frac{dy}{dx}$, then $\frac{dy}{dx}$ can be differentiated (if possible) again and we get the second derivative of y w.r.t. x denoted by $\frac{d^2y}{dx^2}$.

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If $\frac{d^2y}{dx^2}$ is again differentiated w.r.t. x, the result is called the third derivative of y w.r.t. x and is denoted by $\frac{d^3y}{dx^3}$.

Similarly, if y is differentiated successively n times, the result is called n th derivative of y w.r.t. x and is denoted by $\frac{d^ny}{dx^n}$.

The process of differentiating the same function again and again is called *successive differentiation*. The 1st, 2nd, 3rd, ...n h derivatives are also denoted by $f'(x), f''(x), f'''(x), \ldots, f^n(x)$, or $y_1, y_2, y_3, \ldots\ldots\ldots y_n$ or $Dy, D^2y, D^3y, \ldots\ldots\ldots\ldots D^ny$ $\left(\text{where } D = \frac{d}{dx}\right)$

Standard Results

1. *nth derivative of* $(ax+b)^m$

$$\text{Let } y = (ax+b)^m$$

$$y_1 = m(ax+b)^{m-1}.a$$

$$y_2 = m(m-1)(ax+b)^{m-2}.a^2$$

....

$$y_n = [m(m-1)(m-2)\ldots n \text{ factors}](ax+b)^{m-n}a^n.$$

$$= m(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}a^n$$

Corollary 1: If $a = 1$, $b = 0$, then $y = x^m$

$$\text{Then } y_n = m(m-1)(m-2).(m-n+1)\,x^{m-n}$$

$$\text{If } m = n \text{ is a positive integer, } y = x^n.$$

$$\text{Then } y_n = n(n-1)(n-2)\ldots 1$$

$$= 1.2.3\ldots n = n!$$

$$\therefore \frac{d^n(x^n)}{dx^n} = n!$$

Corollary 2: If $m = -1$, then $y = \dfrac{1}{ax+b}$

$$y_n = \frac{(-1)(-2)(-3)\ldots.(-n)\,a^n}{(ax+b)^{n+1}}$$

$$= \frac{(-1)^n\,n!\,a^n}{(ax+b)^{n+1}}$$

Corollary 3: If $y = \log(ax+b)$, then $y_1 = \dfrac{1}{ax+b}.a$

$$\text{Hence } y_n = (n-1)\text{th derivative of } y_1$$

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}.a = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

2. *nth derivatives of sin (ax + b) and cos (ax + b)*

$$\text{Let } y = \sin(ax+b)$$

$$y_1 = \cos(ax+b)\,a = a\sin\left(\frac{\pi}{2} + ax + b\right)$$

$$y_2 = a^2\cos\left(\frac{\pi}{2} + ax + b\right)$$

$$= a^2\sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right) = a^2\sin\left(2.\frac{\pi}{2} + ax + b\right)$$

$$y_3 = a^3\cos\left(2.\frac{\pi}{2} + ax + b\right) = a^3\sin\left(3.\frac{\pi}{2} + ax + b\right)$$

....

$$y_n = a^n\sin\left(\frac{n\pi}{2} + ax + b\right)$$

$$\frac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(\frac{n\pi}{2} + ax + b\right)$$

Similarly, $\dfrac{d^n}{dx^n}[\cos(ax+b)] = a^n\cos\left(\dfrac{n\pi}{2} + ax + b\right)$

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Putting $b = 0, a = 1$ in the above results, we get

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(\frac{n\pi}{2} + x\right), \frac{d^n}{dx^n}(\cos x) = \cos\left(\frac{n\pi}{2} + x\right)$$

3. *nth derivatives of $e^{ax}\sin(bx+c)$ and $e^{ax}\cos(bx+c)$*

$$\text{Let } y = e^{ax}\cos(bx+c)$$

$$y_1 = e^{ax}a\cos(bx+c) - e^{ax}\sin(bx+c).b$$

$$= e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$$

$$\text{Put } a = r\cos\theta \text{ and } b = r\sin\theta$$

$$\text{Then } r^2 = a^2 + b^2 \text{ and } \tan\theta = b/a$$

$$\therefore y_1 = e^{ax}[r\cos\theta.\cos(bx+c) - r\sin\theta.\sin(bx+c)]$$

$$= r e^{ax}.\cos(bx+c+\theta)$$

Similarly, $$y_2 = r^2 e^{ax}.\cos(bx+c+2\theta)$$

....

$$y_n = r^n e^{ax}\cos(bx+c+n\theta) \text{ where } r = (a^2+b^2)^{1/2}, \theta = \tan^{-1} b/a$$

Similarly, $\frac{d^n}{dx^n}\left[e^{ax}\sin(bx+c)\right] = r^n e^{ax}\sin(bx+c+n\theta)$

Examples

Ex.1: If $ax^2 + 2hxy + by^2 = 1$ prove that $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$

Sol: Differentiating the given equation both sides w.r.t. x,

$$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx}(hx+by) = -(ax+hy).$$

$$\therefore \quad \frac{dy}{dx} = -\frac{ax+hy}{hx+by}.$$

$$\frac{d^2y}{dx^2} = -\left[\frac{(hx+by)\left(a+h\frac{dy}{dx}\right) - (ax+hy)\left(h+b.\frac{dy}{dx}\right)}{(hx+by)^2}\right]$$

$$= -\left[\frac{(hx+by)\left\{a - \frac{ahx+h^2y}{hx+by}\right\} - (ax+hy)\left\{h - \frac{abx+bhy}{hx+by}\right\}}{(hx+by)^2}\right]$$

$$= \frac{-(hx+by)(ab-h^2)y + (ax+hy)(h^2-ab)x}{(hx+by)^3}$$

$$= \frac{h^2-ab}{(hx+by)^3}(ax^2+2hxy+by^2)$$

$$= (h^2-ab)/(hx+by)^3 \qquad (\because ax^2+2hxy+by^2 = 1)$$

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Ex.2: If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ find $\frac{d^2y}{dx^2}$ when $\theta = \pi/2$.

Sol.:

$$\frac{dx}{d\theta} = a(1 + \cos\theta) = 2a\cos^2\frac{\theta}{2}$$

$$\frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a\sin\theta/2\cos\theta/2}{2a\cos^2\theta/2} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) = \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\frac{d\theta}{dx}$$

$$= \sec^2\frac{\theta}{2}\cdot\frac{1}{2}\;\frac{1}{2a\cos^2\theta/2} = \frac{1}{4a\cos^4\theta/2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{4a\cos^4\frac{\pi}{4}} = \frac{1}{4a\left(\frac{1}{\sqrt{2}}\right)^4} = \frac{1}{a}$$

Ex.3: Find nth derivative of $\sin x\ \sin 2x\ \sin 3x$.

Sol:

$$\text{Let } y = \sin x\ \sin 2x\ \sin 3x$$

$$= \frac{1}{2}(2\sin x\ \sin 2x)\sin 3x. = \frac{1}{2}(\cos x - \cos 3x)\sin 3x.$$

$$= \frac{1}{4}(2\cos x\ \sin 3x) - \frac{1}{4}(2\sin 3x\ \cos 3x)$$

$$= \frac{1}{4}(\sin 4x + \sin 2x) - \frac{1}{4}\sin 6x$$

$$y = \frac{1}{4}(\cos 2x + \sin 4x - \sin 6x)$$

$$y_n = \frac{1}{4}\left[2^n\sin\left(\frac{\pi n}{2} + 2x\right) + 4^n\sin\left(\frac{n\pi}{2} + 4x\right) - 6^n\sin\left(\frac{n\pi}{2} + 6x\right)\right]$$

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Ex.4: Find the nth derivative of $\frac{x^2}{(x-1)(x-2)}$

Sol: To find the n th derivative of a rational function like the given one, it is to be resolved into partial fractions.

$$\frac{x^2}{(x-1)(x-2)} \equiv 1 + \frac{4}{x-2} - \frac{1}{x-1}$$

$$\frac{d^n}{dx^n}\left[\frac{x^2}{(x-1)(x-2)}\right] = 4\frac{d^n}{dx^n}(x-2)^{-1} - \frac{d^n}{dx^n}(x-1)^{-1},\ n \ge 1$$

$$= 4(-1)^n n!\,(x-2)^{-n-1} - (-1)^n n!\,(x-1)^{-n-1}$$

$$= (-1)^n n!\left[\frac{4}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$

2.6 Leibnitz's theorem

Statement : If u and v are two functions of x possessing n th order derivatives

$$(uv)_n = c(n,0)\,u_n v + c(n,1)\,u_{n-1} v_1 + c(n,2)\,u_{n-2} v_2 + \ldots +$$

$$c(n,r)u_{n-r}\,v_r + \ldots\ldots + c(n,n)uv_n$$ where $c(n, r)$ representes the number of combinations of n objects taken r at a time and is equal to $\dfrac{n!}{r!\,(n-r)!}$

Proof: We prove this by Mathematical Induction.

$$(vu)_1 = u_1 v + uv_1 \text{ (Product Rule)}$$

Thus the theorem is true for $n = 1$.

Assume that the theorem is true for $n = m$. Then,

$$(uv)_m = c(m,0)\,u_m v + c(m,1)\,u_{m-1} v_1 + c(m,2)\,u_{m-2} v_2 + \ldots +$$

$$c(m,r-1)u_{m-r+1}\,v_{r-1} + \ldots\ldots + c(m,r)u_{m-r}v_r + \ldots\ldots + c(m,m)uv_m.$$

Differentiating both sides w.r.t. x, we have

$$(uv)_{m+1} = c(m,0)(u_{m+1}v + u_m v_1) + c(m,1)\,(u_m v_1 + u_{m-1} v_2)$$

$$+ c(m,2)\,(u_{m-1}v_2 + u_{m-2}v_3) + \ldots + c(m,r-1)(u_{m-r+1}\,v_r + u_{m-r+2}\,v_{r-1})$$

$$+ c(m,r)(u_{m-r}\,v_{r+1} + u_{m-r+1}v_r) + \ldots + c(m,m)(u_1\,v_m + uv_{m+1})$$

$$= c(m,0)\,u_{m+1}v + (c(m,0) + c(m,1))u_m v_1 + (c(m,1) + c(m,2))u_{m-1}v_2$$

$$+ \ldots + (c(m,r-1) + c(m,r))\,u_{m-r+1}\,v_r + \ldots\ldots + c(m,m)uv_{m+1}$$

But $c(m,0) = 1 = c(m+1,0)$, $c(m,m) = 1 = c(m+1,m+1)$, $c(m,r-1)+c(m,r) = c(m+1,r)$.

$$(uv)_{m+1} = c(m+1,0)u_{m+1}v + c(m+1,1)u_m v_1 + c(m+1,2)u_{m-1}v_2 + \ldots$$

$$+ c(m+1,r)_{m+1-r}\,v_r + \ldots\ldots + c(m+1,m+1)\,uv_{m+1}.$$

$\therefore$ The theorem is true for $n = m + 1$.

$\therefore$ By the principle of mathematical induction the theorem is true for all positive integral values of n.

Note: When one of the two functions in the product is of the form x^m, it should be chosen as v and second one as u since x^m will have only m non-zero differential coefficients.

Examples :

Ex.1: Find the n th derivative of $x^2 \sin 3x$.

Sol: $$\left(x^2 \sin 3x\right)_n = (\sin 3x)_n\,x^2 + c(n,1)(\sin 3x)_{n-1}\,2x + c(n,2)(\sin 3x)_{n-2}\,.\,2$$

$$= x^2.\,3^n.\sin\left(\frac{n\pi}{2} + 3x\right) + 2nx.3^{n-1}.\sin\left[\frac{(n-1)\pi}{2} + 3x\right]$$

$$+ n\,(n-1)\,3^{n-2}\,\sin\left[\frac{(n-2)\pi}{2} + 3x\right]$$

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Ex. 2: If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)\,y_2 + xy_1 - m^2 y = 0$. Using Leibnitz's theorem, show that $(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0$.

Sol: $$y^{1/m} + \frac{1}{y^{1/m}} = 2x \Rightarrow y^{2/m} - 2xy^{1/m} + 1 = 0$$

$$\therefore y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$\text{or } y = (x \pm \sqrt{x^2-1})^m.$$

$$\text{If } y = (x + \sqrt{x^2-1})^m \text{ then}$$

$$y_1 = m\,(x+\sqrt{x^2-1})^{m-1}\left(1 + \frac{1}{2}\frac{1}{\sqrt{x^2-1}}\cdot 2x\right)$$

$$= m\,(x+\sqrt{x^2-1})^{m-1}\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$= \frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

Similarly if $y = \left(x - \sqrt{x^2-1}\right)^m$, $y_1 = -\dfrac{my}{\sqrt{x^2-1}}$

Squaring, we have, in either case

$$y_1^2 = \frac{m^2y^2}{x^2-1} \text{ and so } (x^2-1)y_1^2 = m^2y^2$$

Differentiating both sides,

$(x^2-1)\,2y_1y_2 + y_1^2\ 2x = 2m^2y\,y_1$

Dividing with $2y_1 (\neq 0)$, $(x^2-1)y_2 + xy_1 = m^2y$

or $(x^2-1)\,y_2 + xy_1 - m^2y = 0$

Differentiating each term n times by Leibnitz's theorem,

$$(x^2-1)\,y_{n+2} + c(n,1)\,y_{n+1}\,2x + c(n,2)\,y_n\,.\,2 + xy_{n+1} + c(n,1)\,y_n\,.\,1 - m^2y_n = 0$$

Hence $(x^2-1)\,y_{n+2} + (2nx+x)\,y_{n+1} + \left[\dfrac{n\,(n-1)}{1.2}\,2 + n - m^2\right]y_n = 0$

$\therefore\quad (x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$

Ex.3: If $y = e^{m\sin^{-1}x}$ show that

(i) $(1-x^2)\,y_2 - xy_1 - m^2y = 0$ and

(ii) $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (m^2+n^2)y_n = 0.$

Also find $(y_n)_0$

Sol: $y = e^{m\sin^{-1}x}$

$$y_1 = e^{m\sin^{-1}x}\ \frac{m}{\sqrt{1-x^2}} = \frac{my}{\sqrt{1-x^2}} \qquad \text{(i)}$$

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$\therefore (1-x^2)^{1/2} y_1 = my$ and so $(1-x^2) y_1^2 = m^2 y^2$ (ii)

$(1-x^2)\, 2y_1 y_2 + y_1^2 \;(-2x) = 2m^2 y y_1$

or $(1-x^2)y_2 - xy_1 - m^2 y = 0$ (iii)

Differentiating each term n times by Leibnitz's theorem,

$(1-x^2)y_{n+2} + c(n,1)\, y_{n+1}(-2x) + c(n,2)\, y_n\, (-2)$

$-[xy_{n+1} + c\,(n,1)\, y_n .1] - m^2 y_n = 0$

$\therefore\; (1-x^2)\, y_{n+2} - x\, y_{n+1}(2n+1) - y_n \left[\frac{n\,(n-1)}{1.2} 2 + n + m^2\right] = 0$

Hence $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2+m^2) y_n = 0$ (iv)

$(y_n)_0$ means the value of y_n when $x = 0$

Putting $x = 0$ in (i), (ii), (iii) and (iv), we get

$$(y)_0 = e^{m \sin^{-1} 0} = e^0 = 1$$

$$(y_1)_0 = m\,(y)_0 = m \,.\, 1 = m$$

$$(y_2)_0 - m^2 (y)_0 = 0 \text{ or } (y_2)_0 = m^2 (y)_0 = m^2$$

$$(y_{n+2})_0 - (n^2+m^2)\,(y_n)_0 = 0$$

$$\therefore\; (y_{n+2})_0 = (n^2+m^2)\,(y_n)_0 \qquad \text{(v)}$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$ in (v)

$$(y_3)_0 = (1^2+m^2)\,(y_1)_0 = (m^2+1^2)\, m$$

$$(y_4)_0 = (2^2+m^2)\,(y_2)_0 = (m^2+2^2)\, m^2$$

$$(y_5)_0 = (3^2+m^2)\,(y_3)_0 = (m^2+3^2)\,(m^2+1^2)\, m$$

$$(y_6)_0 = (4^2+m^2)\,(y_4)_0 = (m^2+4^2)\,(m^2+2^2)\, m^2$$

....

Hence when n is odd.

$$(y_n)_0 = [m^2+(n-2)^2] \ldots.\, (m^2+3^2)\,(m^2+1^2)\, m$$

and when n is even

$$(y_n)_0 = [m^2+(n-2)^2] \ldots.\, (m^2+4^2)\,(m^2+2^2)\, m^2$$

Ex.4: If $y = \frac{\log x}{x}$, show that

$$y_n = \frac{(-1)^n\, n!}{x^{n+1}} \left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots\ldots\ldots - \frac{1}{n}\right)$$

Sol: Let $u = 1/x$, $v = \log x$. Putting $a = 1$, $b = 0$ in cor. 2 and cor. 3 of standard result 1, we get

$$u_n = \frac{(-1)^n\, n!}{x^{n+1}},\; v_n = \frac{(-1)^{n-1}\,.\,(n-1)!}{x^n}$$

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Applying Leibnitz's theorem,

$$y_n = (1/x)_n\ (\log x) + c(n, 1)\,(1/x)_{n-1}\,(\log x)_1 + c(n, 2)(1/x)_{n-2}\,(\log x)_2$$

$$+ \ldots\ldots + c(n, n)(1/x)\,(\log x)_n$$

$$= \frac{(-1)^n\, n!}{x^{n+1}} \log x + \frac{n(-1)^{n-1}\,(n-1)!}{x^n} \cdot \frac{1}{x}$$

$$+ \frac{n(n-1)}{1.2}\ \frac{(-1)^{n-2}\,(n-2)!}{x^{n-1}}\ \frac{(-1)}{x^2} + \ldots$$

$$+ c(n, n) \cdot \frac{1}{x}\ \frac{(-1)^{n-1}\,(n-1)!}{x^n}$$

$$= \frac{(-1)^n\, n!\, \log x}{x^{n+1}} - \frac{(-1)^n\, n!}{x^{n+1}} - \frac{(-1)^n\, n!}{x^{n+1}} \cdot \frac{1}{2} - \ldots - \frac{(-1)^n\, n!}{x^{n+1}} \cdot \frac{1}{n}$$

$$y_n = \frac{(-1)^n\, n!}{x^{n+1}} \left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right)$$

SAQ 3: If $I_n = D^n\,(x^n \log x)$, show that $I_n = n! \left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)$

2.7 Summary

The derivative of a given function f at a point $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The derivative of f at $x = a$ exists iff both the left hand and right hand derivatives exist (2.3) and are equal at $x = a$. A function f derivable at a point a is continuous at that point and the converse is not true (see 2.4 and Example 4).

The process of differentiating the same function again and again is called successive differentiation. We use Leibnitz's theorem for finding the nth derivative of a product of two functions (see. 2.6).

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2.8 Sample Examination Questions

I. *Answer the following questions in detail.*

(i) (a) Define the differential coefficient of f at a.

If f is derivable at a then prove that f is continuous at a.

(b) Prove that $f(x) = x^2 \sin 1/x$ for $x \neq 0$ and

$f(x) = 0$ when $x = 0$ is differentiable at $x = 0$.

(ii) (a) Explain the process of successive differentiation and find the nth derivative of x^n.

(b) If $x^3 + y^3 + 3ax^2 = 0$, show that $\frac{d^2y}{dx^2} + \frac{2a^2 x^2}{y^5} = 0$

(iii) (a) State and prove Leibnitz's theorem.

(b) Find the nth derivative of $x^3 \cos x$.

II. *Briefly answer the following questions.*

(i) Prove that the function $f(x) = x \cos 1/x$ when $x \neq 0$

when $x = 0$

is not derivable at $x = 0$.

(ii) Differentiate $\tan^{-1}\left(\frac{\sin x}{1-\cos x}\right)$ and $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$

(iii) Differentiate $(\cot x)^{\sin x} + (\tan x)^{\cos x}$ and $(\log x)^x$.

(iv) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$ and $\log x$ w.r.t. $\log \sqrt{x}$.

(v) If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, show that

$$\frac{d^2y}{dx^2} = \frac{24}{a\pi} \text{ at } \theta = \frac{\pi}{3}$$

(vi) Find the nth derivatives of $x^3 \log x$ and $e^x \cos x . \cos 2x$.

(vii) If $y = (\sin^{-1} x)^2$ prove that $(1-x)^2 y_{n+2} - (2n+1)x\, y_{n+1} - n^2 y_n = 0$ and hence find the value of y_n at $x = 0$.

(viii) If $\cos^{-1}(y/b) = \log(x/n)^n$, prove that $x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0$.

III. *Answer the following questions in about 5 lines.*

(i) Define the left and right derivatives.

(ii) When do you say that the derivative of a function at a point exists?

(iii) What is the use of the logarithmic differentiation?

2.9 Answers to self assessment questions

SAQ 1:

$$f(x) = \sqrt{x}$$

$$f(x+h) = \sqrt{x+h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

SAQ 2 :

$$f(x) = \frac{x-1}{(2x-5)(x-1)} = \frac{1}{2x-5} \text{ (if } x \neq 1)$$

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2(1-h)-5} - \frac{1}{2(1)-5}}{-h}$$

$$= \lim_{h\to 0} \frac{-2}{3(2h+3)} = \frac{-2}{9}$$

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

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$$= \lim_{h \to 0} \frac{\frac{1}{2(1+h)-5} + \frac{1}{3}}{-h}$$

$$= \lim_{h \to 0} \frac{2h}{3h(2h-3)} = \lim_{h \to 0} \frac{2}{3(2h-3)} = \frac{-2}{9}.$$

Since $Lf'(1) = Rf'(1), f$ is differentiable at $x = 1$ and $f'(1) = -\frac{2}{9}$

SAQ 3 :

$$I_n = D^n(x^n \log x)$$

$$= D^{n-1}[D(x^n \log x)]$$

$$= D^{n-1}(nx^{n-1}\log x + x^n . 1/x)$$

$$= n D^{n-1}(x^{n-1}\log x) + D^{n-1}(x^{n-1})$$

$$I_n = n I_{n-1} + (n-1)!$$

Dividing both sides with n !,

$$\frac{I_n}{n!} = \frac{I_{n-1}}{(n-1)!} + \frac{1}{n}$$

$$\frac{I_n}{n!} - \frac{I_{n-1}}{(n-1)!} = \frac{1}{n}.$$

Giving values $n-1, n-2, \ldots\ldots, 3, 2,$ to n, we get

$$\frac{I_{n-1}}{(n-1)!} - \frac{I_{n-2}}{(n-2)!} = \frac{1}{n-1}$$

$$\frac{I_{n-2}}{(n-2)!} - \frac{I_{n-3}}{(n-3)!} = \frac{1}{n-2}$$

....

$$\frac{I_3}{3!} - \frac{I_2}{2!} = \frac{1}{3}$$

$$\frac{I_2}{2!} - \frac{I_1}{1!} = \frac{1}{2}$$

Adding vertically,

$$\frac{I_n}{n!} - \frac{I_1}{1!} = \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$$

But $I_1 = D(x \log x) = \log x + 1$

$$\therefore \quad \frac{I_n}{n!} = \log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}$$

$$\therefore \quad I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}\right)$$

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Unit – 3 : Mean Value Theorems

3.0 Contents

3.1 Aims and Objectives

After going through this Unit, you will be able to :

(i) state and verify the Rolle's theorem and mean value theorems,

(ii) notice the importance of mean value theorems in various expansions like Taylor's expansion, Maclaurin's expansion etc.,

(iii) expand the given function in Taylor or Maclaurin's series if the given function satisfies the required conditions.

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3.2 Introduction

This Unit deals with mean value theorems and their use in Taylor's and Maclaurin's expansions. The statements of Rolle's theorem, Lagrange's mean value theorem and Cauchy's mean value theorems are given without proofs. The proofs are beyond the scope of the syllabus and you will be learning them in course-III, Analysis. However, the geometrical meaning of theorems are given and the verification of these theorems will be discussed here. Taylor's and Maclaurins expansions with Lagrange's and Cauchy's form of remainders are given.

3.3 Rolle's theorem

If a function f defined on $[a, b]$

(i) is continuous in the closed interval $[a, b]$,

(ii) is derivable in the open interval (a, b),

(iii) $f(a) = f(b)$,

then there exists at least one value c of x in the open interval (a, b) such that $f'(c) = 0$.

3.3.1 Geometrical interpretation

Let AB be the graph of the function f. Let A and B of the graph correspond to a and b of the closed interval $[a, b]$.

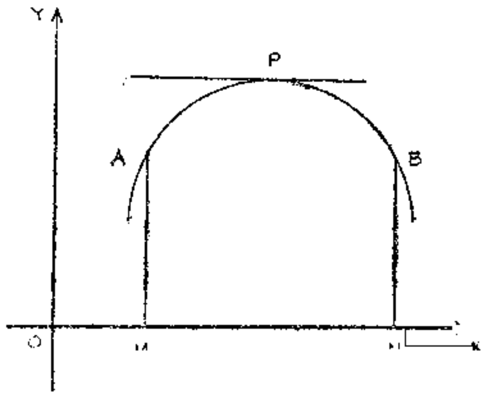

Fig. 1

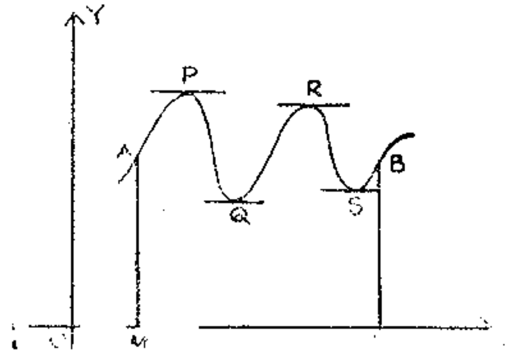

Fig. 2

(i) As f is continuous in $[a, b]$, the graph of f is a continuous curve between A and B.

(ii) As f is derivable in the interval (a, b), the graph of f posseses a tangent at every intermediary point between $x = a$ and $x = b$.

(iii) As $f(a) = f(b)$, AM and BN, the corresponding ordinates at $x = a$ and $x = b$ are equal.

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Further, $f'(c) = 0$ for some c such that $a < c < b$ implies that the tangent to the graph of f at $x = c$ is parallel to x - axis, since the slope of the tangent at $x = c$ is $f'(c)$ and the slope of x - axis is 0.

Examples :

Ex. 1 : Verify Rolle's theorem for $x^3 - 4x$ in $[-2, 2]$.

Sol : Let $f(x) = x^3 - 4x$.

(i) Since f is a polynomial, and every polynomial is a continuous function of x for all x, f is continuous in $[-2, 2]$.

(ii) $f'(x) = 3x^2 - 4$.

It is clear that $f'(x)$ exists in $(-2, 2)$.

(iii) $f(-2) = -8 + 8 = 0$, $f(2) = 8 - 8 = 0$

$f(-2) = f(2) = 0$

$\therefore$ f satisfies all the three conditions of Rolle's theorem.

Then there should exist at least one value c of x in $(-2, 2)$ such that $f'(c) = 0$.

$f'(c) = 3c^2 - 4 = 0 \Rightarrow c^2 = 4/3 \Rightarrow c = \pm 2/\sqrt{3}$.

i.e., $c = \pm 1{\cdot}15$ approximately.

$\because$ Both values of c lie clearly in $(-2, 2)$.

Thus, Rolle's theorem is verified.

Ex. 2 : Verify Rolle's theorem for

$$\log \frac{x^2 + ab}{(a+b)\,x} \text{ in } [a, b]\,;\; a > 0, b > 0.$$

Sol : Let $f(x) = \log \dfrac{x^2 + ab}{(a+b)\,x} = \log\left(x^2 + ab\right) - \log(a+b) - \log x.$

$$f'(x) = \frac{1}{x^2 + ab} \cdot 2x - \frac{1}{x} = \frac{x^2 - ab}{x\left(x^2 + ab\right)}$$

Clearly $f'(x)$ exists for every value of x in $[a, b]$.

$\therefore$ f is derivable in $[a, b]$.

For $x \,\varepsilon\, [a, b]$, $a > 0$, $b > 0$, the given function $f(x) = \log \dfrac{x^2 + ab}{(a+b)\,x}$ is continuous.

$$\text{Also } f(a) = \log(a^2 + ab) - \log(a+b) - \log a = 0$$

$$f(b) = \log(b^2 + ab) - \log(a+b) - \log b = 0$$

$$\therefore\ f(a) = f(b)$$

Hence f satisfies all the three conditions of Rolle's theorem. Then there must exist at least one value c of x in (a, b) such that $f'(c) = 0$.

$$f'(c) = \frac{c^2 - ab}{c\,(c^2 + ab)} = 0 \Rightarrow c^2 = ab = c = \pm\sqrt{ab}$$

$\sqrt{ab}$ lies in (a, b).

Thus, Rolle's theorem is verified.

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SAQ 1: Show that the hypotheses of Rolle's theorem are satisfied for the function

$$f(x) = x^{4/3} - 3x^{1/3} \text{ defined on } [0, 3].$$

3.4 Lagrange's mean value theorem or first mean value theorem

If a function f defined on $[a, b]$:

(i) is continuous in the closed interval $[a, b]$

(ii) is derivable in the open interval (a, b),

then there exists at least one value c of x lying in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Another form : Let $b - a$ = length of the interval $= h$.

Then $b = a + h$.

Then c, which lies between a and $a + h$ may be written as

$$c = a + \theta h \text{ where } 0 < \theta < 1.$$

Then the theorem may be stated as :

If a function f defined on $[a, a+h]$;

(i) is continuous in the closed interval $[a, a+h]$

(ii) is derivable in the open interval $(a, a+h)$,

then there exists at least one number θ lying between 0 and 1 such that

$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

$$\Rightarrow f(a+h)-f(a) = hf'(a+\theta h)$$

$$\Rightarrow f(a+h) = f(a)+hf'(a+\theta h),\ 0<\theta<1.$$

3.4.1 Geometrical interpretation

Let A and B be the points on the graph of the function $y=f(x)$ corresponding to $x=a$ and $x=b$.

$\therefore$ The points A and B are respectively $[a, f(a)]$ and $[b, f(b)]$.

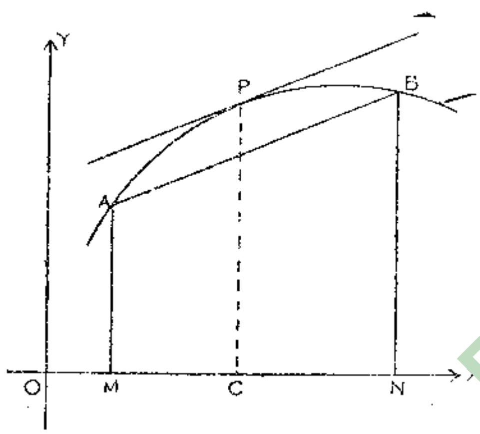

Fig. 3

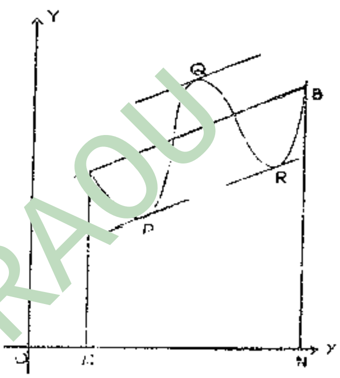

Fig. 4

Slope of the chord $AB = \dfrac{f(b)-f(a)}{b-a}$.

$f'(c)$ = slope of the tangent to the curve at $P\,[c, f(c)]$.

(i) Since $f(x)$ is continuous in $[a, b]$, the graph of $f(x)$ is a continuous curve between A and B.

(ii) Since $f(x)$ is derivable in (a, b), the graph of the function $f(x)$ has a unique tangent at every point between A and B.

Hence the condition $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ for some number c in (a, b) implies that the tangent to the curve at the point P is parallel to the chord AB.

Ex. 3 : Verify Lagrange's mean value theorem for the function

$$f(x) = (x-1)(x-2)(x-3) \text{ in } [0, 4].$$

Sol : $f(x) = (x-1)(x^2-5x+6) = x^3-6x^2+11x-6$

Here $a = 0,\ b = 4.$

(i) $f(x)$ is a polynomial in x. Hence $f(x)$ is continuous in $[0, 4]$.

(ii) $f'(x) = 3x^2 - 12x + 11$.

Obviously this exists in $(0, 4)$.

Two conditions of Lagrange's mean value theorem are satisfied.

$\therefore$ There must exist at least one value c of x in $(0, 4)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 3c^2 - 12c + 11 = \frac{f(4)-f(0)}{4-0}$$

$$\Rightarrow 3c^2 - 12c + 11 = \frac{64 - 6(16) + 11(4) - 6 + 6}{4} = 3$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{3}}{3}$$

i. e., $c = 3.15$ or 0.84

These values lie in $(0, 4)$,

Hence Lagrange's mean value theorem is verified.

Ex. 4 : Compute the value of θ in the first mean value theorem

$$f(x+h) = f(x) + hf'(x+\theta h) \text{ if } f(x) = ax^2 + bx + c.$$

Sol :

$$f(x) = ax^2 + bx + c$$

$$f(x+h) = a(x+h)^2 + b(x+h) + c.$$

$$f'(x) = 2ax + b$$

$$f'(x+\theta h) = 2a(x+\theta h) + b$$

From first mean value theorem,

$$f(x+h) = f(x) + hf'(x+\theta h)$$

$$\Rightarrow ax^2 + ah^2 + 2ahx + bx + bh + c = ax^2 + bx + c + 2ahx + 2a\,\theta\, h^2 + bh$$

$$\Rightarrow ah^2 = 2a\,\theta\, h^2$$

$$\Rightarrow 2\theta = 1 \Rightarrow \theta = \frac{1}{2}.$$

SAQ 2 : Verify Lagranges mean value theorem for $f(x) = \frac{x^2}{6}$ defined on $[2, 6]$.

Note : The mean value theorem assures us of the existence of a solution to problems, such as those in examples 3 and 4 above, but it does not tell us how to find such solution. Indeed, each problem has to be handled according to its own pecularities. In using the mean value theorem one must make certain that all the hypotheses are satisfied. If the hypotheses do not hold the conclusion of the mean value theorem need not hold,see the following example.

Ex. 5 : If $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$ in the interval $[0, 2]$

verify the Lagranges mean value theorem.

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Sol : The given function is differentiable in (0, 2) but fails to be continuous in [0, 2].

In fact $\lim_{x \to 0^+} f(x) = +\infty \neq \frac{1}{2} = f(0)$

Hence the conditions of the mean value theorem are not satisfied.

3.5 Cauchy's mean value theorem

If two functions $f(x)$ and $g(x)$ defined on $[a, b]$

(i) are continuous in the closed interval $[a, b]$,

(ii) are derivable in the open interval (a, b),

(iii) $g'(x) \neq 0$ for any value of x in (a, b),

then there exists at least one value c of x in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Another form : Let $b - a = h$ so that $b = a + h$ and c which lies between a and $a + h$ may be written as $c = a + \theta h$ where $0 < \theta < 1$.

The Cauchy's mean value theorem is

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad \text{where } 0 < \theta < 1$$

Ex. 6 : In Cauchy's mean value theorem

(i) if $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ prove that c is the geometric mean between a and b and

(ii) if $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ prove that c is the harmonic mean between a and b.

Sol : (i) $f(x) = \sqrt{x}, g(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \frac{1}{2\sqrt{x}}, g'(x) = -\frac{1}{2x\sqrt{x}}$$

$$\therefore \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{1}{-2c\sqrt{c}}} \Rightarrow \frac{\sqrt{ab}\left(\sqrt{b} - \sqrt{a}\right)}{-\left(\sqrt{b} - \sqrt{a}\right)} = -c$$

$\Rightarrow \sqrt{ab} = c \Rightarrow c$ is G. M. between a and b.

(ii) $f(x) = \frac{1}{x^2}, g(x) = \frac{1}{x}$

$$f'(x) = \frac{-2}{x^3}, g'(x) = \frac{-1}{x^2}$$

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$$\therefore \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} = \frac{\frac{-2}{c^3}}{\frac{-1}{c^2}}$$

$$\Rightarrow \frac{a^2-b^2}{a^2b^2} \cdot \frac{ab}{a-b} = \frac{2}{c} \Rightarrow \frac{a+b}{ab} = \frac{2}{c}$$

$$\Rightarrow \frac{2}{c} = \frac{1}{a} + \frac{1}{b} \Rightarrow c \text{ is H. M .between } a \text{ and } b.$$

Ex. 7 : Prove that functions f and g defined on $[0, \frac{1}{2}]$ by

$f(x) = x(x-1)(x-2)$ and $g(x) = x(x-2)(x-3)$

satisfy Cauchy's mean value theorem.

Sol.: (i) $f(x), g(x)$ being polynomials in x are continuous in $\left[0, \frac{1}{2}\right]$

(ii)

$$f(x) = x^3 - 3x^2 + 2x$$
$$f'(x) = 3x^2 - 6x + 2$$
$$g(x) = x^3 - 5x^2 + 6x$$
$$g'(x) = 3x^2 - 10x + 6$$

$f'(x)$ and $g'(x)$ exist in $\left(0, \frac{1}{2}\right)$

$\Rightarrow f(x)$ and $g(x)$ are derivable in $\left(0, \frac{1}{2}\right)$

$g'(x) \neq 0$ in $\left(0, \frac{1}{2}\right)$.

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There exists c a value of x in $\left(0, \frac{1}{2}\right)$ such that

$$\frac{f\left(\frac{1}{2}\right) - f(0)}{g\left(\frac{1}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right) - 0}{\frac{1}{2}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right) - 0} = \frac{3c^2 - 6c + 2}{3c^2 - 10c + 6}$$

$$\Rightarrow 15c^2 - 30c + 10 = 3c^2 - 10c + 6$$

$$\Rightarrow 12c^2 - 20c + 4 = 0 \Rightarrow 3c^2 - 5c + 1 = 0$$

$$c = \frac{5 \pm \sqrt{25-12}}{6} = \frac{5 \pm \sqrt{13}}{6}$$

$\therefore \frac{5+\sqrt{13}}{6}$ does not lie in $\left(0, \frac{1}{2}\right)$. the value of c is $\frac{5-\sqrt{13}}{6}$.

This lies in $\left(0, \frac{1}{2}\right)$. Thus, the theorem is verified.

3.6 Deductions from Lagrange's mean value theorem

1. If $f(x)$ is such that $f'(x) = 0$ for all values of x in the open interval (a, b) then $f(x)$ is constant in this interval.

Proof : Let x_1 and x_2 be any two values of x in (a, b) such that

$a < x_1 < x_2 < b$.

$\therefore [x_1, x_2] \subset (a, b)$

Since $f'(x) = 0$ for all values of x in (a, b),

$f'(x) = 0$ for all values of x in $[x_1, x_2]$.

$\therefore$ $f(x)$ is derivable in $[x_1, x_2]$.

Since derivability $\Rightarrow$ continuity, $f(x)$ is continuous in $[x_1, x_2]$.

$\therefore$ $f(x)$ satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$.

$$\therefore \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \text{ where } x_1 < c < x_2$$

But $f'(x) = 0$ for all x in (a, b).

$\therefore f'(c) = 0 \Rightarrow f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$

$\therefore$ $f(x)$ has the same value for every value of x in (a, b).

Hence $f(x)$ is constant in (a, b).

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2. If $f(x)$ is continuous in the closed interval $[a, b]$ and $f'(x)$ is positive for every value of (a, b) then $f(x)$ is a monotonic increasing function of x in the interval $[a, b]$.

Proof : Let x_1 and x_2 be any two numbers between a and b such that $x_2 > x_1$.

By Lagrange's mean value theorem,

$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$ where $x_1 < c < x_2$.

Since $x_2 > x_1$, $x_2 - x_1$ is positive.

Also $f'(c)$ is positive.

$\therefore$ $f(x_2) - f(x_1)$ is positive

$\Rightarrow f(x_2) > f(x_1)$

$\therefore$ $f(x)$ is monotonic increasing in $[a, b]$.

3. If $f(x)$ is continuous in the interval $[a, b]$ and $f'(x)$ is negative for every value of x in (a, b) then $f(x)$ is a monotonic decreasing function of x in $[a, b]$.

(We can prove this in the same lines as in 2 above).

Ex. 8 : If $x > 0$, shows that $x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1 + x)}$

Sol : Let $f(x) = \log(1 + x) - (x - x^2/2)$

$$f'(x) = \frac{1}{1+x} - 1 + \frac{2x}{2} = \frac{1}{1+x} - (1-x) = \frac{x^2}{1+x}$$

Since $x > 0, f'(x) > 0$.

$\therefore f(x)$ is a monotonic increasing function for $x > 0$,

$$\because f(0) = 0,\ f(x) > 0 \text{ for } x > 0.$$

Hence, $\log(1+x) - (x - x^2/2) > 0$ for $x > 0$.

$$\Rightarrow x - x^2/2 < \log(1+x). \qquad ...(1)$$

$$\text{Let } g(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$$

$$g'(x) = 1 - \frac{\frac{1}{2}(1+x)\,2x - x^2}{(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{x^2}{2(1+x)^2} > 0 \text{ for } x > 0.$$

$g(x)$ is a monotonic increasing function

$$\text{since } g(0) = 0,\ g(x) > g(0) \text{ for } x > 0.$$

$$\therefore \log(1+x) < x - \frac{x^2}{2(1+x)} \qquad ...(2)$$

Combining (1) and (2) we get the required result.

3.7 Taylor's theorem with Lagrange's from of remainder

If a function $f(x)$ defined on $[a, a+h]$ is such that

(i) $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continuous in $[a, a+h]$

(ii) $f^n(x)$ exists in $(a, a+h)$,

then there exists at least one number θ laying between 0 and 1 such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h).$$

The $(n+1)$ th term $\frac{h^n}{n!} f^n(a+\theta h)$ is called Lagrange's form of remainder after n terms and is denoted by R_n.

3.8 Taylor's theorem with Cauchy's form of remainder

If a function $f(x)$ defined on $[a, a+h]$ is such that

(i) $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continuous in the closed interval $[a, a+h]$

(ii) $f^n(x)$ exists in the open interval $(a, a+h)$ then there exists at least one number θ between 0 and 1 such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h).$$

The last term $\frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$ is called

 Cauchy's form of the remainder after n terms.

3.9 Maclaurin's theorem with Lagrange's form of remainder

If a function $f(x)$ defined on $[0, x]$ is such that

(i) $f(x), f'(x), f''(x), \dots f^{n-1}(x)$ are continuous in the closed interval $[0, x]$

(ii) $f^n(x)$ exists in the open interval $(0, x)$ then there exists at least one number θ lying between 0 and 1 such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{x^n}{n!} f^n(\theta x).$$

Note : This theorem will be obtained from Taylor's theorem with Lagrange's form of remainder by putting $a = 0, h = x$.

3.10 Maclaurin's theorem with Cauchy's form of remainder

Putting $a = 0$ and $h = x$ in Taylor's theorem with Cauchy's form of the remainder we get,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f(\theta x)$$

3.11 Taylor's infinite series

This is the expansion of the function $f(a+h)$ as an infinite series in ascending powers of h. This expansion holds good if $f(x)$ possesses derivatives of every order in the closed interval $[a, a+h]$ and $R_n \to 0$ as $n \to \infty$.

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots$$

Note: To find the expansion of $f(x)$ in ascending integral powers of $(x-a)$ we write

$f(x) = f(a + \overline{x-a})$. Here $h = x - a$

$$\therefore f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^n(a) + \dots$$

3.12 Maclaurin's infinite series

It is the expansion of the function $f(x)$ as an infinite series in ascending powers of x. The expansion holds good if $f(x)$ possesses derivatives of every order in the closed interval $[0, x]$ and $R_n \to 0$ as $n \to \infty$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f''(0) + \dots$$

Ex 9. Show that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

Sol :

Let $f(x) = \sin x$		$\therefore f(0) = 0$
$f'(x) = \cos x$		$f'(0) = 1$
$f''(x) = -\sin x$		$f''(0) = 0$
$f'''(x) = -\cos x$		$f'''(0) = -1$
$f^{iv}(x) = \sin x$		$f^{iv}(0) = 0$
$f^{v}(x) = \cos x$		$f^{v}(0) = 1$

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Using Maclaurins expansion,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4 f^{iv}(0)}{4!} + \frac{x^5}{5!}f^{v}(0) + \ldots$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Ex 10 : Show that $a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots.$

Sol : Let $f(x) = a^x, f'(x) = a^x \log a, f''(x) = a^x (\log a)^2, \ldots$

$$f(0) = 1, f'(0) = \log a,\ f''(0) = (\log a)^2, \ldots$$

Using Maclaurin's expansion,

$$a^x = 1 + \frac{x}{1!}\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$$

SAQ 3 : Use the above result to show that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Ex 11 : Show that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Sol : Let $f(x) = \log(1+x), f'(x) = \frac{1}{1+x}$

$$f''(x) = -\frac{1}{(1+x)^2},\ f'''(x) = \frac{2}{(1+x)^3}$$

$$f^{iv}(x) = -\frac{6}{(1+x)^4}, \ldots\ldots$$

$$\therefore f(0) = \log(1+0) = 0$$

$$f'(0) = 1,\ f''(0) = -1$$

$$f'''(0) = 2, f^{iv}(0) = -6, - \ldots\ldots$$

Using Maclaurin's expansion,

$$\log(1+x) = 0 + \frac{x}{1!}1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}2 + \frac{x^4}{4!}(-6) + \ldots$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

SAQ 4 : Use the above result to show that

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots$$

Ex 12 : Expand e^x in ascending powers of $(x-1)$.

Sol : Let $f(x) = e^x$

$$e^x = f(x) = f(1 + \overline{x-1}) = f(1+h) \text{ where } h = x - 1.$$

$$f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x, \ldots\ldots$$

$$f(1) = e, f'(1) = e, f''(1) = e, f'''(1) = e, \ldots\ldots$$

$$f(1+h) = f(1) + hf'(1) + \frac{h^2}{2!} f''(1) + \frac{h^3}{3!} f'''(1) + \ldots..$$

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \ldots...$$

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right]$$

Ex. 13 : Expand log sin x in powers of $(x-3)$.

Sol : Let $f(x) = \log \sin x$

$$\log \sin x = f(x) = f(3 + x - 3) = f(3+h) \text{ where } h = x - 3.$$

$$\therefore \quad f(x) = f(3) + hf'(3) + \frac{h^2}{2!} f''(3) + \ldots.$$

$$f'(x) = \frac{1}{\sin x} \cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x, \ldots\ldots$$

$$f(3) = \log \sin 3, f'(3) = \cot 3, f''(3) = -\operatorname{cosec}^2 3, \ldots\ldots$$

$$\therefore \log \sin x = \log \sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2} \operatorname{cosec}^2 3 + \ldots..$$

3.13 Summary

In Rolle's theorem and mean value theorems the general conditions to be satisfied by the given function(s) are :

(i) the given function(s) are continuous in the closed (given) interval, say $[a, b]$,

(ii) differentiable in the open interval (a, b).

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Then for Rolle's theam if $f(a) = f(b)$ there exists a point in the open interval such that the derivative of the function at that point vanishes.

And for Lagrange mean value theorem there exists a point c in the open interval such that the derivative at that point $f'(c) = \frac{f(b) - f(a)}{b - a}$ and for Cauchy's mean value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

To verify the theorem, we have to verify the conditions stated above and find the value of the c.

The expansion of $f(a+h)$ as an infinite series in the ascending powers of h is called Taylor infinite series. The expansion hold good if the deritivatives of all orders exists in the closed interval $[a, a+h]$ and $R_n \to 0$ as $n \to \infty$ and the expansion is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots.. + \frac{h^n}{n!} f^n(a) + \ldots\ldots$$

$f(x)$ can be expanded as a Maclaurin's infinite series in the interval $[0, x]$ provided the derivatives of f of all orders exists in $[0, x]$ and $R_n \to 0$ as $n \to \infty$, and the expansion is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots.. + \frac{x^n}{n!} f^n(0) + \ldots..$$

3.14 Sample Examination Questions

I. *Answer the following questions in detail.*

(i) (a) State Rolle's theorem and give its geometrical interpretation.

(b) Verify the Rolle's theorem for the function

$$f(x) = x^2 - 6x + 8 \text{ in } [2, 4].$$

(ii) (a) State the Lagrange's mean value theorem and give its geometrical interpretation.

(b) Verify Lagrange's mean value theorem for the function.

$$f(x) = \log x \text{ in } \left[\frac{1}{2}, 2\right].$$

II. *Briefly answer the followting questions.*

(i) Verify the Rolle's theorem for

(a) $f(x) = (x-a)^m (x-b)^n$ in $[a, b]$

(b) $f(x) = x(x+3)\, e^{-x/2}$ in $[-3, 0]$

(c) $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$.

(ii) Verify Lagrange's mean value theorem for the function

(a) $f(x) = x(x-1)(x-2), a = 0, b = \frac{1}{2}$

(b) $f(x) = x^3 - 3x - 1, a = -11/7, b = 13/7$

(c) $f(x) = x^3 - 5x^2 - 3x, a = 1, b = 3.$

(iii) Verify Cauchy's mean value theorem and find c for

(a) $f(x) = x^2, g(x) = x, a = 0, b = 1$

(b) $f(x) = \sin x, g(x) = \cos x$ in $[0, \pi/2]$.

(iv) Prove by Maclaurin's theorem.

(a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots..$

(b) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots...$

(c) $\sin^{-1} x = x + 1^2 \frac{x^3}{3!} + 1^2 3^2 \frac{x^5}{5!} + \ldots...$

III. *Answer the following questions in about five lines.*

(i) State Taylor's theorem with Lagrange's form of remainder.

(ii) State Taylor's theorem with Cauchy's form of remainder.

(iii) State Taylor's infinite series.

(iv) State Maclaurin's infinite series.

Answers :

I. (ib) $c = 3$ (iib) $c = \frac{3}{4 \log 2}$

II. (ia) $\frac{mb + na}{m + n}$ (ib) $c = -2$ (ic) f' does not exist at $x = 1$ and the Rolle's theorem is not valid in $[0, 2]$,

(iia) $c = 0.24$ (iib) $c = \pm 1$, (iic) $7/3$

(iiia) $c = \frac{1}{2}$ (iii) $-\pi/4$.

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3.15 Answers to SAQ's.

SAQ 1 : For $x \neq 0$, $f'(x) = \frac{4}{3} x^{1/3} - x^{-2/3} = \frac{1}{3} x^{-2/3} (4x - 3)$

Also $f(x) = x^{4/3} - 3x^{1/3} = x^{1/3}(x - 3)$; Evidently, f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Also, $f(0) = 0 = f(3)$. So, the hypotheses of the Rolle's theorem are satisfied. Solving the equation

$$f'(c) = \frac{1}{3} c^{-2/3} (4c - 3) = 0 \Rightarrow c = \frac{3}{4} \in (0, 3).$$

SAQ 2 : Since f is a polynomial, it is continuous on $[2, 6]$ and differentiable on $(2, 6)$.

Here, $f'(x) = x/3$, $f(6) = 6$ and

$f(2) = \frac{2}{3}$. Hence we must solve the equation

$$\frac{c}{3} = \frac{6 - 2/3}{6 - 2} = \frac{4}{3} \Rightarrow c = 4 \in (2, 6).$$

SAQ 3 : Put $a = e$ in the expansion given in example 9.

SAQ 4 : Writing $-x$ in place of x in the expansion given in example 10, you will get the result.

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Unit-4 : Indeterminate Forms, L-Hospital's Rule

4.0 Contents

4.1 Aims and Objectives

After going through this unit, you will be able to

(i) identify different types of indeterminate forms,

(ii) and evaluate the limits which are in the form of different types of indeterminate forms using L-Hospital's rule.

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4.2 Introduction

If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist and $\lim\limits_{x\to a} g(x) \neq 0$, then we know that $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ exists and is equal to $\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$. If $\lim\limits_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} f(x) \neq 0$, we also know that $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ does not exist. But when $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, it cannot be decided whether $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ exists or not and cannot be taken to be equal to $\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$.

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, then the quotient $\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ is then of the form 0/0 and this form is called an *indeterminate form*.

Note : 1. The other indeterminate forms are ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0.

2. It is to be borne in mind that we are not going to find the value of 0/0 or any other indeterminate forms. But we will find the limits of the combinations of the functions whose limits assume indeterminate forms when the limits of the functions are taken separately.

4.3. The indeterminate form 0/0 ; L– Hospital rule

[G. F. A. De L-Hospital (1661 - 1704) was a French mathematician. He was a student of J. Bernoulli. He wrote the first book on Differential Calculus in 1696 which was based on lectures by J. Bernoulli.]

Theorem : If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ and $f'(a)$, $g'(a)$ both exist and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof : Let $f(x)$ and $g(x)$ be functions of x which can be expanded in the deleted neighbourhood of $x = a$ by Taylor's theorem.

$$\text{Let } f(a) = g(a) = 0.$$

$$\text{We know that } f(x) = f[a + (x - a)]$$

$$= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + R_1,$$

$$\text{where } R_1 = \frac{(x-a)^n}{n!} f^n[a + \theta_1 (x-a)], 0 < \theta_1 < 1$$

$$g(x) = g(a) + (x-a) g'(a) + \frac{(x-a)^2}{2!} g''(a) + \ldots + R_2$$

$$\text{where } R_2 = \frac{(x-a)^n}{n!} g^n[a + \theta_2 (x-a)], 0 < \theta_2 < 1$$

$$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + (x-a) f'(a) + \ldots + R_1}{g(a) + (x-a) g'(a) + \ldots + R_2}$$

$$= \lim_{x \to a} \frac{(x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + R_1}{(x-a) g'(a) + \frac{(x-a)^2}{2!} g''(a) + \ldots + R_2}$$

Cancelling the factor $(x - a)$ throughout,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a) + \frac{(x-a)}{2!} f''(a) + \ldots + \frac{R_1}{(x-a)}}{g'(a) + \frac{(x-a)}{2!} g''(a) + \ldots + \frac{R_2}{(x-a)}}$$

$$= \frac{f'(x)}{g'(x)} \quad \left[\because \text{All the other terms are having } (x-a) \text{ as a factor and vanishes when } x \to a \right]$$

Corollary : If $f'(a)$ and $g'(a)$ are both zero, we can prove similarly that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f''(a)}{g''(a)}$$

Generalisation: If $f(x)$ and $g(x)$ have their first $(n-1)$ derivatives equal to zero at $x = a$ and their nth derivatives are finite and not equal to zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)} = \frac{f^n(a)}{g^n(a)}$$

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Note : L-Hospital rule also holds good when $x \to \infty$.

You may verify it by substituting

$$x = 1/t \text{ so that } t \to 0 \text{ as } x \to \infty .$$

Then,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0} \frac{f(1/t)}{g(1/t)}$$

$$= \lim_{t \to 0} \frac{f'(1/t) \cdot (-1/t^2)}{g'(1/t) \cdot (-1/t^2)}$$

$$= \lim_{t \to 0} \frac{f'(1/t)}{g'(1/t)}$$

Examples

Ex. 1 : Evaluate $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$.

Sol : This is an indeterminate form of the type $\frac{0}{0}$.

Applying L-Hospital rule the given limit

$$= \lim_{x \to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x}.$$

This again takes the form $\frac{0}{0}$.

Applying L-Hospital rule again the given limit

$$= \lim_{x \to 0} \frac{xe^x + 2e^x + \frac{1}{(1+x)^2}}{2} = \frac{3}{2}$$

Ex. 2 : Show that $\lim_{x \to 0} \frac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x} = -15.$

Sol : $\lim_{x \to 0} \frac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$ (form 0/0)

$$= \lim_{x \to 0} \frac{5 \cos x - 14 \cos 2x + 9 \cos 3x}{\sec^2 x - 1} \quad \text{(form 0/0)}$$

$$= \lim_{x \to 0} \frac{-5 \sin x + 28 \sin 2x - 27 \sin 3x}{2 \tan x \sec^2 x} \quad \text{(form 0/0)}$$

$$= \lim_{x \to 0} \frac{-5 \cos x + 56 \cos 2x - 81 \cos 3x}{2(\sec^4 x + 2 \sec^2 x \cdot \tan^2 x)}$$

$$= \frac{-5 + 56 - 81}{2} = \frac{-30}{2} = -15$$

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Ex. 3 : Find the values of a, b and c so that

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$$

Sol : The denominator $x\sin x$ becomes zero when $x = 0$. Hence the given expression will tend to a finite limit only if the numerator also becomes zero when $x = 0$.

$$\text{i.e., } a - b + c = 0. \qquad \ldots(1)$$

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \qquad \text{(form 0/0)}$$

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{x\cos x + \sin x}$$

The denominator becomes zero when $x = 0$. Since the limit has to be finite, the numerator must also become zero when $x = 0$.

$$\therefore a - c = 0 \qquad \ldots(2)$$

The given limit is

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{x\cos x + \sin x} \qquad \text{(form 0/0)}$$

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{-x\sin x + \cos x + \cos x}$$

$= \dfrac{a+b+c}{2}$. This should be equal to 2.

$$\Rightarrow a + b + c = 4 \qquad \ldots(3)$$

Solving (1), (2), (3), we get $a = 1$, $b = 2$, $c = 1$.

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Ex. 4 : Prove that $\displaystyle \lim_{x\to\infty} \frac{a^{1/x} - b^{1/x}}{\log \frac{x}{x-1}} = \log\left(\frac{a}{b}\right), a > 0, b > 0$

Sol : $$\lim_{x\to\infty} \frac{a^{1/x} - b^{1/x}}{\log \frac{x}{x-1}} = \lim_{x\to\infty} \frac{a^{1/x} - b^{1/x}}{\log \frac{1}{1-\frac{1}{x}}} \qquad \text{(form 0/0)}$$

$$= \lim_{x\to\infty} \frac{a^{1/x}\log a\left(-\frac{1}{x^2}\right) - b^{1/x}\log b\left(-\frac{1}{x^2}\right)}{\frac{-1}{x(x-1)}}$$

$$= \lim_{x\to\infty} \frac{a^{1/x}\log a - b^{1/x}\log b}{\frac{x}{(x-1)}}$$

$$= \lim_{x\to\infty} \frac{a^{1/x}\log a - b^{1/x}\log b}{\frac{1}{1-\frac{1}{x}}}$$

$$= \frac{\log a - \log b}{1} = \log\left(\frac{a}{b}\right)$$

SAQ 1: If a, b, c, d are positive numbers, $c \neq d$, then show that

$$\lim_{x \to 0} \left(\frac{a^{-x} - b^{-x}}{c^{-x} - d^{-x}} \right) = \frac{\log b - \log a}{\log d - \log e}$$

4.4 The Indeterminate form ∞/∞

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are both infinite, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Since $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$,

$$\lim_{x \to a} \frac{1}{f(x)} = 0 \text{ and } \lim_{x \to a} \frac{1}{g(x)} = 0$$

Now $\lim_{x \to a} \frac{f(x)}{g(x)}$ (form ∞/∞)

$$= \lim_{x \to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} \qquad \text{(form 0/0)}$$

$$= \lim_{x \to a} \frac{-g'(x)}{g^2(x)} \Big/ \frac{-f'(x)}{f^2(x)}$$

$$= \lim_{x \to a} \left[\frac{g'(x)}{f'(x)} \right] \lim_{x \to a} \left[\frac{f(x)}{g(x)} \right]^2 \qquad \text{... (1)}$$

Let $\lim_{x \to a} \frac{f(x)}{g(x)} = l$ ... (2)

Now three cases arise:

(i) $l \neq 0$ and $l \neq \infty$.

From (1), we get $l = l^2 \lim_{x \to a} \frac{g'(x)}{f'(x)}$

$$\therefore \; l = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Hence by (2), $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

(ii) $l = 0.$

$$\therefore \; l + 1 = \lim_{x \to a} \frac{f(x)}{g(x)} + 1 = \lim_{x \to a} \frac{f(x) + g(x)}{g(x)}$$

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$$= \lim_{x \to a} \frac{f'(x) + g'(x)}{g'(x)} \quad \text{[by case (i)]}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)} + 1$$

$$\therefore \; l = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

(iii) $l = \infty$

$$\therefore \lim_{x \to a} \frac{g(x)}{f(x)} = 0$$

$$\Rightarrow \lim_{x \to a} \frac{g'(x)}{f'(x)} = 0, \text{ [by case (ii)]}$$

$$\Rightarrow \lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Thus, in all the three cases, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Ex 5 : Evalute $\lim_{x \to 0} \frac{\log \sin ax}{\log \sin bx}$ $(a, b > 0)$

Sol : $\lim_{x \to 0} \frac{\log \sin ax}{\log \sin bx}$ (form ∞/∞)

$$= \lim_{x \to 0} \frac{\frac{1}{\sin ax} \cdot \cos ax \cdot a}{\frac{1}{\sin bx} \cos bx \cdot b} = \lim_{x \to 0} \frac{a}{b} \cdot \frac{\cot ax}{\cot bx} \quad \text{(form } \infty/\infty)$$

$$= \lim_{x \to 0} \frac{a}{b} \cdot \frac{\tan bx}{\tan ax} \quad \text{(form 0/0)}$$

$$= \lim_{x \to 0} \frac{a}{b} \cdot \frac{b \sec^2 bx}{a \sec^2 ax} = \frac{a}{b} \cdot \frac{b}{a} \cdot 1 = 1$$

Ex 6 : Show that $\lim_{x \to 0} \log_{\tan x} \tan 2x = 1$.

Sol : $\lim_{x \to 0} \log_{\tan x} \tan 2x = \lim_{x \to 0} \frac{\log \tan 2x}{\log \tan x}$ (form ∞/∞)

$$= \lim_{x \to 0} \frac{\frac{1}{\tan 2x} 2 \sec^2 2x.}{\frac{1}{\tan x} \cdot \sec^2 x}$$

$$= \lim_{x \to 0} \frac{2}{\sin 2x \cdot \cos 2x} \Big/ \frac{1}{\sin x \cdot \cos x}$$

$$= \lim_{x \to 0} \frac{2}{2 \sin x \cdot \cos x \cdot \cos 2x} \cdot \sin x \cos x$$

$$= \lim_{x \to 0} \frac{1}{\cos 2x} = 1$$

SAQ 2 : Show that $\lim_{x \to a} x^m e^{-x} = 0$.

4.5 The indeterminate form 0 x ∞

Let $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$

Then $\lim_{x \to a} f(x).\, g(x) = \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}$ (form 0/0)

$$= \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}} \quad \text{(form } \infty/\infty\text{)}$$

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Hence the form 0 x ∞ can be transformed into one of the forms 0/0 or ∞/∞ and L–Hospital rule can be applied.

Ex : 7 : Prove that $\lim_{x \to 1} (1-x) \cdot \tan \frac{\pi x}{2} = \frac{2}{\pi}$.

Sol : $\lim_{x \to 1} (1-x) \cdot \tan \frac{\pi x}{2}$ $(0 \times \infty)$

$$= \lim_{x \to 1} \frac{1-x}{\cot \frac{\pi x}{2}} \qquad \left(\text{form } \tfrac{0}{0}\right)$$

Applying L– Hospital rule

$$= \lim_{x \to 1} \frac{-1}{\frac{-\pi}{2} \operatorname{cosec}^2 \frac{\pi x}{2}}$$

$$= \lim_{x \to 1} \frac{2}{\pi} \sin^2 \frac{\pi x}{2} = \frac{2}{\pi}.$$

Ex : 8 : Show that $\lim_{x\to 0} x^m (\log x)^n = 0.$

Sol : $\lim_{x\to 0} x^m (\log x)^n$ (form $0 \times \infty$)

$$= \lim_{x\to 0} \frac{(\log x)^n}{x^{-m}} \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{n(\log x)^{n-1}.1/x}{-mx^{-m-1}}$$

$$= \lim_{x\to 0} \frac{-n}{m} \frac{(\log x)^{n-1}}{x^{-m}} \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{-n}{m} \frac{(n-1)\ (\log x)^{n-2}.1/x}{-mx^{-m-1}}$$

$$= \lim_{x\to 0} (-1)^2 \frac{n(n-1)}{m^2} \cdot \frac{(\log x)^{n-2}}{x^{-m}}. \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

....

$$= \lim_{x\to 0} (-1)^n \frac{n\,(n-1)\,(n-2) \ldots (n-\overline{n-1})}{m^n} \cdot \frac{(\log x)^{n-n}}{x^{-m}}$$

$$= \lim_{x\to 0} \frac{(-1)^n n!}{m^n} \cdot x^m = 0.$$

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4.6 The indeterminate form $\infty - \infty$

Let $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$

Then $\lim_{x\to a} [f(x) - g(x)]$ (form $\infty - \infty$)

$$= \lim_{x\to a} \left[\frac{1}{1/f(x)} - \frac{1}{1/g(x)}\right]$$

$$= \lim_{x\to 0} \left[\frac{1/g(x)}{1/f(x)} - \frac{1/f(x)}{1/g(x)}\right] \quad \text{(form 0/0)}$$

Now L– Hospital rule can be applied.

Ex : 9 : Show that $\lim_{x\to \pi/2} (\sec x - \tan x) = 0.$

Sol : $\lim_{x\to \pi/2} (\sec x - \tan x)$ (form $\infty - \infty$)

$$= \lim_{x\to \pi/2} \frac{1 - \sin x}{\cos x} \quad \text{(form 0/0)}$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \lim_{x \to \pi/2} \cot x = 0.$$

Ex : 10 : Show that $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right) = \frac{2}{3}$.

Sol : $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right)$ (form $\infty - \infty$)

$$= \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right) = \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x}$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4} \quad \left[\because \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = 1\right]$$

By Maclaurin's expansion

$$\sin x = x - \frac{x^3}{3!} + \ldots \text{ and } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2 - x^2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^2}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2 - \frac{2x^4}{3!} + \ldots - x^2 \,(1 - x^2 + \ldots)}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2 - x^2 - \frac{2x^4}{6} + x^4 - \ldots}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{4x^4}{6} + \text{terms containing higher powers of } x}{x^4}$$

$$= \lim_{x \to 0} \left(\frac{2}{3} + \text{terms containing } x \text{ and higher powers of } x\right)$$

$$= \frac{2}{3}$$

Ex 11 : Prove that $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right) = \frac{1}{2}$

Sol : $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$ (form $\infty - \infty$)

$$= \lim_{x \to 1} \frac{x \log x - x + 1}{(x-1) \log x} \quad \text{(form 0/0)}$$

$$= \lim_{x\to 1} \frac{x\cdot\left(\frac{1}{x}\right) + \log x - 1}{(x-1)\left(\frac{1}{x}\right) + \log x} = \lim_{x\to 1} \frac{\log x}{1 - \left(\frac{1}{x}\right) + \log x} \quad \text{(form 0/0)}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{\left(\frac{1}{x^2}\right) + \left(\frac{1}{x}\right)} = \frac{1}{1+1} = \frac{1}{2}$$

4.7 The indeterminate forms 0^0, 1^∞ and ∞^0

Let $y = \lim_{x\to a} [f(x)]^{g(x)}$ assumes one of the above three indeterminate forms.

$$\log y = \lim_{x\to a} \log [f(x)]^{g(x)}$$

$$= \lim_{x\to a} g(x) \cdot \log f(x) \qquad \text{(form } 0 \times \infty)$$

This may be transformed to either 0/0 or ∞/∞ and the limit may be evaluated by L–Hospital rule.

Ex 12: Show that $\lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan \pi x/2a} = e^{2/\pi}$

Sol : Let $\lim_{x\to a} y = \lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan \pi x/2a}$ (form 1^∞)

$$\lim_{x\to a} \log y = \lim_{x\to a} \tan \frac{\pi x}{2a} \log\left(2 - \frac{x}{a}\right) \qquad \text{(form } \infty \times 0)$$

$$\lim_{x\to a} \log y = \lim_{x\to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot \frac{\pi x}{2a}} \qquad \text{(form 0/0)}$$

$$= \lim_{x\to a} \left[\frac{1}{2 - \frac{x}{a}} \left(-\frac{1}{a}\right) \Big/ - \left(\operatorname{cosec}^2 \frac{\pi x}{2a}\right) \frac{\pi}{2a}\right]$$

$$= \frac{2}{\pi} \lim_{x\to a} \frac{\sin^2 \frac{\pi x}{2a}}{2 - \frac{x}{a}} = \frac{2}{\pi} \cdot \frac{\sin^2 \frac{\pi}{2}}{2-1} = \frac{2}{\pi}$$

$$\lim_{x\to a} y = e^{2/\pi}$$

$$\therefore \lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan \pi x/2a} = e^{2/\pi}$$

Ex 13 : Show that $\lim\limits_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right) \frac{1}{x} = 1.$

Sol :

$$\text{Let } \lim_{x\to\infty} y = \lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x} \quad (\text{form } 0^0)$$

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{1}{x} \log\left(\frac{\pi}{2} - \tan^{-1} x\right)$$

$$= \lim_{x\to\infty} \frac{\log\left(\frac{\pi}{2} - \tan^{-1} x\right)}{x} \quad (\text{form } \infty/\infty)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\frac{\pi}{2} - \tan^{-1} x} \cdot \frac{-1}{1+x^2}}{1}$$

$$= \lim_{x\to\infty} \frac{\frac{-1}{1+x^2}}{\frac{\pi}{2} - \tan^{-1} x} \quad (\text{form } 0/0)$$

$$= \lim_{x\to\infty} \frac{1}{(1+x^2)^2} \cdot \frac{2x}{\frac{-1}{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{-2x}{1+x^2} \quad (\text{form } \infty/\infty)$$

$$= \lim_{x\to\infty} \frac{-2}{2x} = 0$$

$$\lim_{x\to\infty} y = e^0 = 1$$

$$\therefore \lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x} = 1.$$

Ex 14 : Show that $\lim\limits_{x\to\pi/2} (\sec x)^{\cot x} = 1$

Sol :

$$\text{Let } \lim_{x\to\pi/2} y = \lim_{x\to\pi/2} (\sec x)^{\cot x} \quad (\text{form } \infty^0)$$

$$\lim_{x\to\pi/2} \log y = \lim_{x\to\pi/2} \log (\sec x)^{\cot x}$$

$$= \lim_{x\to\pi/2} \cot x \,.\, \log(\sec x) \quad (\text{form } 0 \times \infty)$$

$$= \lim_{x\to\pi/2} \frac{\log \sec x}{\tan x} \quad (\text{form } \infty/\infty)$$

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$$= \lim_{x\to\pi/2} \frac{\frac{1}{\sec x} \cdot \sec x \cdot \tan x}{\sec^2 x}$$

$$= \lim_{x\to\pi/2} \frac{\tan x}{\sec^2 x} = \lim_{x\to\pi/2} (\tan x \cdot \cos^2 x)$$

$$= \lim_{x\to\pi/2} (\sin x \cdot \cos x) = 0$$

$$\therefore \lim_{x\to\pi/2} y = e^0 = 1 \text{ or } \lim_{x\to\pi/2} (\sec x)^{\cot x} = 1.$$

4.8 Summary

When $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then

$\lim_{x\to a} \frac{f(x)}{g(x)}$ is evaluated using the L – Hospitals rule, which states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \text{ ; provided } g'(a) \neq 0.$$

If $f'(a) = 0 = g'(a)$; we go for the 2nd derivative and $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}$ provided $g''(a) \neq 0$

We proceed this way, differentiating the numerator and the denominator till we are releived of the indeterminate form.

When $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$ also we can use the L–Hospitals rule.

The other indeterminate forms $0 \times \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^∞ and ∞^0 will be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and can be solved.

4.9 Sample Examination Questions

I. Answer the following questions in detail

(i) (a) State and prove L–Hospital rule.

(b) Show that $\lim_{x\to 0} \frac{\cos hx - \cos x}{x \sin x} = 1.$

(ii) (a) How do you evaluate the indeterminate form ∞/∞? Discuss various cases.

(b) Show that $\lim_{x\to 1} \frac{\log (1 - x)}{\cot \pi x} = 0.$

(iii) (a) Show that $\lim_{x\to\pi/2} (1 - \sin x) . \tan x = 0.$

(b) Show that $\lim_{x\to 1} \left(\frac{2}{x^2 - 1} - \frac{1}{x - 1}\right) = \frac{-1}{2}.$

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II. *Prove the following*

(i) $\lim\limits_{x \to 0} \dfrac{x \cos x - \log(1+x)}{x^2} = \dfrac{1}{2}$.

(ii) $\lim\limits_{x \to 1} \dfrac{x^x - 1}{1 - x + \log x} = -2$

(iii) $\lim\limits_{x \to \pi/2} \dfrac{\log(x - \pi/2)}{\tan x} = 0.$

(iv) $\lim\limits_{x \to a} \dfrac{\log(x-a)}{\log(e^x - e^a)} = 1$

(v) $\lim\limits_{x \to 0} \sin x . \log x^2 = 0.$

(vi) $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x) = 0.$

(vii) $\lim\limits_{x \to 1} \left(\dfrac{2}{x^2 - 1} - \dfrac{1}{x-1}\right) = \dfrac{-1}{2}$.

(viii) $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2} = 0.$

(ix) $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{1/x} = 1.$

III. *Answer the following questions in about 5 lines*

(i) What is an Indeterminate form?

(ii) What is the use of L-Hospital rule in evaluating the indeterminate forms?

(iii) How do you evaluate the indeterminate form $0 \times \infty$?

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4.10 Answers to SAQ's

SAQ 1 : The L.H.S. of the given function assumes the indeterminate form $\frac{0}{0}$.

Both the numerator and the denominator of the LHS are differentiated and

$$\frac{d}{dx}\left(a^{-x} - b^{-x}\right) = -a^{-x} \log a + b^{-x}. \log b.$$

Its value at $x = 0$ is $\log b - \log a$.

$$\frac{d}{dx}\left(c^{-x} - d^{-x}\right) = -c^{-x} \log c + d^{-x}. \log d.$$

Its value at $x = 0$ is $\log d - \log c$.

Since $c \neq d$, $\log d - \log c \neq 0$.

$$\lim_{x \to 0} \left(\frac{a^{-x} - b^{-x}}{c^{-x} - d^{-x}}\right) = \frac{\log b - \log a}{\log d - \log c}$$

SAQ 2: $$\lim_{x\to\infty} x^m - e^{-x} = \lim_{x\to\infty} \frac{x^m}{e^x} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{m \cdot x^{m-1}}{e^x} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

Continue the differentiation till all the powers of x exhaust.

Taking the m^{th} derivative of numerator and denominator

$$= \lim_{x\to\infty} \frac{m(m-1)\,(m-2)\,.......1}{e^x} \lim_{x\to\infty} \frac{m!}{e^x} = 0.$$

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BLOCK - 2 : DIFFERENTIAL CALCULUS - II

Unit - 5 : Partial Differentiation

5.0 Contents

5.1 Aims and objectives

After going through this Unit, you will be able to :

(i) differentiate the given function of two or more variables,

(ii) apply the Euler's theorem for the differentiation of homogeneous functions,

(iii) differentiate partially, the implicit and composite functions.

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5.2 Indtroduction

In Unit-2, you have learnt the differentiation of a function of a single real variable; however, there are many practical situations in which functions depend on several variables. For example, the pressure of a gas depends on its temperature and its volume; the demand for a commodity may depend not only on its price but also on the prices of related commodities, on income level, and on time etc.

In this Unit you will study functions of more than one variable. You will find that the concepts of limit, continuity are applicable to such functions and you can differentiate them partially.

5.3 Functions of two variables

If a real value of z is defined for every pair of values of x and y where the point (x, y) lies within a certain region in xy - plane, z is called a function of x and y . z is called the dependent variable and x, y are called the independent variables. The region in which the point (x, y) varies is called the domain of the point (x, y).

For example, the relation $z = \sqrt{1 - x^2 - y^2}$ determines a value of z corresponding to every pair of values x, y such that $x^2 + y^2 \leq 1$. That is the points (x, y) lie on or within the circle with centre origin and radius 1. Thus the domain of the function is the set

$\{(x, y)/x^2 + y^2 \leq 1\}$ and $f: (x, y) \rightarrow z$.

The set of all points (x, y) such that $|x - a| < \delta$, $|y - b| < \delta$ where $\delta > 0$, is called a rectangular δ neighbourhood of (a, b).

The set $0 < |x - a| < \delta$, $0 < |y - b| < \delta$ which excludes (a, b) is called a rectangular deleted δ neighbourhood of (a, b).

5.4 Limit and continuity of a function of two variables

Limit : Let $f(x, y)$ be defined in deleted δ_1 neighbourhood (a, b). The function $f(x, y)$ is said to tend to the limit l as x tends to a and y tends of b if for any positive number ϵ we can find some positive number δ (depending on ϵ) such that $|f(x, y) - l| < \epsilon$ whenever $0 < |x - a| < \delta$ and $0 < |y - b| < \delta$, or $\sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Symbolically, $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = l$ or $\lim\limits_{(x, y) \to (a, b)} f(x, y) = l$

Continuity : Let f be defined in a δ neighbourhood of (a, b). $f(x, y)$ is said to be continuous at (a, b) if for any positive number ϵ we can find some positive number δ (depending on ϵ) such that $|f(x, y) - f(a, b)| < \epsilon$ whenever $|x - a| <$ and $|y - b| < \delta$.

It follows that three conditions must be satisfied in order that $f(x, y)$ is continuous at (a, b). They are :

(i) $f(x, y)$ is defined at (a, b), i.e. $f(a, b)$ must exist.

(ii) $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = l$

(iii) $f(a, b) = l$

5.5 Partial derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant, is called the partial derivative of the function with respect to that variable.

Def. 1: Let $z = f(x, y)$ be a function of two independent variables x and y. Let δx be an increment in x and let y remain constant. Then z changes from $f(x, y)$ to $f(x + \delta x, y)$

If $\lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x, y) - f(x, y)}{\delta x}$ exists, then it is called the partial derivative of $f(x, y)$

w.r.t. x at the point (x, y) and is denoted by $\left(\dfrac{\partial z}{\partial x}\right)_{(x, y)}$ or f_x

$$\therefore \frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, if $\lim\limits_{\delta y \to 0} \dfrac{f(x, y + \delta y) - f(x, y)}{\delta y}$ exists, then it is called the partial derivative of

$f(x, y)$ w.r.t. y at the point (x, y) and is denoted by $\left(\dfrac{\partial z}{\partial y}\right)_{(x, y)}$ or f_y

$$\therefore \frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

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SAQ : 1 Use definition to find out $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ given that $z = x^2 y$.

5.6 Partial derivatives of 2nd and higher order

If $f(x, y)$ has partial derivatives at each point (x, y) in a region, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are themselves functions of x and y and these may also have partial derivatives. Thus the second order partial derivative of z w.r.t. x is defined as

$$\lim_{\delta x \to 0} \frac{f_x(x + \delta x, y) - f_x(x, y)}{\delta x} \text{ and is denoted by}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \text{ or } \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx} \text{ provided the limit exists.}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \lim_{\delta x \to 0} \frac{f_x(x + \delta x, y) - f_x(x, y)}{\delta x}.$$

$$\text{Similarly, } \frac{\partial^2 z}{\partial y^2} = f_{yy} = \lim_{\delta y \to 0} \frac{f_y(x, y + \delta y) - f_y(x, y)}{\delta y}$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = f_{yx} = \lim_{\delta y \to 0} \frac{f_x(x, y + \delta y) - f_x(x, y)}{\delta y}$$

$$\text{and } \frac{\partial^2 z}{\partial x\, \partial y} = f_{xy} = \lim_{\delta x \to 0} \frac{f_y(x + \delta x, y) - f_y(x, y)}{\delta x} \text{ provided all these limits exist.}$$

If f_{yx} and f_{xy} are continuous in a region, then $f_{xy} = f_{yx}$ and the order of differentiation is immaterial. Otherwise f_{xy}, f_{yx} may not be equal.

Note : 1. $\frac{\partial^2 z}{\partial x\, \partial y}$ is obtained by first differentiating z partially w.r.t. y and then differentiating the result partially w.r.t. x.

2. $\frac{\partial^2 z}{\partial y\, \partial x}$ is obtained by first differentiating z partially w.r.t. x and then differentiating the result partially w.r.t. y.

5.6.1 Techniques of calculating partial derivatives

Since partial differentiation is the same as the ordinary differentiation with other variables considered as constant, the following rules hold good for partial differentiation.

1. Differential coefficient of a sum.

   If u, v are functions of x and y and $z = u + v$

   then $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$.

2. Differential coefficient of a product.

   If $u = f(x, y)$, $v = g(x, y)$ and $z = uv$

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$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial (xv)}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}.$$

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial (uv)}{\partial y} = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}.$$

3. Differential coefficient of a quotient.

If $u = f(x, y)$, $v = g(x, y)$ and $z = \frac{u}{v}$

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}, \quad \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

4. Function of a function.

If z is a function of t and t is a function of x and y

$$\frac{\partial z}{\partial x} = \frac{dz}{dt} \cdot \frac{\partial}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{dz}{dt} \cdot \frac{\partial t}{\partial y}$$

Note : Since z is a function of single variable t, $\frac{\partial z}{\partial t} = \frac{dz}{dt}$.

Eamples :

Ex.1 : If $f(x, y) = \frac{x + y}{x^2 + y^2}$; find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Sol : Using the Quotient rule, assuming y to be a constant and differentiating w.r.t. x,

we have $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x + y}{x^2 + y^2}\right]$

$$= \frac{(x^2 + y^2)\frac{\partial}{\partial x}(x + y) - (x + y)\frac{\partial}{\partial x}(x^2 + y^2)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{(x^2 + y^2)(1 + 0) - (x + y)(2x + 0)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{y^2 - 2xy - x^2}{\left(x^2 + y^2\right)^2}$$

Similarly, $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x + y}{x^2 + y^2}\right)$

$$= \frac{(x^2 + y^2)\frac{\partial}{\partial y}(x + y) - (x + y)\frac{\partial}{\partial y}(x^2 + y^2)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{x^2 - 2xy - y^2}{\left(x^2 + y^2\right)^2}$$

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Ex. 2 : Find $f_z(x, y, z)$ given $f(x, y, z) = e^{xy^2} z^3$

$$\frac{\partial}{\partial z} f(x, y, z) = \frac{\partial}{\partial z}\left(e^{xy^2} z^3\right) = e^{xy^2} \frac{\partial}{\partial z} z^3 = e^{xy^2}\, 3z^2$$

Ex. 4 : If $f(x, y) = 2e^x \cos y$, find f_{xx}, f_{yy}, f_{xy} and f_{yx}.

Sol : $f_x(x, y) = \frac{\partial}{\partial x}(2e^x \cos y) = 2e^x \cos y,$

$$f_y = \frac{\partial}{\partial y}(2e^x \cos y) = -2e^x \sin y,$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x(x, y)) = \frac{\partial}{\partial x}(2e^x \cos y) = 2e^x \cos y.$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x(x, y)) = \frac{\partial}{\partial y}(2e^x \cos y) = -2e^x \sin y.$$

$$f_{yx} = \frac{\partial}{\partial x}(f_x(x, y)) = \frac{\partial}{\partial x}(-2e^x \sin y)$$

$$= -2e^x \sin y$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y(x, y)) = \frac{\partial}{\partial y}(-2e^x \sin y) = -2e^x \cos y.$$

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5.7 Homogeneous functions

A function $f(x, y)$ is said to be homogeneous of degree n if on replacing x by kx and y by ky the function is multiplied by k^n, i.e., $f(kx, ky) = k^n f(x, y)$.

Examples

1.
$$\begin{aligned} f(x, y) &= a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_n y^n, \\ f(kx, ky) &= a_0 k^n x^n + a_1 k^{n-1} x^{n-1} ky + a_2 k^{n-2} x^{n-2} k^2 y^2 + \ldots + a_n k^n y^n \\ &= k^n (a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_n y^n) \\ &= k^n f(x, y) \end{aligned}$$

$\therefore$ $f(x, y)$ is homogeneous of degree n.

2. $\log x - \log y$ is homogeneous of zero degree since

$$\log kx - \log ky = \log \frac{kx}{ky} = \log \frac{x}{y} = \log x - \log y = k^0 (\log x - \log y)$$

3. $\sqrt{x^2 - y^2} \sin^{-1} \frac{y}{x}$ is homogeneous of degree 1 since

$$\sqrt{(kx)^2 - (ky)^2} \sin^{-1} \frac{ky}{kx} = k^1 \sqrt{x^2 - y^2} \sin^{-1} \frac{y}{x}.$$

4. $\sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ is homogeneous of degree 0.

Another definition : If a function can be expressed in the form $x^n f\left(\frac{y}{x}\right)$, it is called a homogeneous function of degree n in x and y.

Examples

1. $ax + by = x\left(a + b\frac{y}{x}\right) = xf\left(\frac{y}{x}\right)$.

   $\therefore$ $ax + by$ is a homogeneous function of degree one.

2. $f(x, y) = ax^2 + 2hxy + by^2 = x^2\left(a + 2h\ \frac{y}{x} + b\ \frac{y^2}{x^2}\right)$

   $$= x^2 f\left(\frac{y}{x}\right)$$

   $\therefore$ $f(x, y)$ is a homogeneous function of degree two.

3. $f(x, y) = \dfrac{\sqrt{x} + \sqrt{y}}{x + y} = \dfrac{x^{1/2}\left(1 + \sqrt{\frac{y}{x}}\right)}{x\left(1 + \frac{y}{x}\right)} = x^{-1/2} f\left(\frac{y}{x}\right)$

   $\therefore$ $f(x, y)$ is a homogeneous function of degree $-\frac{1}{2}$.

5.8 Euler's theorem on homogeneous functions

If $z = f(x, y)$ is a homogeneous function of x, y of degree n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \ \forall (x, y) \ \varepsilon \text{ the domain of the function.}$$

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Proof : Since $z = f(x, y)$ is a homogeneous function of degree n, $z = x^n f\left(\frac{y}{x}\right)$.

$$\frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot \left(\frac{-y}{x^2}\right)$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$= nx^n\ f\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Corollary 1 : If u is a homogeneous function of degree n in x, y, z,

then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = nu.$

$$\text{Let } u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$$

$$= x^n f(v, w) \text{ where } v = \frac{y}{x}, w = \frac{z}{x}$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f(v, w) + x^n \left(\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}\right)$$

$$= nx^{n-1} f(v, w) + x^n \left(-\frac{y}{x^2}\frac{\partial f}{\partial v} - \frac{z}{x^2}\frac{\partial f}{\partial w}\right)$$

$$\frac{\partial u}{\partial y} = x^n \left(\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}\right) = x^n \left(\frac{1}{x} \frac{\partial f}{\partial v} + 0\right) = x^{n-1} \frac{\partial f}{\partial v}$$

$$\frac{\partial u}{\partial z} = x^n \left(\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z}\right) = x^n \left(0 + \frac{1}{x} \frac{\partial f}{\partial w}\right) = x^{n-1} \frac{\partial f}{\partial w}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nx^n f(v, w) - x^{n-1} y \frac{\partial f}{\partial v}$$

$$- zx^{n-1} \frac{\partial f}{\partial w} + yx^{n-1} \frac{\partial f}{\partial v} + zx^{n-1} \frac{\partial f}{\partial w}$$

$$= nx^n f(v, w)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu.$$

Note : Euler's theorem is true for homogeneous functions of any number of variables.

Corollary 2 : If z is a homogeneous function of degree n, then

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)x.$$

Since z is a homogeneous function of degree n, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are homogeneous functions of degree $(n-1)$.

Applying Euler's theorem to the functions $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + y \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = (n-1) \frac{\partial z}{\partial x}$$

$$x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) + y \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = (n-1) \frac{\partial z}{\partial y}$$

$$x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y \partial z} = (n-1) \frac{\partial z}{\partial x} \quad (1)$$

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$$x\frac{\partial^2 z}{\partial x\,\partial y}+y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial x} \qquad (2)$$

Multiplying (1) by x and (2) by y and adding

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

$$= n(n-1)\,z$$

$$\left(\because x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = nz \text{ and } \frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial^2 z}{\partial x\,\partial y}\right)$$

Examples :

Ex. 1 : If $z(x+y) = x^2+y^2$, show that

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

Sol :

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)\,2x-(x^2+y^2).\,1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)\,2y-(x^2+y^2).\,1}{(x+y)^2} = \frac{y^2+2xy-y^2}{(x+x)^2}$$

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$$L.H.S. = \left[\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2 = 4\left(\frac{x-y}{x+y}\right)^2$$

$$R.H.S. = 4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+y^2+2xy-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= 4\frac{x^2-2xy+y^2}{(x+y)^2} = 4\left(\frac{x-y}{x+y}\right)^2$$

Ex. 2 : If $u = f(r)$ where $r^2 = x^2+y^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)+\frac{1}{r}f'(r)$.

Sol : $u = f(r)$ ... (1) $\qquad r^2 = x^2+y^2$ ... (2)

From (1), $\frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x}$

From (2), $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{r} f'(r)$$

Differentiating w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) + x\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) f'(r) + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \qquad (3)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r) \qquad (4)$$

Adding (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r} f'(r) - \frac{x^2+y^2}{r^3} f'(r) + \frac{x^2+y^2}{r^2} f''(r)$$

$$= \frac{2}{r} f'(r) - \frac{1}{r} f'(r) + f''(r)$$

$$= \frac{1}{r} f'(r) + f''(r).$$

Ex. 3 : If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$

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Sol :

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(\frac{-x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2} = x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 1 - \frac{2y}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{x^2-y^2}{x^2+y^2}$$

5.9 Composite functions and total derivative

Def : If u is given as a function of two variables x and y and these variables are themselves given to be the functions of the variable t, then u is called a composite function of the variable t.

Thus, the relations $u = f(x, y)$, $x = g_1(t)$, $v = g_2(t)$ define u as a composite function of t.

Similarly, the relations $z = f(x, y)$, $x = g_1(u, v)$, $y = g_2(u, v)$ define z as a composite function of u and v.

Def : Let u be a composite function of t defined by the relations $u = f(x, y)$, $x = g_1(t)$, $y = g_2(t)$. If u possesses an ordinary derivative $\frac{du}{dt}$, then $\frac{du}{dt}$ is called the total differential coefficient of u.

Theorem : If u is a composite function of t defined by the relations $u = f(x, y)$, $x = x(t)$, $y = y(t)$ where u possesses continuous first order partial derivatives with respect to x and y; x and y possess continuous derivatives with respect to t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Proof: Let δt be an increment in t. Let $\delta x, \delta y$ and δu be the corresponding increments in x, y and u respectively.

$$\text{Then } u + \delta u = f(x + \delta x, y + \delta y)$$

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]$$

Applying Lagrange's mean value theorem to R. H. S., we have

$$\delta u = \delta x \cdot f_x(x + \theta_1 \delta x, y + \delta y) + \delta y \cdot f_y(x, y + \theta_2 \delta y)$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$.

$$\frac{\delta u}{\delta t} = \frac{\delta x}{\delta t} \cdot f_x(x + \theta_1 \delta x, y + \delta y) + \frac{\delta y}{\delta t} \cdot f_y(x, y + \theta_2 \delta y)$$

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} \cdot \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_x(x + \theta_1 \delta x, y + \delta y) + \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \cdot \lim_{\delta y \to 0} f_y(x, y + \theta \delta y).$$

As $\delta t \to 0, \delta x \to 0, \delta y \to 0$.

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_x(x + \theta_1 \delta x, y + \delta y) = f_x(x, y) = \frac{\partial u}{\partial x}$$

$$\lim_{\delta y \to 0} f_y(x, y + \theta_2 \delta y) = f_y(x, y) = \frac{\partial u}{\partial y},$$

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt}, \quad \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \quad \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dx}.$$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \qquad (1)$$

Corollary 1: If in (1), $t = x$, we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

Since $u = f(x, y)$ the above result can be stated as follows :

The differential coefficient of $f(x, y)$ w.r.t. x is given by

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{dy}{dx}$$

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Corollary 2: If $u = f(x, y, z)$ and x, y, z are functions of t, then

$$\frac{du}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} + \frac{df}{\partial z}\frac{dz}{dx}$$

i.e., The differential coefficient of $f(x, y, z)$ w.r.t. x is

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} + \frac{df}{\partial z} \cdot \frac{dz}{dx}$$

Corollary 3: If $u = f(x, y)$ and x, y be functions of two variables t_1, t_2, then (1) gives

$$\frac{\partial u}{\partial t_1} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t_1} \quad \text{and} \quad \frac{\partial u}{\partial t_2} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

5.10 Differentiation of implicit functions

If an equation such as $F(x, y, u) = 0$ defines one variable u as a function of the other two variables x and y, then u is called an implicit function of x and y and is written as $u = f(x, y)$. Similarly, if the equation $f(x, y) = 0$ defines one variable y as a function of the other variable x, then y is called an implicit function of x. In such situations $\frac{dy}{dx}$ can be found as follows.

Let $f(x, y) = c$ be an implicit relation between x and y where c is a constant and y is a function of x. Since x is a function of x itself, we may consider $f(x, y)$ as a composite function of x.

Differentiating the given equation $f(x, y) = c$ w.r.t. x, we get

$$0 = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Corollary : If f is a function of n variables $x_1, x_2, x_3, \ldots\ldots, x_n$ and $x_2, x_3, \ldots\ldots, x_n$ are all functions of x_1, then the total differential coeffecient of f w.r.t. x_1, is given by

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dx_1} + \frac{\partial f}{\partial x_3} \cdot \frac{dx_3}{dx_1} + \ldots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dx_1}.$$

Examples

Ex.1: If $f(x, y) = (\cos x)^y - (\sin y)^x = 0$, find $\frac{dy}{dx}$.

Sol:

$$f(x, y) = (\cos x)^y - (\sin y)^x = e^{\log(\cos x)^y} - e^{\log(\sin y)^x}$$

$$= e^{y \log(\cos x)} - e^{x \log(\sin y)}$$

$$\frac{\partial f}{\partial x} = e^{y \log(\cos x)} \cdot y \frac{1}{\cos x}(-\sin x) - e^{x \log(\sin y)} \log(\sin y)$$

$$= -y \tan x \,.\, (\cos x)^y - (\sin y)^x \log(\sin y)$$

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$$\frac{\partial f}{\partial y} = e^{y \log(\cos x)} . \log(\cos x) - e^{x \log(\sin y)} \frac{x}{\sin y} \cos y$$

$$= (\cos x)^y \log(\cos x) - (\sin y)^x x \cot y.$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{y \tan x (\cos x)^y + (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - (\sin y)^x x \cot y}$$

$$= \frac{y \tan x + \log(\sin y)}{\log(\cos x) - x \cot y} \quad \left[\text{since } (\cos x)^y = (\sin y)^x\right]$$

Ex.2: If $x^y + y^x = c$, find $\frac{dy}{dx}$.

Sol: Let $f(x, y) = x^y + y^x$.

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Ex. 3: If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol: Let $y - z = X, z - x = Y, x - y = Z$

Then, $u = f(X, Y, Z)$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial X} . \frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y} \frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z} \frac{\partial Z}{\partial x}$$

$$= \frac{\partial f}{\partial X} . 0 + \frac{\partial f}{\partial Y} (-1) + \frac{\partial f}{\partial Z} . (1)$$

$$= -\frac{\partial f}{\partial Y} + \frac{\partial f}{\partial Z} \qquad (1)$$

Similarly, $$\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial Z} + \frac{\partial f}{\partial X} \qquad (2)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial X} + \frac{\partial f}{\partial Y} \qquad (3)$$

Adding (1), (2), (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

SAQ 2: Find the total differential coefficient of $f(x, y) = 3x^3 y^2 - 2xy^3 + xy - 1$, x, y are functions of t.

SAQ 3: Find $\frac{dy}{dx}$ when $x = 1$ $y = 1$, given that $x^3 y^2 + 3xy^2 + 5x^4 = 2y + 7$, where y is a function of x.

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5.11 Summary

Given a function of two or more variables, we adopt partial differentiation, for the differentiation of such functions.

To differentiate $f(x, y)$ with respect to x, we treat y as a constant and,

$$f_x = \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

f_{xy} need not be equal to f_{yx}. These two are equal if they are continuous in a region.

If z is a homogeneous function of x, y of degree n, Eulers theorem states that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$. If u is a function of x, y and each x, y are functions of t and if u possesses a derivative with respect to t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$ is called the total derivative of u.

If $f(x, y) = c$ is an implicit relation between x and y then $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

5.12 Sample Examination Questions

I. Answer the following questions in detail.

(i) (a) Explain what is a partial derivative.

(b) If $u = \sqrt{x^2 + y^2 + z^2}$, prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$

(ii) (a) If u is a composite function of t defined by the relations $u = f(x, y)$, $x = g_1(t)$, $y = g_2(t)$ where u possesses continuous first order partial derivatives with respect to x and y ; x and y possess continuous derivatives with respect to t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

(b) If $u = \sin(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$, show that

$$\frac{du}{dx} = \frac{2(b^2-a^2)}{b^2}x\cos(x^2 + y^2).$$

II. Briefly answer the following.

(i) If $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(ii) If $z = \tan^{-1}(y/x)$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

(iii) If $z = e^{ax}\sin by$, show that $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$

(iv) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

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(v) If $e^x + e^y = 2xy$, prove that $\frac{dy}{dx} = \frac{e^x - 2y}{2x - e^y}$

(vi) If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, then prove that the value of $\frac{dz}{dx}$, at $x = a, y = b$ is zero.

(vii) If $u = \sin^{-1}(x-y)$, $x = 3t$ and $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

III. Answer the following questions in about 5 lines.

(i) Define a homogeneous function of degree n and state the Euler's theorem on homogeneous functions.

(ii) Define the total differential coefficient.

(iii) What is an implicit function and how do you differentiate it ?

5.13 Answers to SAQ's.

SAQ 1:

$$z = x^2y$$

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 y - x^2 y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{x^2 + 2x\ \delta x + (\delta x)^2\} y - x^2 y}{\delta y}$$

$$= \lim_{\delta x \to 0} 2xy + y\,\delta x = 2xy$$

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{x^2(y+\delta y) - x^2 y}{\delta y} = \lim_{\delta y \to 0} \frac{x^2\,\delta y}{\delta y} = x^2$$

SAQ 2:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = 9x^2y^2 - 2y^3 + y;\quad \frac{\partial f}{\partial y} = 6x^3y - 6xy^2 + x$$

$$\frac{df}{dt} = (9x^2y^2 - 2y^3 + y)\frac{dx}{dt} + (6x^3y - 6xy^2 + x)\frac{dy}{dt}$$

SAQ 3: Taking all the terms to one side, we get the equation of the form $f(x, y) = 0$, where

$$f(x, y) = x^3y^2 + 3xy^2 + 5x^4 - 2y - 7 = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2y^2 + 3y^2 + 20x^3}{2x^3y + 6xy - 2};\ \text{at } x = 1, y = 1$$

$$\frac{dy}{dx} = -\frac{3 + 3 + 20}{2 + 6 - 2} = \frac{-13}{3}$$

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Unit – 6 : Curvature, Concavity, Convexity and Points of Inflexion

6.0 Contents

6.1 Aims and Objectives

After going through this unit, you are expected to (i) find the radius of curvature of a curve at a given point,

(ii) decide whether the given curve at a given point has the convexity, concavity or the point of the inflexion.

6.2 Introduction

One of the most distinctive features of a graph is the direction in which it bends. The graph on the left in fig. 1 bends upward and the point that traces it moves from left to right and the graph on the right bends downwards.

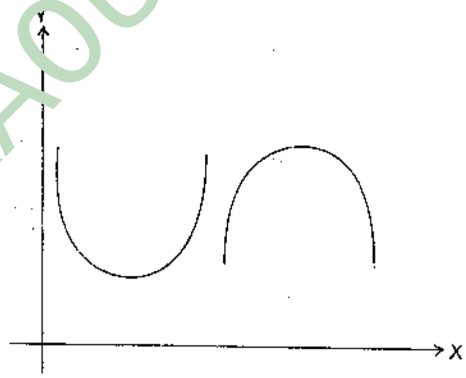

Fig. 1

If you are given a function, How do you find out whether the graph of a given function bends downwards or upwards? The derivatives of the given function will give us the required information. In this unit you will be knowing about the relation between the curvature and the derivatives of a given function. Also we will derive the formulas for the length of a curve and radius of curvature of a curve. This information is mostly useful in tracing of curves.

6.3 Length of an Arc

The definition of the length of the arc measured from some fixed point on a given curve is based on the following axiom.

Axiom : If P, Q are any two points on a curve such that the arc PQ, throughout the length is concave to the chord PQ, then chord PQ < arc $PQ < PR + QR$ where PR, QR are any two lines enclosing the curve.

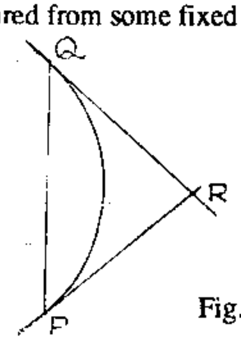

Fig. 2

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Deduction from the axiom : If P, Q are any two points on a curve then

$$\lim_{Q \to P} \frac{\text{arc } PQ}{\text{Chord } PQ} = 1$$

Let PT be the tangent at P to the curve and let M be the projection of Q on PT. Let $\angle QPM = \alpha$. By the axiom we have

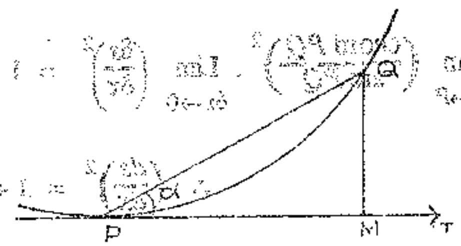

Fig. 3

$$\text{Chord } PQ < \text{arc } PQ < PM + QM$$

$$\Rightarrow 1 < \frac{\text{arc } PQ}{\text{Chord } PQ} < \frac{PM}{\text{Chord } PQ} + \frac{QM}{\text{Chord } PQ}$$

$$\Rightarrow 1 < \frac{\text{arc } PQ}{\text{Chord } PQ} < \cos\alpha + \sin\alpha$$

When $Q \to P$, chord $PQ \to$ tangent PM and $\alpha \to 0$.

$$\text{i.e., } 1 < \lim_{Q \to P} \frac{\text{arc } PQ}{\text{Chord } PQ} < \lim_{\alpha \to 0} (\cos\alpha + \sin\alpha)$$

$$\therefore \quad \lim_{Q \to P} \frac{\text{arc } PQ}{\text{Chord } PQ} = 1.$$

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6.4 Derivative of the length of arc of a curve, Cartesian Coordinates

Let $y = f(x)$ be the equation of a curve and let A be a fixed point on the curve. Let $P(x, y)$ be any point on the curve and $Q(x + \delta x, y + \delta y)$ be a point very near P. If s be the length of the arc AP, then $s + \delta s$ is the length of the arc AQ so that arc $PQ = \delta s$. Let M and N be the projections of P and Q on x-axis. Let L be the proejction of P on QN.

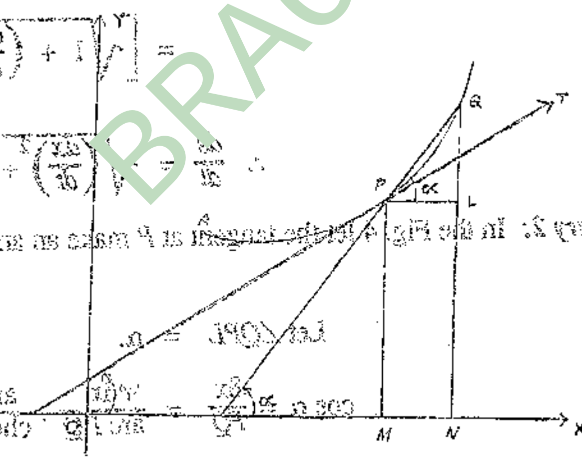

Fig. 4

From right angle ΔPLQ, (fig.4)

$$PQ^2 = PL^2 + QL^2 = (\delta x)^2 + (\delta y)^2$$

$$\left(\frac{PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\text{arc } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

As $Q \to P$ we have $\dfrac{\text{chord } PQ}{\text{arc } PQ} = 1, \delta x \to 0.$

$$\lim_{Q \to P} \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \lim_{\delta x \to 0} \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right)^2$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{ds}{dx} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Positive or negative sign is taken according as s increases or decreases along the curve with x. We will take always that s is measured in the direction in which x increases. Hence we take positive sign for $\dfrac{ds}{dx}$.

Corollary 1 : If x and y are given in terms of a parameter t as $x = f(t)$, $y = g(t)$ then we have

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dt}$$

$$= \sqrt{1 + \left(\frac{dy}{dt}\,\frac{dt}{dx}\right)^2} \cdot \frac{dx}{dt}$$

$$= \left[\sqrt{1 + \left(\frac{dy}{dt}\right)^2 \Big/ \left(\frac{dx}{dt}\right)^2}\,\right] \cdot \frac{dx}{dt}$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Corollary 2: In the Fig. 4 let the tangent at P make an angle ψ with the positive direction of x–axis.

$$\text{Let } \angle QPL = \alpha.$$

$$\cos \alpha = \frac{\delta x}{PQ} = \frac{\delta x}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\text{chord } PQ}$$

As $Q \to P$, $\alpha \to \psi$, chord $PQ \to$ tangent at P

$$\lim_{\alpha \to \psi} \cos \alpha = \lim_{Q \to P} \frac{\delta x}{\delta s} \cdot \lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ}.$$

$$\cos \psi = \frac{dx}{ds}.$$

$$\text{Similarly, } \sin \psi = \frac{dy}{ds}.$$

6.5 Curvature

Let the tangents to the given curve at P and Q make angles ψ, $\psi + \delta\psi$ respectively with the positive direction of the x-axis. The direction of motion along the arc PQ is turned through an angle $\delta\psi$.

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Then $\frac{\delta\psi}{\delta s}$ is defined as the average curvature along the arc PQ and as $Q \to P$ $\lim \frac{\delta\psi}{\delta s}$ i.e. $\frac{d\psi}{ds}$

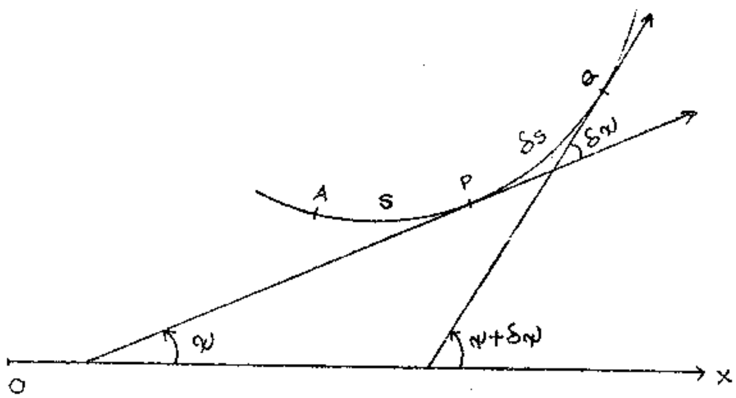

Fig . 5

is called the curvature of the curve at P. If the curvature at P is not equal to zero, the reciprocal of the curvature at P is defined as the radius of curvature of the curve at P and is denoted by ρ.

$$\text{Thus} \quad \rho = \frac{ds}{d\psi}.$$

6.5.1 The curvature of a Circle

Consider any circle with centre C and radius r. Let P and Q be the neighbouring points whose distances from a fixed point A along the arc of the circle be s and $s + \delta s$ respectively. Let the tangents at P and Q make angles ψ and $\psi + \delta\psi$ with x-axis so that the angle between the tangents is $\delta\psi$.

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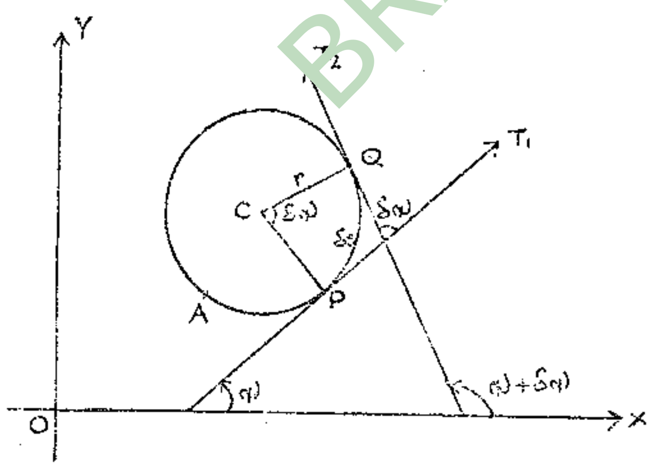

Fig. 6

$$\text{Then } \angle PCQ = \frac{\text{arc } PQ}{CP} \Rightarrow \delta\psi = \frac{\delta s}{r}$$

$$\text{Curvature of the circle at } P = \lim_{Q\to P} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\delta s}{r.\delta s} = \frac{1}{r}$$

$\therefore$ radius of curvature of the circle at $P = r$ radius of the circle.

6.6 Radius of curvature of a curve

Let the equation of the curve be $y = f(x)$. Let $P(x, y)$ be any point on the curve. Let the tangent at P to the curve make an angle ψ with x-axis in the positive direction.

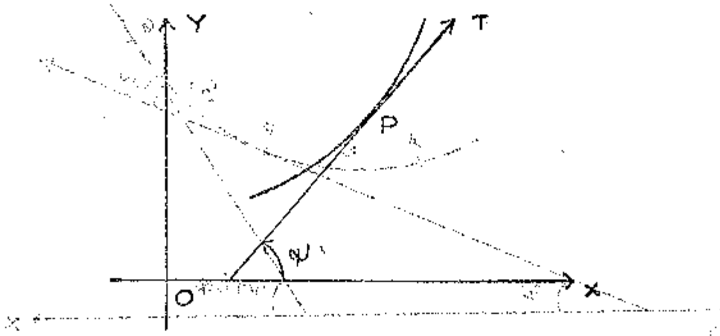

Fig. 7

$$\therefore \tan\psi = \frac{dy}{dx}$$

Differentiating $w.\ r.\ t.\ s$; $\sec^2\psi \,\frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{ds}$$

$$= \frac{d^2y}{dx^2}\cdot\frac{dx}{ds}$$

$$\sec^2\psi\,\frac{1}{\rho} = \frac{d^2y}{dx^2}\cdot\frac{dx}{ds}$$

$$\therefore \rho = \frac{\sec^2\psi\,\frac{ds}{dx}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} \quad \text{where } y_2 \neq 0.$$

Note : From the definition, the radius of curvature depends only on the curve and is independent of the axes. Hence we can interchange the axes of x and y.

$$\therefore \rho = \left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2} \Big/ \frac{d^2x}{dy^2} \quad \text{where } \frac{d^2x}{dy^2} \neq 0.$$

This formula is useful when the tangent is parallel to y–axis.

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6.6.1 Radius of Curvature when x and y are function of s

We know that $\cos\psi = \frac{dx}{ds}$ ...... (1) and $\sin\psi = \frac{dy}{ds}$ ....(2)

Differentiating (1) w.r.t. s,

$$-\sin\psi \frac{d\psi}{ds} = \frac{d^2x}{ds^2} \text{ and so } -\sin\psi \cdot \frac{1}{\rho} = \frac{d^2x}{ds^2} \quad ....(3)$$

Differentiating (2) w.r.t. s,

$$\cos\psi \frac{d\psi}{ds} = \frac{d^2y}{ds^2} \text{ and so } \cos\psi \cdot \frac{1}{\rho} = \frac{d^2y}{ds^2} \quad ....(4)$$

Squaring and adding (3) and (4) we get

$$\frac{1}{\rho^2}(\sin^2\psi + \cos^2\psi) = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

$$\therefore \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

$$\text{Consider } \frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{d}{dx}(\sin\psi)$$

$$= \cos\psi \frac{d\psi}{dx} = \cos\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$= \cos\psi \cdot \frac{1}{\rho} \cdot \sec\psi = \frac{1}{\rho}$$

$$\therefore \frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{ds}\right)$$

6.6.2 Radius of Curvature when implicit equation $f(x, y) = 0$ is given

From partial differentiation we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$ where $f_y \neq 0$

and
$$\frac{d^2y}{dx^2} = -\frac{f_{xx}(f_y)^2 - 2 f_x f_y f_{xy} + (f_x)^2 f_{yy}}{(f_y)^3}$$

$$\text{But } \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$\therefore \rho = \frac{-\left[1 + \frac{(f_x)^2}{(f_y)^2}\right]^{3/2}(f_y)^3}{f_{xx}(f_y)^2 - 2 f_x f_y f_{xy} + (f_x)^2 f_{yy}}$$

$$= \frac{\left[(f_y)^2 + (f_x)^2\right]^{3/2}}{(f_x)^2 f_{yy} - 2 f_x f_y f_{xy} + (f_y)^2 f_{xx}} \text{ in magnitude.}$$

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Examples

Ex.1: For the curve $y = \frac{ax}{a + x}$, if ρ is the radius of curvature at any point (x, y) show that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

Sol : $y = \frac{ax}{a + x}, y_1 = \frac{(a + x) a - ax}{(a + x)^2} = \frac{a^2}{(a + x)^2}$

$$y_2 = \frac{-2a^2}{(a + x)^3}$$

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \left|\left[1 + \frac{a^4}{(a + x)^4}\right]^{3/2} \frac{(a + x)^3}{-2a^2}\right|$$

$$= \frac{\left[(a + x)^4 + a^4\right]^{3/2} (a + x)^3}{2a^2 (a + x)^6}$$

$$\frac{2\rho}{a} = \frac{\left[(a + x)^4 + a^4\right]^{3/2}}{a^3 (a + x)^3}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{\left[(a + x)^4 + a^4\right]}{a^2 (a + x)^2} = \frac{(a + x)^2}{a^2} + \frac{a^2}{(a + x)^2}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 \quad \left(\because \frac{y}{x} = \frac{a}{a + x}\right)$$

Ex.2. Show that the radius of curvature at any point of the curve $x = a(\cos\theta + \log\tan\theta/2)$, $y = a \sin\theta$ is inversely proportional to the length of the normal intercepted between the point of the curve and the axis of x.

Sol :

$$x = a \left(\cos\theta + \log\tan\theta/2\right)$$

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\theta/2}\sec^2\theta/2 \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{1}{2\sin\theta/2\cos\theta/2}\right) = a\left(-\sin\theta + \frac{1}{\sin\theta}\right)$$

$$= a\left(\frac{1 - \sin^2\theta}{\sin\theta}\right) = \frac{a\cos^2\theta}{\sin\theta}$$

$$y = a\sin\theta, \quad \therefore \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a\cos\theta}{a\cos^2\theta} \cdot \sin\theta = \tan\theta$$

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$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta) = \sec^2\theta.\frac{d\theta}{dx} = \frac{1}{\cos^2\theta}\cdot\frac{\sin\theta}{a\cos^2\theta}$$

$$= \frac{\sin\theta}{a\cos^4\theta}$$

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{(1+\tan^2\theta)^{3/2}}{\sin\theta}\cdot a\cos^4\theta.$$

$$= \frac{a\sec^3\theta\,.\,\cos^4\theta}{\sin\theta} = a\cot\theta.$$

L.N. = (length of the normal) = $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$= a\sin\theta\sqrt{1+\tan^2\theta}$$

$$= a\sin\theta\,.\sec\theta = a\tan\theta.$$

$$= a\,.\,a/\rho = a^2/\rho.$$

$$\therefore \rho = \frac{a^2}{\text{L.N}}$$

$\therefore$ ρ varies inversely as L.N.

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Ex.3.: Show that the radius of curvature at a point of the curve $x = a e^{\theta}(\sin\theta - \cos\theta)$, $y = a e^{\theta}(\sin\theta + \cos\theta)$ is twice the distance of the tangent at the point from the origin.

Sol : $$\frac{dx}{d\theta} = a e^{\theta}(\cos\theta + \sin\theta) + a e^{\theta}(\sin\theta - \cos\theta) = 2a e^{\theta}\sin\theta.$$

$$\frac{dy}{d\theta} = a e^{\theta}(\cos\theta - \sin\theta) + a e^{\theta}(\sin\theta + \cos\theta) = 2a e^{\theta}\cos\theta.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{2a\,e^{\theta}\cos\theta}{2a\,e^{\theta}\sin\theta} = \cot\theta.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cot\theta) = -\operatorname{cosec}^2\theta.\frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\theta.\frac{1}{2a\,e^{\theta}\sin\theta} = -\frac{1}{2a\,e^{\theta}\sin^3\theta}$$

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \left|\frac{\left(1+\cot^2\theta\right)^{3/2}.\,2a\,e^{\theta}\sin^3\theta}{-1}\right|$$

$$= \operatorname{cosec}^3\theta.\,2a\,e^{\theta}\sin^3\theta$$

$$\therefore \rho = 2a\,e^{\theta}.$$

Equation of the tangent at $[a\,e^{\theta}(\sin\theta-\cos\theta), a\,e^{\theta}(\sin\theta+\cos\theta)]$

is $y - a\,e^{\theta}(\sin\theta+\cos\theta) = \dfrac{\cos\theta}{\sin\theta}[x - a\,e^{\theta}(\sin\theta-\cos\theta)]$

i.e., $y\sin\theta - a\,e^{\theta}(\sin^2\theta+\sin\theta\cos\theta)$

$$= x\cos\theta - a\,e^{\theta}(\sin\theta\cos\theta-\cos^2\theta)$$

$x\cos\theta - y\sin\theta + a\,e^{\theta}(\sin^2\theta+\sin\theta\cos\theta-\sin\theta\cos\theta+\cos^2\theta) = 0.$

or $x\cos\theta - y\sin\theta + a\,e^{\theta} = 0$

p = perpendicular distance from (0, 0) on the tangent

$$= \left|\frac{0 - 0 + a\,e^{\theta}}{\sqrt{\cos^2\theta+\sin^2\theta}}\right| = a\,e^{\theta}$$

$\therefore \rho = 2(a\,e^{\theta}) = 2.p$ = twice the distance of the tangent at the point from the origin.

Ex.4. Show that for the curve in which $s = a\,e^{x/c}$, $c\,\rho = s\sqrt{s^2-c^2}$.

Sol : $\quad s = ae^{x/c};\ \dfrac{ds}{dx} = a\,e^{x/c}\dfrac{1}{c} = s/c.$

We have, $\left(\dfrac{ds}{dx}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2 \Rightarrow \dfrac{s^2}{c^2} = 1 + \left(\dfrac{dy}{dx}\right)^2$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{s^2}{c^2} - 1 = \frac{s^2-c^2}{c^2} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{s^2-c^2}}{c}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{c}\cdot\frac{1}{2}(s^2-c^2)^{-1/2}\cdot 2s\cdot\frac{ds}{dx}$$

$$= \frac{1}{c}\cdot\frac{s}{\sqrt{s^2-c^2}}\cdot\frac{s}{c} = \frac{s^2}{c^2\sqrt{s^2-c^2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \left(1+\frac{s^2-c^2}{c^2}\right)^{3/2} \Big/ \frac{s^2}{c^2\sqrt{s^2-c^2}}$$

$$= \frac{s^3}{c^3}\times\frac{c^2\sqrt{s^2-c^2}}{s^2} = \frac{s}{c}\sqrt{s^2-c^2}$$

$$\Rightarrow c.\rho = s\sqrt{s^2-c^2}.$$

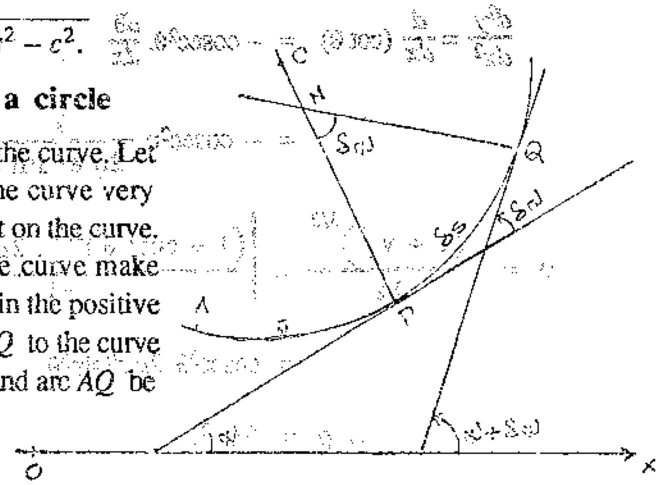

Fig.8.

6.7 Centre of curvature of a circle

Let $P(x, y)$ be any point on the curve. Let $Q(x+\delta x, y+\delta y)$ be a point on the curve very near to P. Let A be the fixed point on the curve. Let the tangents at P and Q to the curve make angles ψ and $\psi+\delta\psi$ with x—axis in the positive direction. Let the normals at P and Q to the curve intersect at N. Let arc AP be s and arc AQ be $s+\delta s$ so that arc PQ is δs.

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from fig. 8, $\angle PNQ = \delta\psi$ and

We have $$\frac{PN}{\sin \angle PQN} = \frac{\text{chord } PQ}{\sin \angle PNQ}.$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \frac{\text{arc } PQ}{\sin \delta\psi}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi}.$$

as $Q \to P$, $\angle PQN \to (\pi/2)$ and $\text{Lim} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$,

$\text{Lim} \frac{\sin \delta\psi}{\delta\psi} = 1$, $\lim_{\delta\psi \to 0} \frac{\delta s}{\delta\psi} = \frac{ds}{d\psi}$.

Let $N \to C$, as $Q \to P$.

$$\lim_{Q \to P} \frac{PN}{\sin \angle PQN} = \lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \lim_{\delta\psi \to 0} \frac{\delta s}{\delta\psi} \cdot \lim_{\delta\psi \to 0} \frac{\delta\psi}{\sin \delta\psi}$$

$$\text{i.e., } \frac{PC}{\sin \pi/2} = 1 \frac{ds}{d\psi}$$

$$\therefore PC = \frac{ds}{d\psi} = \rho \quad \text{(by definition)}$$

The circle with C as centre and ρ as radius has the same tangent and the same curvature as the curve has at P.

This circle is called the circle of curvature. C is called the center of the circle of curvature.

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6.7.1 Coordinates of the centre of circle of curvature

Let $C(x, y)$ be the centre of curvature corresponding to any point $P(x, y)$ on the curve $y = f(x)$. Then PC becomes the radius of curvature at P. Let the tangent at P to the curve make an angle ψ with x - axis in the positive direction. Let M, N be the projections of C and P respectively on x - axis. Let Q be the projection of P on CM.

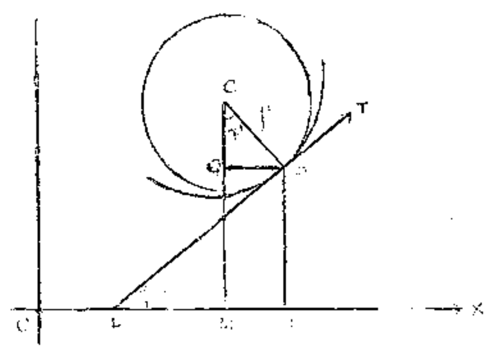

Fig.9.

$\angle PCQ = 90° - \angle QPC = 90 - (90 - \psi) = \psi$

$X = OM = ON - MN = ON - PQ = x - \rho \sin \psi.$

$Y = CM = CQ + QM = PN + CQ = \rho \cos \psi + y.$

$\tan \psi = \frac{dy}{dx} = y_1$

$\sin \psi = \frac{y_1}{\sqrt{1 + {y_1}^2}}$, $\cos \psi = \frac{1}{\sqrt{1 + {y_1}^2}}$

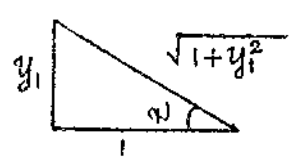

Fig. 9. (a)

$$\therefore X = x - \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{1 + y_1^2}} = x - \frac{y_1\left(1 + y_1^2\right)}{y_2}$$

$$Y = y + \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{1 + y_1^2}} = y + \frac{1 + y_1^2}{y_2}.$$

Corollary **:** The equation of the circle of curvature at the point P on the curve is

$$(x - X)^2 + (y - Y)^2 = \rho^2.$$

6.8 Evolute

The locus of the centres of curvature of a given curve is called the evolute of the curve. The curve itself is called an involute of its evolute.

Examples

Ex. 1 : Find the equation of the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Sol: $\sqrt{x} + \sqrt{y} = \sqrt{a}$, Differentiating w.r.t. x,

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0 \quad \text{i.e.,} \quad \frac{dy}{dx} = -\frac{y^{1/2}}{x^{1/2}}.$$

$$\left(\frac{dy}{dx}\right)_{(a/4,\, a/4)} = \frac{-\sqrt{a}/2}{\sqrt{a}/2} = -1.$$

$$\frac{d^2y}{dx^2} = -\left[\frac{x^{1/2} \cdot 1/2\, y^{-1/2}\, y_1 \cdot - y^{1/2}\, 1/2\, x^{-1/2}}{x}\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(a/4,\, a/4)} = -\frac{\frac{1}{2}\left[\frac{\sqrt{a}}{2} \cdot \frac{2}{\sqrt{a}}(-1) - \frac{\sqrt{a}}{2} \cdot \frac{2}{\sqrt{a}}\right]}{a/4} = 4/a.$$

$$\rho_{(a/4,\, a/4)} = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + 1)^{3/2}}{4}.a = \frac{2\sqrt{2}\, a}{4} = \frac{a}{\sqrt{2}}.$$

$$X = x - \frac{y_1\,(1 + y_1^2)}{y_2} = \frac{a}{4} - \frac{-1\,(1 + 1)\,a}{4} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$$Y = y + \frac{1 + y_1^2}{y_2} = \frac{a}{4} + \frac{(1 + 1)\,a}{4} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}.$$

Equation of the circle of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$ is $(x - X)^2 + (y - Y)^2 = \rho^2$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}.$$

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Ex. 2 : Find the coordinates of the centre of curvature of the ellipse $x = a\cos\theta$, $y = b\sin\theta$. Hence show that the equation of the evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

Sol : $x = a\cos\theta$, $\frac{dx}{d\theta} = -a\sin\theta$; $y = b\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$.

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta.$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}\,\text{cosec}^2\theta.\frac{d\theta}{dx}$$

$$= \frac{b}{a}\,\text{cosec}^2\theta.\frac{1}{-a\sin\theta} = \frac{-b}{a^2\sin^3\theta}$$

$$X = x - \frac{(1+y_1^2)\,y_1}{y_2}$$

$$= a\cos\theta - \frac{\left(1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)\left(-\frac{b\cos\theta}{a\sin\theta}\right)}{\frac{-b}{a^2\sin^3\theta}}$$

$$= a\cos\theta - \frac{(a^2\sin^2\theta + b^2\cos^2\theta)\cos\theta}{a}$$

$$= \frac{a^2\cos\theta - a^2\cos\theta\sin^2\theta - b^2\cos^3\theta}{a}$$

$$= \frac{a^2\cos\theta\,(1-\sin^2\theta) - b^2\cos^3\theta}{a} = \frac{(a^2-b^2)\cos^3\theta}{a} \qquad \text{...(i)}$$

$$Y = y + \frac{(1+y_1^2)}{y_2} = b\sin\theta + \frac{\left(1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)a^2\sin^3\theta}{-b}$$

$$= b\sin\theta - \frac{(a^2\sin^2\theta + b^2\cos^2\theta)\sin\theta}{b}$$

$$= \frac{b^2\sin\theta - (a^2\sin^2\theta + b^2\cos^2\theta)\sin\theta}{b}$$

$$= \frac{\left[b^2(1-\cos^2\theta) - a^2\sin^2\theta\right]\sin\theta}{b}$$

$$= \frac{(b^2-a^2)\sin^3\theta}{b}. \qquad \text{...(ii)}$$

From (i) and (ii),

$$\cos\theta = \left(\frac{aX}{a^2-b^2}\right)^{1/3},\ \sin\theta = \left(\frac{-bY}{a^2-b^2}\right)^{1/3}$$

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squaring and adding,

$$1 = \left(\frac{aX}{a^2 - b^2}\right)^{2/3} + \left(\frac{bY}{a^2 - b^2}\right)^{2/3}$$

$\therefore$ The equation of the evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

Ex. 3 : Find the coordinates of the centre of curvature of the rectangular hyperbola $xy = c^2$ and deduce the equation of its evolute.

Sol : $xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$

$$y_1 = \frac{-c^2}{x^2},\ y^3 = \frac{+2c^2}{x^3}$$

$$X = x - \frac{y_1(1 + {y_1}^2)}{y_2} = x - \frac{\left(-\dfrac{c^2}{x^2}\right)\left(1 + \dfrac{c^4}{x^4}\right)}{2\,c^2/x^3}$$

$$= x + \frac{x}{2}\left(1 + \frac{y^2}{x^2}\right) \qquad \left(\because y^2 = \frac{c^4}{x^2}\right)$$

$$= \frac{2x^2 + x^2 + y^2}{2x} = \frac{3x^2 + y^2}{2x}$$

$$Y = y + \frac{1 + {y_1}^2}{y_2} = y + \frac{1 + c^4/x^4}{2c^2/x^3} = y + \frac{x^3}{2c^2}\left(1 + \frac{y^2}{x^2}\right)$$

$$= y + \frac{x^2}{2y}\left(1 + \frac{y^2}{x^2}\right) = \frac{2y^2 + x^2 + y^2}{2y} = \frac{3y^2 + x^2}{2y}$$

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$\therefore$ Centre of curvature $= \left[\dfrac{3x^2 + y^2}{2x}, \dfrac{3y^2 + x^2}{2y}\right]$

$$X + Y = \frac{3x^2 + y^2}{2x} + \frac{3y^2 + x^2}{2y} = \frac{3x^2y + y^3 + 3y^2x + x^3}{2xy}$$

$$= \frac{(x + y)^3}{2c^2} \qquad (\because xy = c^2)$$

$$X - Y = \frac{3x^2 + y^2}{2x} - \frac{3y^2 + x^2}{2y} = \frac{3x^2y + y^3 - 3y^2x - x^3}{2xy}$$

$$= \frac{(x + y)^3}{2c^2} = -\frac{(x - y)^3}{2c^2}$$

$$(X + Y)^{2/3} - (X - Y)^{2/3} = \frac{(x + y)^2}{(2c^2)^{2/3}} - \frac{(x - y)^2}{(2c^2)^{2/3}}$$

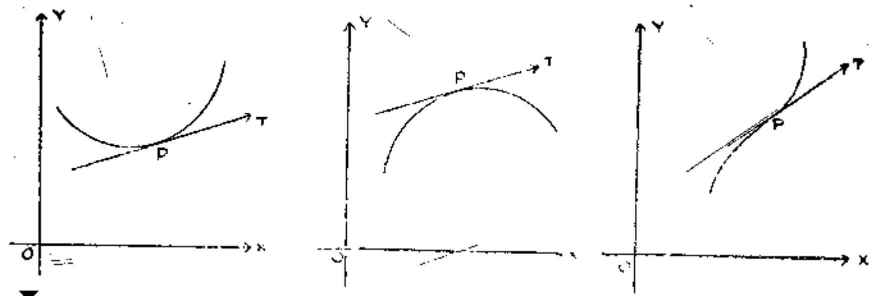
$$= \frac{4xy}{(2c^2)^{2/3}} = \frac{4c^2}{2^{2/3}.\,c^{4/3}} = (4c)^{2/3}.$$

$\therefore$ The equation of the evolute is

$$(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}.$$

6.9 Concavity, Convexity, points of inflexion

Definitions : Let P be a point on a curve $y = f(x)$. Let the tangent at P be not parallel to y - axis.



(i) (ii) (iii)

Fig. 10

If a part of the curve on both sides of P, however small, lies above the tangent {Fig. 10 (i)} the curve is said to be concave updwards or convex downwards at P; if the curve is below the tangent at P {Fig. 10 (ii) } the curve is said to be concave downwards or convex upwards at P.

If a part of the curve on one side of P lies below the tangent and a part on the other side of P lies above the tangant {Fig. 10 (iii) } then P is called a point of inflexion. At such a point the curve changes from concavity to convexity or vice versa.

6.9.1 Criteria for concavity, convexity and point of inflexion with respect to x - axis

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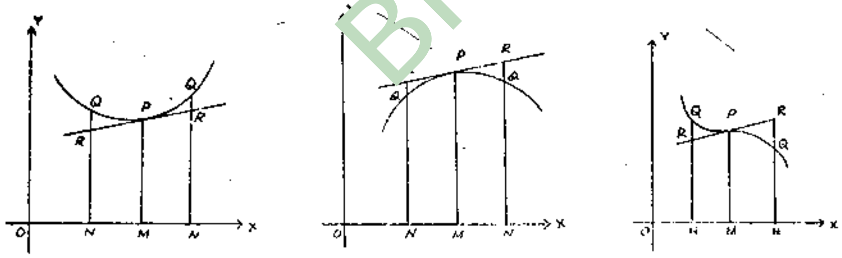
Let the equation of the curve be $y = f(x)$. Let $P\,[a, f(a)]$ be a point on the curve. Let the tangent drawn at P to the curve be not parallel to y - axis.



(i) (ii) (iii)

Fig. 11.

Let $Q\,[a + h, f(a + h)]$ be a point on the curve near the point P. Q will lie to the right or the left of the point P according as h is positive or negative. Let QN be the perpendicular to x - axis meeting the tangent at P and to the curve at R.

The equation of the tangent at P to the curve is

$$y - f(a) = f'(a)(x - a)$$

$$\text{or} \quad y = f(a) + (x - a) f'(a)$$

The tangent meets QN where $x = a + h$ in R.

$$\therefore RN = y = f(a) + (a + h - a) f'(a)$$

$$= f(a) + h\, f'(a).$$

Also QN = ordinate of Q corresponding to the abscissa $(a+h)$

$$= f(a+h)$$

$$QR = QN - RN = f(a+h) - f(a) - h f'(a).$$

From Taylor's theorem,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

where $0 < \theta < 1$.

$$QN - RN = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$+ \frac{h^n}{n!} f^n(a+\theta h) - f(a) - h f'(a)$$

$$= h^2 \left[\frac{1}{2!} f''(a) + \frac{h}{3!} f'''(a) + \ldots \right.$$

$$\left. + \frac{h^{n-3}}{(n-1)!} f^{(n-1)}(a) + \frac{h^{n-2}}{n!} f^{(n)}(a+\theta h)\right]$$

Now if $f''(a) \neq 0$ and h be chosen sufficiently small the sign of the right hand side will be same as that of $f''(a)$ whether h is positive or negative.

$\therefore$ The sign of $QN - RN$ depends on $f''(a)$.

(i) The curve will be concave upwards or convex downwards at P as in Fig. 11 (i), if $QN > RN$ (when h is positive or negative) i.e, $QN - RN$ is positive i.e., if $f''(a)$ is positive.

(ii) The curve will be concave downwards or convex upwards at P as in Fig. 11 (ii) if $QN < RN$ (when h is + ve or – ve)

i.e., if $QN - RN$ is negative

i.e., if $f''(a)$ is negative.

(iii) For the point of inflexion :

If at the point $P, f''(a) = 0$ and $f'''(a) \neq 0$ then

$$QN - RN = \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f^{(iv)}(a)$$

$$+ \ldots + \frac{h^n}{n!} f^n(a+\theta h)$$

for sufficiently small values of h, the sign of the right hand side is same as that of $h^3 f'''(a)$. This sign changes with the sign of h. Hence $QN - RN$ changes sign with h as we move from left to right through P. Hence the curve is convex w.r.t. x – axis on one side and concave on the other.

$\therefore$ The point P will be a point of inflexion of the curve if $f''(a) = 0$ and $f'''(a) \neq 0$

Generalisation :

If $f''(a) = f'''(a) = f^{iv}(a) = \ldots\ldots = f^{(n-1)}(a) = 0$ and $f^n(a) \neq 0$ then (i) the curve has a point of inflexion at P if n is odd (ii) the curve is concave upwards or concave downwards according as $f^n(a) > 0$ or < 0 and n is even.

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Examples

Ex.1. Find the range of values of x for which the curve $y = x^{*} - 6x^3 + 12x^2 + 5x + 7$ is concave upwards or downwards. Find also the point of inflexion.

Sol : The equation of the curve is $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$$\therefore \frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24 = 12(x-1)(x-2)$$

We observe that,

when $x < 1, \frac{d^2y}{dx^2} > 0$; when $x = 1, \frac{d^2y}{dx^2} = 0$

when $1 < x < 2, \frac{d^2y}{dx^2} < 0$; when $x = 2, \frac{d^2y}{dx^2} = 0$.

When $x > 2$, $\frac{d^2y}{dx^2} > 0$. Hence the curve is concave upwards in the intervals $[-\infty, 1)$, $(2, \infty]$ and concave downwards in the interval $(1, 2)$.

Also $\frac{d^3y}{dx^3} = 24x - 36$. At $x = 1$, $\frac{d^3y}{dx^3} = 24 - 36 \neq 0$

At $x = 2$, $\frac{d^3y}{dx^3} = 48 - 36 \neq 0$.

$\therefore x = 1$ and $x = 2$ are points of inflexion.

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Ex.2. Discuss the concavity and convexity of the curve $y = (\sin x + \cos x)e^x$ when $0 \leq x \leq 2\pi$. Find also the point of inflexion.

The equation of the curve is $y = (\sin x + \cos x)\, e^x$.

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)e^x = 2e^x \cos x.$$

$$\frac{d^2y}{dx^2} = 2e^x(-\sin x) + 2e^x \cos x = 2e^x(\cos x - \sin x)$$

$\because e^x \neq 0$ for any x, $\frac{d^2y}{dx^2} = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$ in $[0, 2\pi]$.

$$\frac{d^2y}{dx^2} = 2e^x(\cos x - \sin x) = 2\sqrt{2}\, e^x \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$

$$= 2\sqrt{2}\, e^x \left(\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x\right)$$

$$= 2\sqrt{2}\, e^x \cos\left(x + \frac{\pi}{4}\right).$$

Since e^x is positive for any x, $\frac{d^2y}{dx^2} > 0$ when $0 \le x < \pi/4$

$\therefore$ The curve is concave upwards when $0 \le x < \pi/4$.

When $\frac{\pi}{4} < x \le \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} < 0$.

$\therefore$ The curve is concave downwards.

When $\frac{5\pi}{4} < x \le 2\pi$, $\frac{d^2y}{dx^2} > 0$.

$\therefore$ The curve is concave upwards.

When $x = \pi/4$ or $\frac{5\pi}{4}$, $\frac{d^2y}{dx^2} = 0$.

$$\frac{d^3y}{dx^3} = 2\sqrt{2}\, e^x \cos\left(x + \frac{\pi}{4}\right) - 2\sqrt{2} e^x \sin\left(x + \frac{\pi}{4}\right)$$

$$= 2\sqrt{2}\, e^x \left[\cos\left(x + \frac{\pi}{4}\right) - \sin\left(x + \frac{\pi}{4}\right)\right]$$

$$= 2\sqrt{2}\,\sqrt{2}\, e^x \left[\frac{1}{\sqrt{2}} \cos\left(x + \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} \sin\left(x + \frac{\pi}{4}\right)\right]$$

$$= 4\, e^x \cos\left(\frac{\pi}{4} + x + \frac{\pi}{4}\right) = 4\, e^x \cos\left(\frac{\pi}{2} + x\right)$$

$$= -4\, e^x \sin x$$

When $x = \pi/4$, $\frac{d^3y}{dx^3} = -4e^{\pi/4} . \sin \pi/4 \neq 0$

When $x = \frac{5\pi}{4}$, $\frac{d^3y}{dx^3} = -4e^{5\pi/4} . \sin \frac{5\pi}{4} \neq 0$

$\therefore$ There are points of inflextion at $x = \pi/4$ and $5\pi/4$.

When $x = \pi/4, y = (\sin \pi/4 + \cos \pi/4)\, e^{\pi/4}$.

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) e^{\pi/4} = \sqrt{2}\, e^{\pi/4}.$$

When $x = 5\,\pi/4,\ y = (\sin 5\,\pi/4 + \cos 5\,\pi/4)\, e^{\pi/4}$.

$$= \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) e^{5\pi/4} = -\sqrt{2}\, e^{5\pi/4}$$

Hence the points of infexion are

$$\left(\frac{\pi}{4}, \sqrt{2}\, e^{\pi/4}\right) \text{and} \left(\frac{5\pi}{4}, -\sqrt{2}\, e^{5\pi/4}\right)$$

Note : In the article we have assumed $\frac{dy}{dx}$ is finite i.e., the tangent at P to the curve is not

 parallel to y – axis. If $\frac{dy}{dx}$, becomes infinite, we find the points of inflextion by considering $\frac{d^2x}{dy^2}$.

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Ex : 3 Find the points of inflexion of the curve $xy = a^2 \log\left(\frac{y}{a}\right)$

Sol : Equation of the curve is $xy = a^2 \log\left(\frac{y}{a}\right)$

$$\therefore x = \frac{a^2}{y} \log \frac{y}{a}$$

$$\frac{dx}{dy} = \frac{a^2}{y} \cdot \frac{a}{y} \cdot \frac{1}{a} - \frac{a^2}{y^2} \log \frac{y}{a}$$

$$= \frac{a^2}{y^2} - \frac{a^2}{y^2} \log \frac{y}{a}$$

$$= \frac{a^2}{y^2} \left(1 - \log\frac{y}{a}\right).$$

$$\frac{d^2x}{dy^2} = \frac{a^2}{y^2} \left(0 - \frac{a}{y} \cdot \frac{1}{a}\right) + \left(1 - \log\frac{y}{a}\right)\left(-\frac{2a^2}{y^3}\right)$$

$$= -\frac{a^2}{y^3} - \frac{2a^2}{y^3}\left(1 - \log\frac{y}{a}\right).$$

$$= -\frac{a^2}{y^3}\left(3 - 2\log\frac{y}{a}\right).$$

For points of inflexion $\frac{d^2x}{dy^2} = 0$ and $\frac{d^3x}{dy^3} \neq 0.$

$$\frac{d^2x}{dy^2} = 0 \Rightarrow -\frac{a^2}{y^3}\left(3 - 2\log\frac{y}{a}\right) = 0 \Rightarrow \log\frac{y}{a} = \frac{3}{2}$$

$$\Rightarrow \frac{y}{a} = e^{3/2} \Rightarrow y = a.e^{3/2}.$$

$$\frac{d^3x}{dy^3} = -\frac{a^2}{y^3}\left(-2 \cdot \frac{a}{y} \cdot \frac{1}{a}\right) + \left(3 - 2\log\frac{y}{a}\right) \cdot \frac{3a^2}{y^4}$$

$$= \frac{2a^2}{y^4} + \frac{3a^2}{y^4}\left(3 - 2\log\frac{y}{a}\right) = \frac{a^2}{y^4}\left(11 - 6\log\frac{y}{a}\right)$$

When $y = a.e^{3/2}$, $\frac{d^3x}{dy^3} = \frac{a^2}{a^4 \cdot e^6}(11 - 6 \cdot \log e^{3/2})$

$$= \frac{1}{a^2 \cdot e^6}\left(11 - 6 \cdot \frac{3}{2}\right) = \frac{2}{a^2 \cdot e^6} \neq 0.$$

Hence $y = a\,e^{3/2}$ is a point of inflexion.

$\therefore$ The point of inflexion is $\left(\frac{a^2}{a\,e^{3/2}} \cdot \log\frac{a\,e^{3/2}}{a},\ a\,e^{3/2}\right)$

$$= \left(\frac{3}{2} a\, e^{-3/2},\ a\, e^{3/2}\right)$$

6.10 Summary

The derivative of the length s of an arc $y = f(x)$ is given by,

$$\frac{ds}{dx} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

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If ψ is the angle made by the tangent to a given curve at a point P with the positive direction of the x – axis, then $\frac{d\psi}{ds}$ is called the curvature of the curve at the point P. The reciprocal of the curvature i.e., $\frac{ds}{dx}$ is called the radius of curvature of the curve at P.

The radius of curvature at any point of a curve $y = f(x)$ is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \text{ when } \frac{d^2y}{dx^2} \neq 0.$$

The coordinates of the centre of circle of curvature

$$\left(x - \frac{y_1\,(1 + y_1^{\,2})}{y_2},\; y + \frac{1 + y_1^{\,2}}{y_2}\right)$$

The locus of all these centres of curvature of a given curve is called the evolute of the curve. The curve itself is called an involute of its evolute.

A curve $y = f(x)$ is said to be concave upwards, at the point $[a, f(a)]$ if $f''(a)$ is positive; concave downwards if $f''(a)$ is negative. $[a, f(a)]$ is an inflexion point if $f''(a) = 0$ and $f'''(a) \neq 0$.

6.11 Sample Examination Questions

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I *Answer the following in detail*

(i) (a) Obtain the derivative of the length of arc of a curve in cartesian coordinates.

(b) Define the radius of curvature and obtain the radius of curvature of a curve at any point.

(ii) (a) Define circle of curvature. Obtain the co-ordinates of the centre of circle of curvature.

(b) Prove that the equation of circle of curvature for the curve $y = x^3 + 2x^2 + x + 1$ at the point (0,1) is $x^2 + y^2 + x - 3y + 2 = 0$.

(iii) (a) Discuss the nature of the curve $y = f(x)$ at a point in reference to convexity, concavity, and points of inflexion to the curve.

(b) Find out the points of inflextion for the curve $y = e^{-x^2}$

II *Briefly answer the following*

(i) Obtain the radius of curvature for the curve $\sqrt{x} + \sqrt{y} = 1$ at (1/4, 1/4).

(ii) Prove that $\rho = a \sec^2\theta$. $\operatorname{cosec}\theta$ for the curve $y = a\,e^{x/a}$ by taking $\theta = \tan^{-1}(y/a)$.

(iii) Prove that the radius of curvature for the curve

$$x = c \log\left(s + \sqrt{c^2 + s^2}\right),\; y = \sqrt{c^2 + s^2} \text{ is } \frac{c^2 + s^2}{c}.$$

(iv) Show that the equation of the circle of curvature at the origin of the parabola $y = mx + \frac{x^2}{a}$ is $x^2 + y^2 = a\,(1 + m^2)\,(y - mx)$.

(v) Obtain the centre of curvature of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at the point $(a \cos^3 \theta, a \sin^3 \theta)$.

(vi) Find the coordinates of the centre of curvature for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$. Hence show that the equation of the evolute of the hyperbola is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$.

(vii) Find the range of values of x for which the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards or downwards.

(viii) Find the points of inflexion for the curve $x = a(2^2 - \sin\theta)$, $y = a(2 - \cos\theta)$.

III Answer the following in about 5 lines.

(i) What is the curvature of a circle at a point?

(ii) Define the evolute of a curve.

(iii) When do you say that a curve has convexity, concavity or points of inflextion at a given point?

Answers

(I) (iiib) $\left(\pm \frac{1}{\sqrt{2}}, e^{-1/2}\right)$

(II) (i) $\frac{1}{\sqrt{2}}$, (v) $x = a \cos\theta.(\cos^2\theta + 3\sin^2\theta)$.

$y = a \sin\theta.(\sin^2\theta + 3\cos^2\theta)$.

(vii) at $(-2, 0), (2, \infty)$ concave upwards

at $(0, 2), (-\infty, -2)$ concave downwards

(viii) $\left[a\left(4n\pi \pm \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right]$

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Unit – 7 : Errors and Approximations

7.0 Contents

7.1 Aims and objectives

After going through this unit you must be able to :

(i) evaluate the errors involved in the value of the result, when you are given the error in the measured values.

(ii) find the maximum and minimum values of a given function (of two variables).

7.2 Introduction

Among the most striking applications of calculus are those that depend on finding the errors involved in the result and finding the maximum and minimum values of functions. Practical everyday life is filled with such problems and it is natural that mathematicians and others should find them interesting and important.

In this unit you will be studying about the errors and approximations and the problems of maxima and minima of functions of two variables.

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7.3 Differentials.

Infinitesimals: Def: If A is any finite quantity and h is a quantity smaller than any arbitrarily chosen small quantity, then $Ah, Ah^2, Ah^3, ...$ are small and each is smaller than the preceeding. These small quanitities are called infinitesimals of first, second and third order and so on.

Differential of a function

Theorem : If a function f is differentiable at x in an interval, we have

$f(x+h)-f(x)=hf'(x)+h\in(h)$ and $\lim\limits_{h\to 0} \in(h)=0.$

Proof : Since f is differentiable at x,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\text{If } h\neq 0 \text{ let } \in(h) = \frac{f(x+h)-f(x)}{h} - f'(x) \qquad ...(1)$$

$$\lim_{h\to 0}\in(h) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} - f'(x)\right]$$

$$= f'(x) - f'(x) = 0.$$

From (1), we get

$$\frac{f(x+h)-f(x)}{h} = f'(x)+\epsilon(h)$$

$$\therefore f(x+h)-f(x) = hf'(x)+h.\epsilon(h) \qquad ...(2)$$

where $\lim\limits_{h\to 0} \epsilon(h) = 0$

$$\text{Let } y = f(x)$$

$$y+\delta y = f(x+\delta x)$$

$$\delta y = f(x+\delta x)-f(x)$$

$$\text{From (2), } \delta y = \delta x . f'(x)+\delta x . \epsilon(\delta x) \qquad ...(3)$$

where $\epsilon(\delta x)$ is some function of δx.

$f'(x) . \delta x$ is an infinitesimal of first order whereas $\delta x . \epsilon(\delta x)$ is an infinitesimal of higher order.

If $\frac{dy}{dx} \neq 0$, value of $\delta x . \epsilon(\delta x)$ is very small when compared to $f'(x) . \delta x$.

Hence $f'(x) . \delta x$ can be taken as an approximate value of δy.

$$\therefore \delta y \simeq f'(x) . \delta x \qquad ...(4)$$

Definition : $f'(x) . \delta x$ is defined as the differential of y and it is denoted by dy.

From this definition, the differential of x is

$$dx = \frac{d(x)}{dx} . \delta x = 1 . \delta x = \delta x .$$

Thus, differential dx of x coincides with its increment δx.

$$\therefore dy = f'(x) . dx . \qquad ...(5)$$

From (4) and (5), it is observed that dy is very nearly equal to δy,

Thus, we write $\delta y \simeq dy$.

From (5), we may write $f'(x) = \frac{dy}{dx}$.

Hence the derivative $f'(x)$ may be regarded as the ratio of the differential of y to the differential of x.

It may be noted that the advantage of this notation is the increased freedom in algebraic manipulation.

The differentials dx and dy are to be distinguished from the increments δx and δy.

In fact, (4) is only an approximate relation wheras (5) represents an exact relation.

3. The distinction between differentials and increments can be best illustrated graphically.

Let $P(x, y)$ be any point on the curve. Let $Q(x+\delta x, y+\delta y)$ be a neighbouring point on the curve $y = f(x)$.

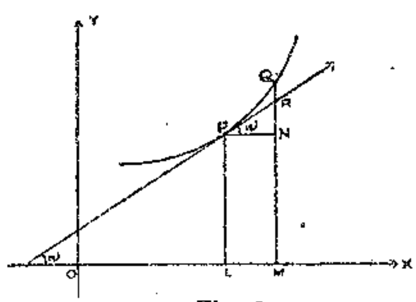

Fig. 1.

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Let L, M be the projections of P and Q on x - axis. Let N be the projection of P on QM.

$$LM = \delta x = PN, \; \delta y = QN$$

$$\tan \angle QPN = \frac{\delta y}{\delta x}.$$

As $Q \to P$, chord $QP \to$ tangent at P.

Let the tangent at P meet QN in R and make an angle ψ with OX then

$$\angle QPN \to \psi \text{ as } Q \to P.$$

$$\therefore \tan \psi = \lim_{h \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}.$$

$$\frac{RN}{PN} = \tan \psi$$

$$\therefore RN = PN . \tan \psi = \delta x . \frac{dy}{dx} = \delta x . f'(x)$$

$$RN = f'(x) . \delta x \text{ represents the differential of } y.$$

$$\text{The length } QR = \delta x . \in (\delta x).$$

But δy, the increment in y is QN.

$\therefore \delta y - dy = QN - RN = QR$ which tends to zero as $Q \to P$.

7.4 Small errors

The area of a circle is calculated from a measured value of its radius. Volume of a right circular cylinder is calculated from the measured values of its base radius and height. In practice, all measurements are subject to errors. In such cases, any error or errors in the measured value or values introduces a corresponding error in the estimated value of the result. Errors are usually small. In the function, $y = f(x)$, y is calculated with the measurement of the value of x. If there is an error of δx in the value of x, then there will also be an error δy in the value of y. The error δy in y is called absolute error. $\frac{\delta y}{y}$ is called relative error in y; $\frac{\delta y}{y} \times 100$ is called percentage error in y.

We have proved in the previous article when δx is very small, δy and dy are approximately equal. When error δx in x is given, to find an approximate δy in y, the formula $\delta y \approx \frac{dy}{dx} \delta x$ is used.

7.4.1 Approximations in the case of two or more variables

Let u be a function of x and y. i.e.,

$$u = f(x, y)$$

Let δx, δy be the errors in the measurement of x and y. Let δu be the corresponding error in u.

$$\therefore \delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]$$

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Applying Lagrange's mean value theorem to each of the two differences in the brackets,

$$\delta u = \delta x \,.\, f_x (x + \theta_1 \delta x, y + \delta y) + f_y (x, y + \theta_2 \delta y)\, \delta y \quad \dots (1)$$

$$\text{where } 0 < \theta_1 < 1,\ 0 < \theta_2 < 1.$$

[Lagrange's mean value theorem is $f(a+h) - f(a) = hf'(a + \theta h)$ where $0 < \theta < 1$.]

In the 1st bracket put $a = x$, $h = \delta x$ and in the 2nd bracket $a = y$, $h = \delta y$.

$$\therefore f(x + \delta x, y + \delta y) - f(x, y + \delta y)] = \delta x \,.\, f_x [f(x + \theta_1 \delta x, y + \delta y)$$

$$f(x, y + \delta y) - f(x, y) = \delta y \,.\, f_y (x, y + \theta_2 \delta y)]$$

$$\text{Let } f_x (x + \theta_1 \delta x, y + \delta y) = f_x (x, y) + t_1 \text{ and}$$

$$f_y (x, y + \theta_2 \delta y) = f_y (x, y) + t_2 \quad \dots (2)$$

As $\delta x\ \delta y \to 0$, $f_x (x + \theta_1 \delta x, y + \delta y) \to f_x (x, y)$ and

$$f_y(x, y + \theta_2 \delta y) \to f_y(x, y)$$

and consequently t_1 and $t_2 \to 0$.

From (1) and (2), we get $\delta u = \delta x [f_x (x, y) + t_1] + [f_y (x, y) + t_2]$

This can be expressed as $\delta u = f_x (x, y).\delta x + f_y(x, y)\, \delta y$ approximately since $t_1\, \delta x$, $t_2\, \delta y$ are infinitesimals compared to other terms.

$$\therefore \delta u = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y.$$

7.5 Maxima and minima of functions of two variables

Definition : For every point $(a + h, b + k)$ of a neighbourhood of the point (a, b), if $f(a, b) > f(a + h, b + k)$, then $f(a, b)$ is said to be a maximum value of $f(x, y)$ at (a, b) and if $f(a, b) < f(a + h, b + k)$, then $f(a, b)$ is said to be a minimum value of $f(x, y)$ at (a, b).

Necessary condition to have a maximum or minimum value

A necessary condition for $f(x, y)$ to have a maximum or minimum value is that the partial derivatives of f w.r.t x and y vanish at $x = a$ and $y = b$.

$$\text{i.e.,} \quad f_x (a, b) = 0,\ f_y (a, b) = 0.$$

Extreme value of the function

The point (a, b) is called a stationary point if $f_x = f_y = 0$ at (a, b).

A stationary point at which the function is either a maximum or a minimum is called an extreme point.

The value of the function at extreme point is called an extreme value.

A stationary point is not necessarily an extreme point. It may be maximum or minimum or neither maximum nor munimum. To decide whether a stationary point is either maximum or minimum, a further investigation is necessary.

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Sufficient conditions for the maximum or minimum of $f(x, y)$ at (a, b)

By Taylor's theorem for a function of two variables,

$$f(a+h, b+k) - f(a, b) = (hf_x + kf_y) + \frac{1}{2}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) - \ldots\ldots$$

for an extreme value, $f_x = 0 = f_y$.

$$\therefore\ f(a+h, b+k) - f(a, b) = \frac{1}{2}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) - \ldots\ldots$$

The sign depends on the first term.

Let $h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} = \Delta$.

Let $f_{xx} = A,\ f_{xy} = B,\ f_{yy} = C$.

$$\therefore\ \Delta = h^2 A + 2hkB + k^2 C$$

$$= \frac{h^2 A^2 + 2hkAB + k^2 AC}{A}$$

$$= \frac{(hA + kB)^2 + k^2 (AC - B^2)}{A} \quad \text{if } A \neq 0.$$

If $AC - B^2 > 0$, the expression $(hA + kB)^2 + k^2 (AC - B^2)$ is positive.

The sign of Δ will depend upon that of A.

If A is negative, Δ is negative and $f(x, y)$ will have a maximum at $x = a, y = b$.

If A is positive, Δ is positive and $f(x, y)$ will have a minimum at $x = a, y = b$.

If $AC - B^2$ is negative, the sign of Δ will depend upon values of h and k, then there will be neither a maximum nor a minimum.

Working rule for finding the maximum and minimum values of $f(x, y)$.

(i) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate them to zero, find the values of x and y.

(ii) Find $A = f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $B = f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, $C = f_{yy} = \frac{\partial^2 f}{\partial y^2}$

at the values of x and y obtained in (i)

(iii) If $A \neq 0$, find $AC - B^2$.

Verify whether $AC - B^2$ is positive, negative or zero.

If $AC - B^2 > 0$, then (a) $f(x, y)$ is minimum if $A > 0$,

(b) $f(x, y)$ is maximum if $A < 0$.

If $AC - B^2 < 0$, no maximum or minimum exists.

If $AC - B^2 = 0$, cannot be decided and further investigation is necessary.

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Examples

Ex. 1 : Given $\log_{10} 4 = 0.6021$, calculate approximately $\log_{10} 404$ being given that $\log_{10} x = 0.4343 \log_e x$.

Sol :

$$\text{Given that } \log_{10} 4 = 0.6021$$

$$\text{Then } \log_{10} 400 = \log_{10} 4 \times 10^2 = \log_{10} 4 + 2\log_{10} 10 = \log_{10} 4 + 2$$

$$= 2 + 0.6021 = 2.6021$$

$$\text{We have } df = f'(x)\, dx \quad \therefore\ d(\log_e x) = \frac{1}{x} . dx$$

$$d(\log_{10} x) = d(.4343 \log_e x) = .4343\, d(\log_e x)$$

$$= 0.4343 . \frac{1}{x}\, dx$$

$$\text{Let } x = 400,\ dx = 4.$$

$$\therefore\ d(\log_{10} x) = 0.4343 \times \frac{1}{400} \times 4 = 0.0043.$$

$$\text{We have } \delta f = f(x + \delta x) - f(x) \simeq f'(x)\, \delta x$$

$$\therefore\ f(x + \delta x) = f(x) + f'(x)\, dx$$

$$\log_{10} 404 \simeq \log_{10} 400 + 0.0043.$$

$$\simeq 2.6021 + 0.0043 = 2.6064.$$

Ex. 2 : If the length of a simple pendulum is decreased by 2%, find the percentage error in its period T.

Sol :

$$\text{We know that } T = 2\pi\sqrt{\frac{l}{g}}.$$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

Differentiating w. r. t. l,

$$\frac{1}{T} . \frac{dT}{dl} = \frac{1}{2} . \frac{1}{l} \Rightarrow \frac{dT}{dl} = \frac{T}{2l}.$$

$$\text{We have, } \delta T \simeq \frac{dT}{dl} . \delta l \simeq \frac{T}{2l} . \delta l.$$

$$\frac{\delta T}{T} \times 100 \simeq \frac{1}{T} . \frac{T}{2l} \times \delta l \times 100$$

$$\simeq \frac{1}{2}\, \frac{\delta l}{l} \times 100$$

$$\simeq \frac{1}{2}(-2) = -1$$

$\therefore$ Period T is decreased by 1%.

Ex. 3 : The area of a triangle ABC is determined from the side a and the two angles B and C. If there are small errors in the values of B and C, show that the resulting error in the calculated value of the area will be $\frac{1}{2}\left(b^2\, \delta C + c^2\, \delta B\right)$.

Sol : Let Δ denote the area of triangle ABC.

$$\therefore\ \Delta = \frac{1}{2}\, ab \sin C.$$

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$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ we have } b = \frac{a \sin B}{\sin A}$$

$$\text{also } \sin A = \sin[180-(B+C)] = \sin(B+C).$$

$$\therefore \Delta = \frac{1}{2} a \frac{a \sin B \cdot \sin C}{\sin(B+C)} = \frac{1}{2}\frac{a^2 \sin B \cdot \sin C}{\sin(B+C)}$$

$$\delta\Delta = \frac{\partial\Delta}{\partial B}\cdot\delta B + \frac{\partial\Delta}{\partial C}\cdot\delta C$$

$$= \frac{1}{2} a^2 \sin C\left[\frac{\sin(B+C)\cos B - \sin B\cos(B+C)}{\sin^2(B+C)}\right]\delta B$$

$$+ \frac{1}{2} a^2 \sin B\left[\frac{\sin(B+C)\cos C - \sin C\cos(B+C)}{\sin^2(B+C)}\right]\delta C$$

$$= \frac{1}{2}\frac{a^2 \sin C \cdot \sin(B+C-B)}{\sin^2(B+C)}\delta B$$

$$+ \frac{1}{2}\frac{a^2 \sin B \cdot \sin(B+C-C)}{\sin^2(B+C)}\delta C$$

$$= \frac{1}{2}\frac{a^2}{\sin^2(B+C)}(\delta B \sin^2 C + \sin^2 B \cdot \delta C)$$

$$\delta\Delta \simeq \frac{a^2}{2\sin^2 A}(\sin^2 C \cdot \delta B + \sin^2 B \cdot \delta C)$$

$$\simeq \frac{a^2}{2}\left(\frac{\sin^2 C}{\sin^2 A}\delta B + \frac{\sin^2 B}{\sin^2 A}\delta C\right)$$

$$\simeq \frac{a^2}{2}\left(\frac{c^2}{a^2}\delta B + \frac{b^2}{a^2}\delta C\right)$$

$$\simeq \frac{1}{2}(c^2\,\delta B + b^2\,\delta C).$$

Ex. 4 : Find the maximum or minimum values of the function $x^3 + y^3 - 3axy$.

Sol : Let $f(x, y) = x^3 + y^3 - 3axy$.

$$f_x = 3x^2 - 3ay,\ f_y = 3y^2 - 3ax.$$

For maximum or minimum, the necessary condition is

$$f_x = 0 = f_y$$

$$3x^2 - 3ay = 0,\ 3y^2 - 3ax = 0$$

$$x^2 - ay = 0 \quad \ldots.(1), \quad y^2 - ax = 0 \quad \ldots.(2)$$

Solving (1) and (2), $y = \frac{x^2}{a}$,

$$\frac{x^4}{a^2} - ax = 0 \Rightarrow x(x^3 - a^3) = 0 \Rightarrow x = 0 \text{ or } x = a.$$

When $x = 0, y = 0$ and when $x = a, y = a$.

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Stationary points are (0, 0) and (a, a).

Now, $A = f_{xx} = 6x,\ B = f_{xy} = -3a,\ C = f_{yy} = 6y.$

$$AC - B^2 = 36xy - 9a^2$$

At (0, 0), $AC - B^2 = -9a^2$, a negative quantity.

$\therefore$ The function has neither a maximum nor a minimum at (0, 0).

At (a, a), $AC - B^2 = 36a^2 - 9a^2 = 27a^2$, a positive quantity.

Hence there is either a maximum or a minimum at (a, a).

But $A = 6x = 6a$.

If a is positive, A is also positive.

Hence $f(x, y)$ has a minimum at (a, a).

If a is negative, A is also negative.

Hence $f(x, y)$ has a maximum at (a, a).

$\therefore$ Maximum or minimum value of $f(x, y) = a^3 + a^3 - 3a^3 = -a^3$.

Ex. 5 : Find the minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = 1$.

Sol : $ax + by + cz = 1 \Rightarrow cz = 1 - ax - by \Rightarrow z = \dfrac{1 - ax - by}{c}$.

$$\therefore\ x^2 + y^2 + z^2 = x^2 + y^2 + \frac{(1 - ax - by)^2}{c^2}$$

$$\text{Let } f(x, y) = x^2 + y^2 + \frac{(1 - ax - by)^2}{c^2}.$$

For maximum or minimum $f_x = 0 = f_y$

$$f_x = 2x + \frac{2}{c^2}(1 - ax - by)(-a) = 0.$$

$$f_y = 2y + \frac{2}{c^2}(1 - ax - by)(-b) = 0.$$

$$\Rightarrow x = \frac{a}{c^2}(1 - ax - by) \Rightarrow \frac{c^2 x}{a} = 1 - ax - by$$

$$y = \frac{b}{c^2}(1 - ax - by) \Rightarrow \frac{c^2 y}{b} = 1 - ax - by.$$

$$\frac{c^2 x}{a} = \frac{c^2 y}{b} \Rightarrow \frac{x}{a} = \frac{y}{b} \Rightarrow x = \frac{ay}{b}.$$

$$\frac{c^2 y}{b} = 1 - \frac{a^2 y}{b} - by \Rightarrow c^2 y = b - a^2 y - b^2 y.$$

$$y(a^2 + b^2 + c^2) = b \Rightarrow y = \frac{b}{a^2 + b^2 + c^2} \text{ and } x = \frac{a}{b} \cdot \frac{b}{a^2 + b^2 + c^2}$$

$$\therefore\ x = \frac{a}{a^2 + b^2 + c^2},\ y = \frac{b}{a^2 + b^2 + c^2}.$$

$$A = f_{xx} = 2 + \frac{2a^2}{c^2},\ B = f_{xy} = \frac{2ab}{c^2},\ C = f_{yy} = 2 + \frac{2b^2}{c^2}.$$

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$$AC - B^2 = \left(2 + \frac{2a^2}{c^2}\right)\left(2 + \frac{2b^2}{c^2}\right) - \frac{4a^2 b^2}{c^4}.$$

$$= 4 + \frac{4b^2}{c^2} + \frac{4a^2}{c^2} + \frac{4a^2 b^2}{c^4} - \frac{4a^2 b^2}{c^4}.$$

$$= \frac{4\,(c^2 + b^2 + a^2)}{c^2}.$$

Since $AC - B^2$ is a positive quantity and also A is positive $f(x, y)$ is minimum.

Minimum value of $x^2 + y^2 + z^2$ = minimum value of $f(x, y)$

$$= \frac{a^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2}{(a^2 + b^2 + c^2)^2}$$

$$+ \left(1 - \frac{a^2}{a^2 + b^2 + c^2} - \frac{b^2}{a^2 + b^2 + c^2}\right)^2 \frac{1}{c^2}$$

$$= \frac{a^2 + b^2}{(a^2 + b^2 + c^2)^2} + \frac{(a^2 + b^2 + c^2 - a^2 - b^2)^2}{c^2\,(a^2 + b^2 + c^2)^2}$$

$$= \frac{1}{a^2 + b^2 + c^2}$$

7.6 Summary

The derivative may be viewed as the ratio of differentials for the sake of convenience.

The distinction between differentials and increments has been illustrated. Defined the terms 'absolute error' 'relative error' and 'percentage error'. Stated the necessary and sufficient conditions for the maximum and minimum of $f(x, y)$ at a point (a, b), and explained the working rule for finding the maximum or minimum values of $f(x, y)$.

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7.7 Sample Examination Questions

I. Answer the following questions in detail.

(i) (a) Graphically illustrate the distinction between differentials and increments.

(b) Calculate tan 46° approximately, given that tan 45° = 1,

$\sec 45° = \sqrt{2}$, $1° = 0.01745$ radians.

(ii) (a) How do you calculate the approximations for the case of two or more variables?

(b) State the necessary and sufficient conditions to have a maximum or minimum value of a function of two variables.

II. Briefly answer the following questions

(i) Compute δy and dy for the values of $y = x^3 + 2x$, $x = -1$ and $\delta x = 0.02$.

(ii) The angle A of a ΔABC is found by measurement to be 63° and the area is calculated by the formula $\frac{1}{2} bc \sin A$. If the error involved in calculating A is 15', find out the percentage error involved in calculating the area of the triangle.

(iii) The refractive index of a medium $\mu = \frac{\sin x}{\sin y}$. If the error '$\epsilon$', in measuring x is twice that of the error in the measurement of y, find the error in measuring μ in terms of x, y and ϵ.

(iv) The area of a triangle is measured by the lengths a, b and c. If a is decreased by a small quantity α and b is increased by the same show that the corresponding change in the area is given by

$$\frac{\delta\Delta}{\Delta} = \frac{2\,(a-b)\,\alpha}{c^2-(a-b)^2}$$

(v) Find the maximum and minimum values of the function

$$f(x, y) = x^3 + y^3 - 63\,(x+y) + 12xy.$$

(vi) Find the extreme value of $xy\,(a-x-y)$.

(vii) Find the minimum value of $x^2 + y^2 + 6x + 12$.

Answers

I. (ib) 1.0349.

II. (i) $\delta y = 0.0988,\ dy = 0.1$

(ii) $\frac{5\pi}{36} \cot 63^\circ$ (iii) $\frac{\epsilon \sin x}{2 \sin y}\,(2 \cot x - \cot y)$.

(v) Minimum at (3, 3) and maximum at (−7, −7).

(vi) Maximum $\frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}\right)$.

(vii) Minimum = 3 at (−3, 0)

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BLOCK – 3 : INTEGRATION

Introduction

In Block - 1 and Block - 2 you have studied how to find the differential coefficient or rate of change of a given function of x. In many branches of mathematics, you are required to solve the inverse problem, i. e., given the rate of change of a function, to find the function. This process of finding a function which has a given rate of change is known as integration.

The integral of a function $f(x)$ with respect to x is the function whose differential coefficient with respect to x is $f(x)$, and is written $\int f(x)\,dx$. The symbol $\int$ is an elongated 'S' the first letter of the word 'Sum'.

$\int f(x)dx$ stands for the integral of $f(x)$ with respect to x. An integral may be defined in two distinct ways, either as the inverse of a differential coefficient or as the limit of the sum of a certain series. The former of these two definitions is the one which leads to the methods of evaluating integrals, the latter is the one upon which many of the applications depend.

In Unit-8 we discuss four principle methods of integration for the evaluation of indefinite integrals. In Unit-9, the theory of definite integrals is presented in two distinct ways, the first one being a geometrical approach and the second one an analytical approach. Double and Triple integrals are treated in Unit-10.

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Unit – 8 : Methods of Integration

8.0 Contents

8.1 Aims and objectives

After going through this Unit, you will be able to find the integral of a given function (if it exists) using appropriate technique.

8.2 Introduction

There are two distinct ways in which we may approach the problem of integration. In the first approach we regard integration as the reverse process of differentiation, which we will use for the evaluation of indefinite integrals. In the second approach we regard integration as the limit of an algebraic summation, which we use for the evaluation of definite integrals. In this Unit we discuss indefinite integrals.

Definition : If the equation

$$\frac{d}{dx}[F(x)] = f(x) \qquad (1)$$

is satisfied, then $F(x)$ is said to be an indefinite integral of $f(x)$, and we write

$$F(x) = \int f(x)\,dx \qquad (2)$$

where $f(x)$ is called Integrand and $F(x)$ as the Integral.

Observe that if $F(x)$ satisfies (1), so does the function $F(x) + C$, where C is an arbitrary constant. $F(x) + C$ is also called an indefinite integral of $f(x)$. C is called constant of integration. *We call (2) as indefinite integral because the integral is not unique.*

We Consider x to be real. Therefore our problem is to find a real function $F(x)$ whose derivative w. r. t. 'x' is a given real function $f(x)$. Given a function $f(x)$, has it necessarily an indefinite integral of $F(x)$? If some functions have indefinite integrals and others not, what property of $f(x)$ will ensure that the function has an indefinite integral? We take up such generalities in Unit-9, where we discuss necessary, and sufficient conditions for integrability. In this unit we take up the work of finding an explicit formula for integral, if we could.

The process of integration is not so systematic as that of differentiation. In general, experience is the guide for suggesting the quickest and simplest method for integrating a given function.

8.3 Evaluation of Indefinite integrals

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All the standard forms, i.e., formulae for integrals which it is necessary to remember have been mentioned here. Each formula is of the type $\int f(x)\,dx = F(x)$, and its validity can be established by showing that $\frac{d}{dx}[F(x)] = f(x)$. Arbitrary constants are to be added in each case.

1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2) $\int \frac{dx}{x} = \log|x|$

3) $\int e^x\,dx = e^x$

4) $\int e^x\,dx = \frac{a^x}{\log a} \quad (a > 0)$

5 $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (a > |x| \geq 0)$

6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

$$= \log \frac{x + \sqrt{x^2 + a^2}}{a} \quad (x,\ a > 0).$$

7) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

$$= \log \frac{x + \sqrt{x^2 - a^2}}{a} \quad (x,\ a > 0)$$

8) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \quad (a \neq 0)$

9) $\int \sin x \, dx = -\cos x$

10) $\int \cos x \, dx = \sin x$

11) $\int \sinh x \, dx = \cosh x$

12) $\int \cosh x \, dx = \sinh x$

13) $\int \sec^2 x \, dx = \tan x$

14) $\int \operatorname{cosec}^2 x \, dx = -\cot x$

15) $\int \tan x \, dx = \log |\sec x|$

16) $\int \cot x \, dx = \log |\sin x|$

17) $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} \quad (n \neq -1)$

18) $\int \frac{f'(x)}{f(x)} \, dx = \log |f(x)|$

19) $\int \sec x \, dx = \log (\sec x + \tan x)$

20) $\int \log x \, dx = x \log x - x$

If $f_1(x)$ and $f_2(x)$ are functions whose indefinite integrals are $F_1(x)$ and $F_2(x)$, then

$$\int [f_1(x) \pm f_2(x)] dx = F_1(x) \pm F_2(x).$$

If K is a constant, then $\int K f_1(x) \, dx = K F_1(x)$.

Examples

Ex. 1 : $\int \left(e^x + x^2\right) dx = \int e^x \, dx + \int x^2 \, dx = e^x + \frac{x^3}{3}$

Ex. 2 : $\int 5x^n \, dx = 5 \int x^n \, dx = \frac{5x^{n+1}}{n+1}$.

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The four principle methods of integration are :

1) Decomposition of the integrand as a sum of integrands with known integrals.
2) Integration by substitution.
3) Integration by parts.
4) Integration by successive reduction.

Now let us go through the above methods one by one.

8.4 Decomposition of the integrand

Some times, integrals cannot be found out directly by the known formulae. Then we will decompose the intregrand as a sum of integrands with known integrals and then we integrate term by term.

Examples :

Ex. 1 : Evaluate $\int (x^3 + \cos x + \sin x)\, dx$

Sol : $\int (x^3 + \cos x + \sin x)\, dx = \int x^3\, dx + \int \cos x\, dx + \int \sin x\, dx$

$$= \frac{x^4}{4} + \sin x - \cos x$$

Ex. 2 : Find $\int \sec^2 x \operatorname{cosec}^2 x\, dx.$

Sol : $\int \sec^2 x \operatorname{cosec}^2 x\, dx = \int \frac{1}{\sin^2 x \cos^2 x}\, dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx \quad (\because \sin^2 x + \cos^2 x = 1)$$

$$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{1}{\sin^2 x}\, dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x.$$

Ex. 3 : Evaluate $\int \tan^2 x\, dx.$

Sol : $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$$= \int \sec^2 x\, dx - \int dx$$

$$= \tan x - x.$$

Ex. 4 : Integrate $\sqrt{1 + \sin 2x}$

Sol : $\int \sqrt{1 + \sin 2x}\, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}\, dx$

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$$= \int \sqrt{(\sin x + \cos x)^2}\, dx$$

$$= \int \sin x\, dx + \int \cos x\, dx$$

$$= -\cos x + \sin x$$

SAQ 1 : Integrate $\dfrac{1 + \sin^2 x}{1 + \cos^2 x}$.

8.5 Integration by Substitution

In this method we express the integral $\int f(x)\, dx$, (where x is the independent variable), in terms of another integral where some other variable, say 't' is independent variable, x and t being connected by some suitable relation $x = g(t)$. Then

$$\int f(x)\, dx = \int f[g(t)]\, g'(t)\, \mathrm{dt}$$

To prove the above result, let $q = \int f(x)\, dx$

$$\therefore \frac{dq}{dx} = f(x)$$

$$\frac{dq}{dt} = \frac{dq}{dx} \cdot \frac{dx}{dt} = f(x) \frac{dx}{dt}$$

Integrating both sides w. r. t. 't', we get

$$q = \int f(x) \frac{dx}{dt}\, dt = \int f[g(t)]\, g'(t)\, dt \text{ for } x = g(t).$$

Therefore, integral of a function $f(x)$ w. r. t. 'x' is equal to the integral of $f(x)\dfrac{dx}{dt}$ w. r. t. 't'. This method is useful when a relation $x = g(t)$ can be so selected that the new integrand $f(x)\dfrac{dx}{dt}$ is of a form whose integral is known.

Examples :

Ex. 1 : Evaluate $\int \sin mx\, dx$

Sol : Put $mx = t$

Differentiating on both sides, we get $m\, dx = dt$.

$$\therefore \int \sin mx\, dx = \int \sin t \frac{dt}{m} = \frac{1}{m} \int \sin t\, dt$$

$$= -\frac{1}{m} \cos t = -\frac{1}{m} \cos mx$$

Ex. 2 : Integrate $\sec^2 5x \tan 5x$

Sol : Put $\tan 5x = t$.

$$\therefore 5 \sec^2 5x\, dx = dt$$

$$\therefore \int \sec^2 5x \tan 5x \, dx = \frac{1}{5} \int t \, dt = \frac{1}{10} t^2 = \frac{1}{10} \tan^2 5x.$$

Ex. 3 : Find $\int \cos^4 x \, dx$

Sol :

$$\cos^4 x = [\cos^2 x]^2 = \left[\frac{1 + \cos 2x}{2}\right]^2$$

$$= \frac{1}{4} [\cos^2 2x + 2 \cos 2x + 1]$$

$$= \frac{1}{4} \left[\frac{\cos 4x + 1}{2} + 2 \cos 2x + 1\right]$$

$$= \frac{\cos 4x}{8} + \frac{1}{2} \cos 2x + \frac{3}{8}$$

$$\therefore \int \cos 4x \, dx = \frac{1}{8} \int \cos 4x \, dx + \frac{1}{2} \int \cos 2x \, dx + \frac{3}{8} \int dx$$

$$= \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + \frac{3x}{8}$$

Ex. 4 : Evaluate $\int \frac{4x^3}{1 + x^8} dx$

Sol : Put $x^4 = t, \therefore 4x^3 \, dx = dt$

$$\therefore \int \frac{4x^3}{1 + x^8} dx = \int \frac{dt}{1 + t^2} = \tan^{-1} t = \tan^{-1} x^4.$$

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SAQ 2 : Integrate $e^x \cos e^x$.

Result -1 : The integral of a function whose numerator is the derivative of its denominator, is equal to the logarithm of its denominator.

i.e., $$\int \frac{f'(x)}{f(x)} dx = \log |f(x)|$$

To prove the above result, put $f(x) = t$.

$$\therefore f'(x) \, dx = dt$$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log |t| = \log |f(x)|$$

Examples

Ex. 1 : Integrate $\frac{6x}{3x^2 - 2}$

Sol : Let $3x^2 - 2 = f(x)$

$$\therefore f'(x) = 6x$$

$$\int \frac{6x}{3x^2 - 2} dx = \int \frac{f'(x)}{f(x)} dx$$

$$= \log |f(x)| = \log |3x^2 - 2|$$

Ex. 2: Find $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

Sol : Let

$$e^x - e^{-x} = t$$

$$\therefore \left(e^x + e^{-x}\right) dx = dt$$

$$\therefore \int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \int \frac{dt}{t} = \log|t| = \log|e^x - e^{-x}|$$

Ex. 3: Find the integrals of (a) $\tan x$ and (b) $\sec x$.

Sol : (a)

$$\int \tan x\, dx = \int \frac{\tan x \sec x}{\sec x}\, dx$$

$$\text{Put } \sec x = t$$

$$\therefore \sec x \tan x\, dx = dt$$

$$\int \tan x\, dx = \int \frac{dt}{t} = \log|t| = \log|\sec x|$$

(b)

$$\int \sec x\, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\, dx$$

$$\text{Let } \sec x + \tan x = t$$

$$\therefore (\sec x \tan x + \sec^2 x)\, dx = dt$$

$$\therefore \int \sec x\, dx = \int \frac{dt}{t} = \log|t| = \log|(\sec x + \tan x)|$$

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SAQ 3: Integrate $\frac{\cot x}{\log \sin x}$

Result 2 :

$$\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} \quad (n \neq -1)$$

To prove the above result, put $f(x) = t$

$$\therefore f'(x)\, dx = dt.$$

$$\int [f(x)]^n f'(x)\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$$

Result 3:

$$\int f'(ax+b)\, dx = \frac{f(ax+b)}{a}$$

To prove the above result, put $ax + b = t.$

$$\therefore a\, dx = dt$$

$$\int f'(ax+b)\, dx = \int \frac{f'(t)\, dt}{a} = \frac{1}{a}\int f'(t)\, dt$$

$$= \frac{1}{a} f(t) = \frac{1}{a} f(ax+b)$$

For example, consider $\int \sin(ax+b)\,dx$

By taking $ax+b=t$, we get $\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a}$

Ex. 5 : Integrate $\frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}$

Sol : Let $\sin^{-1}x = t$, then $\frac{1}{\sqrt{1-x^2}}\,dx = dt$.

$$\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx = \int t^2\,dt = \frac{t^3}{3} = \frac{(\sin^{-1}x)^3}{3}$$

Ex. 6: Integrate $\frac{1}{a\sin x + b\cos x}$

Sol : Let r and α be two constants such that

$a = r\cos\alpha,\ b = r\sin\alpha$

$$\therefore\ r = \sqrt{a^2+b^2} \text{ and } \tan\alpha = \frac{b}{a} \Rightarrow \tan^{-1}\left(\frac{b}{a}\right) = \alpha.$$

$$\therefore\ a\sin x + b\cos x = r(\sin x\cos\alpha + \cos x\sin\alpha)$$

$$= r\sin(x+\alpha)$$

$$\therefore \int \frac{1}{a\sin x + b\cos x} = \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\alpha)\,dx$$

$$= \frac{1}{r}\log\tan\left(\frac{x+\alpha}{2}\right)\left(\because \int \operatorname{cosec} x\,dx = \log\tan\left(\frac{x}{2}\right)\right)$$

Some Important Integrals

1. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$

Sol : Put $x = a\sin\theta$

$$\therefore\ dx = a\cos\theta\,d\theta$$

$$\therefore\ a^2-x^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta.$$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int d\theta = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

2. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)$

Sol : Put $x = a\sinh\theta$

$$\therefore\ dx = a\cosh\theta\,d\theta$$

$$\therefore\ a^2+x^2 = a^2(1+\sinh^2\theta) = a^2\cosh^2\theta.$$

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$$\therefore \int \frac{1}{\sqrt{a^2+x^2}}\,dx = \int \frac{1}{a\cosh\theta}\,a\cosh\theta\,d\theta.$$

$$= \int d\theta = \theta = \sinh^{-1}\left(\frac{x}{a}\right)$$

3. $$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)$$

Sol : Let $x = a\cosh\theta$

$\therefore \quad dx = a\sinh\theta\,d\theta$

$\therefore x^2 - a^2 = a^2(\cosh^2\theta - 1) = a^2\sinh^2\theta.$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \int \frac{1}{a\sinh\theta}\,a\sinh\theta\,d\theta.$$

$$= \int d\theta = \theta = \cosh^{-1}\left(\frac{x}{a}\right)$$

4. $$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

Sol : Let $x = a\sin\theta$

$\therefore \quad dx = a\cos\theta\,d\theta$

$$\therefore \int \sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta.$$

$$= \frac{a^2}{2}\int (\cos 2\theta + 1)\,d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]$$

$$= \frac{a^2}{2}[\sin\theta.\cos\theta + \theta]$$

$$= \frac{a^2}{2}\left[\sin\theta\sqrt{1-\sin^2\theta} + \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} + \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

5. $$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$

Sol : Let $x = a\sinh\theta$

$\therefore \quad dx = a\cosh\theta\,d\theta$

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$$\therefore \int \sqrt{a^2 + x^2}\, dx = \int a^2 \cosh^2\theta \; d\theta.$$

$$\text{We know that } \cosh 2\theta \, d\theta = \cosh^2\theta + \sinh^2\theta$$

$$\text{and } \cosh^2\theta - \sinh^2\theta = 1$$

$$\therefore \cosh^2\theta = \frac{1 + \cosh\ 2\theta}{2}$$

$$\therefore \int\sqrt{a^2 + x^2}\, dx = \frac{a^2}{2}\int (\cosh 2\theta + 1)\, d\theta.$$

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} + \theta\right]$$

$$= \frac{a^2}{2}\left[\sinh\theta . \cosh\theta + \theta\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1 + \frac{x^2}{a^2} +}\ \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$

SAQ 4: Show that $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$

(*Hint* : take $x = a\cosh\theta$)

8.6 Integration by parts

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If the integrand is a product of two functions then we adopt this method of integration.

Let U and V be two functions of x, then

we know that $\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}$

Integrating both sides, we get

$$UV = \int U\frac{dV}{dx}\, dx + \int V\frac{dU}{dx}\, dx$$

$$\therefore \int U\frac{dV}{dx}dx = UV - \int V\frac{dU}{dx}dx \qquad (1)$$

Let $U = f(x)$ and $\frac{dV}{dx} = \phi(x)$

$$\therefore \frac{dU}{dx} = f'(x) \text{ and } V = \int \phi(x)\, dx$$

$\therefore$ (1) may be written as

$$\int f(x)\,\phi(x)\, dx = f(x)\int \phi(x)\, dx - \int\left[\int \phi(x)\, dx\right] f'(x)\, dx$$

i.e., The integral of the product of two functions = First function x Integral of second function – integral of (differential coefficient of first function x integral of second function)

Care should be taken to choose the two functions so as to complete the integration in a few steps.

Examples :

Ex.1. Evaluate $\int x^2 \sin x \, dx$

Sol : Let $f(x) = x^2, \phi(x) = \sin x$

$$\therefore \int x^2 \sin x \, dx = x^2(-\cos x) - \int (-\cos x) 2x \, dx$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx \qquad (1)$$

Now we evaluate $\int x \cos x \, dx = x \sin x - \int \sin x \, . \, 1 \, dx$

$$= x \sin x + \cos x \qquad (2)$$

From (1) and (2)

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x)$$

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Ex.2. Evaluate $\int x \log x \, dx$

Sol : Let the first function be $\log x$ and the second be x.

$$\therefore \int x \log x \, dx = \log x \int x \, dx - \int \left[\frac{d}{dx}(\log x) \int x \, dx\right] dx$$

$$= \frac{x^2}{2} \log x - \int \frac{1}{x} \frac{x^2}{2} dx$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4}$$

$$= \frac{1}{2} x^2 \left(\log x - \frac{1}{2}\right)$$

Ex. 3. Integrate $e^{3x} \sin 5x$

Sol : Let $I = \int e^{3x} \sin 5x \, dx$

Let us take up $\sin 5x$ as first function and e^{3x} as second function.

$$\therefore I = \sin 5x \int e^{3x} \, dx - \int 5 \cos 5x \, \frac{1}{3} e^{3x} \, dx$$

$$= \frac{1}{3} \sin 5x \; e^{3x} - \frac{5}{3} \int \cos 5x \, e^{3x} \, dx \qquad (1)$$

Now consider $\int \cos 5x \, e^{3x} \, dx$

$$\int \cos 5x \, e^{3x} \, dx = \frac{1}{3} \cos 5x \; e^{3x} - \int \left(-5 \sin 5x \; \frac{e^{3x}}{3} \right) dx$$

$$= \frac{1}{3} e^{3x} \cos 5x + \frac{5}{3} \int \sin 5x \, e^{3x} \, dx$$

$$= \frac{1}{3} e^{3x} \cos 5x + \frac{5}{3} I \qquad (2)$$

$\therefore$ From (1) and (2)

$$I = \frac{1}{3} \sin 5x \; e^{3x} - \frac{5}{9} e^{3x} \cos 5x - \frac{25}{9} I$$

$$\therefore \frac{34}{9} I = \frac{1}{9} \left(3 \sin 5x \; e^{3x} - 5 e^{3x} \cos 5x \right)$$

$$\therefore I = \frac{1}{34} \left(3 \sin 5x - 5 \cos 5x \right) e^{3x}$$

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Ex.4. Evaluate $\int \sin^{-1} x \, dx$

Sol: Let us take up unity as second function.

$$\therefore \int \sin^{-1} x \, dx = \sin^{-1} x \int 1 dx - \int \left(\frac{d}{dx} (\sin^{-1} x) \int 1 \, dx \right) dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx \qquad (1)$$

Consider $\int \frac{2x}{\sqrt{1 - x^2}} dx$

Let $1 - x^2 = t$, $\quad \therefore -2x \, dx = dt$.

$$\therefore \int \frac{2x}{\sqrt{1-x^2}} dx = -\int \frac{dt}{\sqrt{t}} = -\frac{t^{-1/2+1}}{-\frac{1}{2}+1}$$

$$= -2\sqrt{t} = -2\sqrt{1-x^2} \qquad (2)$$

$\therefore$ From (1) and (2)

$$\int \sin^{-1}x \, dx = x \sin^{-1} x - \frac{1}{2}\left(-2\sqrt{1-x^2}\right)$$

$$= x \sin^{-1} x + \sqrt{1-x^2}.$$

Ex.5. Evaluate $\int e^{ax} \sin(bx+c)\, dx$

Sol : Let $f(x) = e^{ax}$ and $g(x) = \sin(bx+c)$

$$f'(x) = a e^{ax} \text{ and } \int g(x)\, dx = -\frac{1}{b} \cos(bx+c)$$

$$\text{Let } I = \int e^{ax} \sin(bx+c)\, dx$$

$$= e^{ax}\left(-\frac{1}{b}\cos(bx+c)\right)$$

$$- \int a e^{ax}\left(-\frac{1}{b}\cos(bx+c)\right) dx$$

$$= -\frac{1}{b} e^{ax}.\cos(bx+c) + \frac{a}{b}\int e^{ax}\cos(bx+c) \qquad (1)$$

Now to evaluate $\int e^{ax} \cos(bx+c)\, dx,$

let $f(x) = e^{ax}$ and $g(x) = \cos(bx+c)$

$$\therefore \int e^{ax}\cos(bx+c)\, dx = \frac{1}{b} e^{ax}.\sin(bx+c) - \frac{a}{b}.I \qquad (2)$$

$\therefore$ From (1) and (2)

$$I = \frac{e^{ax}}{a^2+b^2}[a \sin(bx+c) - b\cos(bx+c)]$$

SAQ 5 : Prove that $\int e^x \left[f(x) + f'(x)\right] dx = e^x f(x)$ and

hence find $\int \frac{(1 + x \log x)e^x}{x} dx$

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8.7 Successive Reduction

A formula which expresses an integral interms of another which is simpler, is a reduction formula for the first integral. The successive application of the reduction formula enables us to express the integral of the general member of the class of functions interms of that of simplest member of the class for which integral is known. Now let us consider some important examples for illustration of the method.

Examples :

Ex.1. Integrate $x^n e^x$ and hence find the value of $\int x^4 e^x \, dx$

Sol : Let $I_n = \int x^n e^x \, dx$.

Using the method of integration by parts, we get

$$I_n = x^n e^x - \int n\, x^{n-1} e^x \, dx$$

$$= x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - n\, I_{n-1}$$

Put $n = 4$, we get

$$I_4 = \int x^4 e^x \, dx = x^4 e^x - 4 I_3$$

$$= x^4 e^x - 4 \int x^3 e^x \, dx$$

$$= x^4 e^x - 4\left[x^3 e^x - 3 \int x^2 e^x \, dx\right]$$

$$= x^4 e^x - 4x^3 e^x + 12 \int x^2 e^x \, dx$$

$$= x^4 e^x - 4x^3 e^x + 12\left[x^2 e^x - 2 \int x\, e^x \, dx\right]$$

$$= x^4 e^x - 4x^3 e^x + 12\, x^2\, e^x - 24\left[x e^x - \int e^x \, dx\right]$$

$$= x^4 e^x - 4x^3 e^x + 12\, x^2 e^x - 24\, x\, e^x + 24\, e^x$$

Ex.2. Evaluate $\int \sin^n x \, dx$ and hence find the value of $\int \sin^4 x \, dx$

Sol : Let $I_n = \int \sin^n x \, dx$

$$= \int \sin^{n-1}x \,.\, \sin x \, dx$$

Using the method of integration by parts, we get

$$I_n = \sin^{n-1}x \,.\, (-\cos x) - \int (n-1)\sin^{n-2}x \,.\, \cos x \,.\, (-\cos x)\, dx$$

$$= -\sin^{n-1}x \,.\, \cos x + (n-1)\int \sin^{n-2}x \cos^2 x \, dx \,.$$

$$= -\sin^{n-1}x \,.\, \cos x + (n-1)\int \sin^{n-2}x \, dx - (n-1)\int \sin^n x \, dx$$

$$(\because \cos^2 x = 1 - \sin^2 x)$$

$$= -\sin^{n-1}x \,.\, \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n = -\frac{\sin^{n-1}x \,.\, \cos x}{n} + \frac{n-1}{n} I_{n-2} \qquad \text{— (A)}$$

When n is even, after successive reduction,

we get $I_0 = \int \sin^0 x \, dx = x$ and if n is odd, we get

$$I_1 = \int \sin x \, dx = -\cos x$$

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Now put $n = 4$, to get $\int \sin^4 x \, dx$

$$I_4 = \int \sin^4 x \, dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

$$= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left(\frac{-\sin x \cos x}{2} + \frac{I_0}{2}\right)$$

$$= -\frac{\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}x$$

Ex.3. Integrate $\sec^n\theta$ and hence find $\int \sec^3 x \, dx$

Sol : Let $I_n = \int \sec^n x \, dx$

$$= \int \sec^{n-2}x \sec^2 x \, dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int \left[\sec^{n-3}x \sec x \tan x \int \sec^2 x \, dx\right] dx$$

$$= \sec^{n-2}x \, \tan x - (n-2) \int \sec^{n-2}x \, \tan^2 x \, dx$$

$$= \sec^{n-2}x \, \tan x - (n-2) \int \sec^{n}x \, dx + (n-2) \int \sec^{n-2}x \, dx$$

$$\left(\because 1 + \tan^2 x = \sec^2 x \right)$$

$$= \sec^{n-2}x \, \tan x - (n-2) I_n + (n-2) \, I_{n-2}.$$

$$\therefore \; I_n = \frac{\sec^{n-2}x \, \tan x}{n-1} + \frac{n-2}{n-1} \cdot I_{n-2}. \qquad \text{—(A)}$$

Put $n = 3$, we get from (A)

$$I_3 = \int \sec^3 x \, dx = \frac{\sec x \, \tan x}{2} + \frac{1}{2} \int \sec x \, dx \quad \left(\because I_1 = \int \sec x \, dx \right)$$

$$= \frac{\sec x \, \tan x}{2} + \frac{1}{2} \log |\sec x + \tan x|$$

Ex .4. Obtain the reduction formula for $\int x^m (\log x)^n \, dx$ and apply it to evaluate

$$\int x^4 (\log x)^3 \, dx$$

Sol : Using the method of integration by parts, we get

$$\int x^m (\log x)^n \, dx = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{1}{(m+1)} \int x^{m+1} \cdot n \, (\log x)^{n-1} \frac{1}{x} dx$$

$$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx \qquad \text{—(A)}$$

which is the required reduction formula.

Put $m = 4, n = 3$ and using (A), we get

$$\int x^4 (\log x)^3 dx = \frac{x^5}{5} (\log x)^3 - \frac{3x^5}{25} (\log x)^2 + \frac{6x^5}{125} \log x - \frac{6}{625} x^5.$$

Note 1: If the integrand is an algebraic rational function of the form $f(x)/g(x)$ where degree of $f(x)$ is smaller than the degree of $g(x)$, then if possible use the method of partial fractions for the decomposition of the integrand as a sum of integrands with known integrals.

2. If the integrand is of the form $\frac{1}{a + b\cos x}$ or $\frac{1}{a + b \sin x}$, then use the method of substitution by taking $\tan \frac{x}{2} = t$

3. If the integrand is of the form $\frac{a \cos x + b \sin x}{c \cos x + d \sin x}$, then write the numerator in the form A (Denominator) + B. $\frac{d}{dx}$ (Denominator). Find A and B, by equating the coefficients of like terms and then integrate.

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Examples

Ex.1. Integrate $\dfrac{2x+3}{x^2+x-30}$

Sol : Integrand is of the form $f(x)/g(x)$, where the degree of $f(x)$ is smaller than the degree of $g(x)$ and $x^2+x-30=(x+6)(x-5)$. Using the method of partial fractions,

$$\frac{2x+3}{(x+6)(x-5)} = \frac{A}{x+6}+\frac{B}{x-5}$$

$$\therefore 2x+3 = A(x-5)+B(x+6) \qquad (1)$$

Equating the coefficients of like terms, we get

$A+B=2,\ -5A+6B=3.$

$\therefore A=\frac{9}{11},\ B=\frac{13}{11}$

$$\therefore \int \frac{2x+3}{x^2+x-30}\,dx = \frac{9}{11}\int\frac{1}{(x+6)}\,dx+\frac{13}{11}\int\frac{dx}{(x-5)}$$

$$= \frac{9}{11}\log|x+6|+\frac{13}{11}\log|x-5|$$

Ex. 2: Integrate $\dfrac{1}{x(x^3+1)}$.

Sol:
$$\frac{1}{x(x^3+1)} = \frac{1}{x(x+1)(x^2-x+1)}$$

$$\frac{1}{x(x^3+1)} = \frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}$$

$$1 = A(x+1)(x^2-x+1)+Bx(x^2-x+1)+(Cx+D)x(x+1).$$

Equating the coefficients of like terms we get

$A+B+C=0,\ -B+C+D=0,\ B+D=0$ and $A=1$

$A=1,\ B=-\frac{1}{3},\ C=-\frac{2}{3},\ D=\frac{1}{3}$

$$\therefore \int\frac{1}{x(x^3+1)}\,dx = \int\frac{1}{x}\,dx-\frac{1}{3}\int\frac{1}{x+1}\,dx$$

$$-\frac{1}{3}\int\frac{2x-1}{x^2-x+1}\,dx.$$

$$=\log|x|-\frac{1}{3}\log|x+1|-\frac{1}{3}\log|x^2-x+1|.$$

$$\left(\because\ 2x-1=\frac{d}{dx}(x^2-x+1)\right)$$

 Ex. 3: Evaluate $\displaystyle\int\frac{1}{5+4\sin x}\,dx$

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Sol: Integrand is of the form $\dfrac{1}{a+b\sin x}$ therefore

$$\text{put } \tan\frac{x}{2} = t$$

Differentiating, we get $\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$.

$$\therefore\ dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+t^2}$$

$$\text{we know } \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\therefore \int\frac{1}{5+4\sin x}\,dx = \int\frac{1}{5+\frac{8t}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$

$$= \int\frac{2}{5t^2+8t+5}\,dt$$

$$= \frac{2}{5}\int\frac{dt}{t^2+\frac{8}{5}t+1}$$

$$= \frac{2}{5}\int\frac{dt}{t^2+\frac{8}{5}t+\frac{16}{25}+\frac{9}{25}}$$

$$= \frac{2}{5}\int\frac{1}{\left(t+\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}\,dt$$

$$= \frac{2}{5}\cdot\frac{5}{3}\tan^{-1}\left[\frac{5t+4}{5\times\frac{3}{5}}\right]$$

$$= \frac{2}{3}\tan^{-1}\left[\frac{5\tan\frac{x}{2}+4}{3}\right]$$

Ex.4. Integrate $\dfrac{2\sin x+3\cos x}{3\sin x+4\cos x}$

Sol : Integrand is of the form $\dfrac{a\cos x+b\sin x}{c\cos x+d\sin x}$, therefore we express the *numerator* as A (Denominator) + $B\dfrac{d}{dx}$ (Denominator)

$\therefore\ 2\sin x + 3\cos x = A(3\sin x + 4\cos x) + B(3\cos x - 4\sin x)$.

Equating the coefficients of like terms $3A - 4B = 2,\ 4A + 3B = 3$.

Solving, we get $A = \dfrac{18}{25},\ B = \dfrac{1}{25}$

$$\therefore \int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}\cdot dx = \frac{18}{25}\int dx + \frac{1}{25}\int\frac{\frac{d}{dx}(3\sin x+4\cos x)}{3\sin x+4\cos x}\,dx$$

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$$= \frac{18}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x|$$

Ex. 5 : Evaluate $\int \frac{dx}{1 + \tan x}$

Sol : $\frac{1}{1 + \tan x} = \frac{\cos x}{\sin x + \cos x}$

Let $\cos x = A (\sin x + \cos x) + B \frac{d}{dx} (\sin x + \cos x)$

Equating the coefficients, we get

$A + B = 1,\ A - B = 0 \Rightarrow A = B = \frac{1}{2}$

$$\therefore \int \frac{1}{1 + \tan x} dx = \int \frac{\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\frac{d}{dx}(\sin x + \cos x)}{\sin x + \cos x} dx$$

$$= \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x|$$

8.8 Summary

Integration is treated as the reverse process of differentiation. $F(x) = \int f(x)\, dx$ is said to be an indefinite integral of $f(x)$, if $\frac{d}{dx}[F(x)] = f(x)$ is satisfied. Depending upon the given function to be integrated, we choose a method of integraion suitable for it. Various methods of integration have been discussed.

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8.9 Sample Examination Questions

I. Answer the following questions in detail.

i) a) Explain the process of integration and state various methods of integration.

b) Use the method of decomposition of integrand to evaluate the indefinite integrals of the following functions.

1) $\sqrt{x} + \frac{1}{x} + \frac{1}{x^2} + e^x$ 2) $\cot^2 x$ 3) $\sqrt{1 + \cos 2x}$

ii) a) Explain the method of substitution.

b) Use the method of substitution to evaluate the integrals of the following functions.

1) $2x.\ e^{x^2}$ 2) $e^{\tan x} \sec^2 x$ 3) $\frac{e^{\log x}}{x}$ 4) $\frac{x^2 \tan^{-1} x^2}{1 + x^6}$

iii) a) Explain the method of integration by parts

b) Evaluate the integral of the following functions using the method of by parts

1) $x^2 e^x$ 2) $x^n \log x$ 3) $\cos^{-1} x$ 4) $\frac{x}{\sin^2 x}$

iv) a) Explain the method of integration by successive reduction.

b) Use successive reduction to :

1) evaluate the integral of $x^n\ e^{ax}$ and hence find $\int x^3 e^{ax}\, dx$

2) evaluate $\tan^n x$ and hence obtain $\int \tan^4 x\, dx$.

II. *Use various methods of integration in the evaluation of the integrals of the following functions.*

1) $\dfrac{1}{1 + \sin x}$ 2) $\dfrac{1 + \cos^2 x}{1 - \cos 2x}$ 3) $\dfrac{1}{e^x + e^{-x}}$

4) $\dfrac{\tan^2 x \sec^2 x}{1 + \tan^6 x}$ 5) $\dfrac{1}{\sqrt{x}} \cos \sqrt{x}$ 6) $\dfrac{e^x}{1 + e^x}$

7) $\dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$ 8) $\dfrac{1 - \tan x}{1 + \tan x}$ 9) $\dfrac{1}{\sqrt{x} + x}$

10) $\sin^2 x \cos^3 x$ 11) $\dfrac{1}{\sqrt{a^2 + (bx + c)^2}}$

12) $\dfrac{1}{\sqrt{16 - x^2}}$ 13) $\cos^{-1} x$ 14) xe^x

15) $\dfrac{xe^x}{(x + 1)^2}$ 16) $\dfrac{x^2 + 1}{(x + 1)^2}$

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17) If $I_n = \int (\log x)^n dx$ show that $I_n = (\log x)^n - nI_{n-1}$

and evaluate $\int (\log x)^n dx$

18) Show that $\int x^m (1 - x)^{n-1} dx = \dfrac{x^{m+1} (1 - x)^{n-1}}{m + n}$

$+ \dfrac{n - 1}{m + n} \int x^m (1 - x)^{n-2} dx$

19) $\dfrac{x + 2}{x^2 + 2x + 3}$ 20) $\dfrac{9 \cos x - \sin x}{4 \sin x + 5 \cos x}$

III. *Answer the following questions in about 5 lines.*

i) Define indefinite integral

ii) Find $\int \dfrac{f'(x)}{f(x)}\, dx$

iii) Show that $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} \quad (n \neq -1)$

iv) Show that $\int f'(ax+b)\,dx = \frac{f(ax+b)}{a}$

Answers

(ib) 1) $\frac{2}{3}x^{3/2} + \log|x| - \frac{1}{x} + e^x$ 2) $-\cot x - x$ 3) $\mp \sqrt{2}\sin x$

(iib) 1) e^{x^2} 2) $e^{\tan x}$ 3) $e^{\log x}$ 4) $\frac{(\tan^{-1}x^3)^2}{6}$

(iiib) 1) $(x^2 - 2x + 2)e^x$, 2) $\frac{x^{n+1}}{(n+1)^2}\log\left(\frac{x^{n+1}}{\quad}\right)$

3) $x\cos^{-1}x - \sqrt{1-x^2}$, 4) $-x\cot x + \log(\sin x)$

(ivb) 1) $\frac{e^{ax}}{a^4}\,[a^3x^3 - 3a^2x^2 + 6ax - 6]$

2) $I_n = \frac{\tan^{n-1}x}{n-1} I_{n-2}$, $\frac{\tan^3 x}{3} - \tan x + x$

II 1) $\tan x - \sec x$, 2) $-\cot x - \frac{x}{2}$; 3) $\tan^{-1} e^x$.

4) $\frac{1}{3}\tan^{-1}(\tan^3 x)$ 5) $-\sin\sqrt{x}$ 6) $\log(1+e^x)$

7) $\frac{1}{2}(\sin^{-1}x)^2$ 8) $\log|\sin x + \cos x|$

9) $2\log(1+\sqrt{x})$, 10) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}$

11) $\frac{1}{b}\sinh^{-1}\frac{bx+c}{a}$, 12) $\sin^{-1}\left(\frac{x}{4}\right)$

13) $x\cot^{-1}x + \log\sqrt{x^2+1}$ 14) $(x-1)e^x$.

15) $\frac{e^x}{x+1}$; 16) $\frac{x-1}{x+1}e^x$.

17) $x(\log x)^4 - 4x(\log x)^3 + 12x(\log x)^2 - 24\log x + 24x$.

19) $\frac{1}{2}\log|x^2+2x+3|$ 20) $x + \log|4\sin x + 5\cos x|$

8.10 Answers to Self Assessment Questions

SAQ 1: $\int \frac{1+\sin^2 x}{1+\cos 2x}\,dx = \int \frac{1+\sin^2 x}{2\cos^2 x}\,dx = \frac{1}{2}\int \sec^2 x\,dx + \frac{1}{2}\int \tan^2 x\,dx$

$= \int \sec^2 x\,dx - \frac{1}{2}\int dx \quad [\because \tan^2 x = \sec^2 x - 1]$

$= \tan x - \frac{x}{2}$

SAQ 2 : $\int e^x \cos e^x \, dx$; put $e^x = t$; $e^x dx = dt$

$$\therefore \int e^x \cos e^x \, dx = \int \cos t \, dt = \sin t = \sin e^x.$$

SAQ 3: $\int \frac{\cot x \, dx}{\log \sin x}$ put $\log \sin x = t$; $\cot x \, dx = dt$.

$$\therefore \int \frac{\cot x \, dx}{\log (\sin x)} = \int \frac{dt}{t} = \log |t| = \log |(\log \sin x)|$$

SAQ 4 : $\int \sqrt{x^2 - a^2} \, dx$; put $x = a \cosh \theta$; $dx = \sinh \theta \, d\theta$.

$$\therefore \int \sqrt{x^2 - a^2} \, dx = a^2 \int \sinh^2 \theta \, d\theta$$

$$= \frac{a^2}{2} \int (\cosh 2\theta - 1) d\theta.$$

$$= \frac{a^2}{2} \left[\frac{\sinh 2\theta}{2} - \theta\right] = \frac{a^2}{2} [\sinh \theta \, . \cosh \theta - \theta]$$

$$= \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \left(\frac{x}{a}\right)$$

SAQ 5: $\int e^x [f(x) + f'(x)] \, dx = \int e^x f(x) dx + \int e^x f'(x) \, dx$

But $\int e^x f'(x) \, dx = e^x \int f'(x) \, dx - \int \left[e^x \int f'(x) \, dx \right] dx$

$$= e^x f(x) - \int e^x f(x) \, dx$$

or $\int e^x f'(x) \, dx + \int e^x f(x) \, dx = e^x f(x)$ (proved)

$$\int \left(\frac{1 + x \log x}{x} \right) e^x \, dx = \int \left(\frac{1}{x} + \log x \right) e^x \, dx.$$

This integral is of the form $\int [f(x) + f'(x)] \, e^x \, dx$; where $f(x) = \log x$ and $f'(x) = \frac{1}{x}$

$\therefore$ The given integral is equal to $\log (x) \, e^x$.

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Unit – 9 : Definite Integral

9.1 Aims and Objectives

After going through this unit, you must be able to (i) evaluate the given definite integral (ii) use the properties of definite integrals while evaluating them.

9.2 Introduction

In Unit - 8 we discussed indefinite integral and various methods of integration. The notion of definite integral is useful in the determination of areas, length of curves, center of gravity and other geometrical and physical quantities.

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The theory of definite integral can be presented in two distinct ways, the first one being a geometrical approach and the second one an analytical approach. In geometrical approach, we presume the notion of area and in the analytical approach we try to know the conditions under which we can make precise the notion of the definite integral as the limit of an algebraic sum.

9.3 The Geometrical approach

Let the function $f(x)$ take only positive or zero values in the interval $a \le x \le b$ and let the graph $y = f(x)$ be finite and continuous. Let 'ξ' be a fixed number in this interval and let $A(\xi)$ denote the area bounded by the x-axis, the graph $y = f(x)$ and the ordinates $x = a$ and $x = \xi$ (see fig. 1).

Let the interval (a, ξ) of the x-axis be divided into n sub intervals of lengths $\delta_1, \delta_2, \ldots\ldots, \delta_n$, where n is any positive integer.

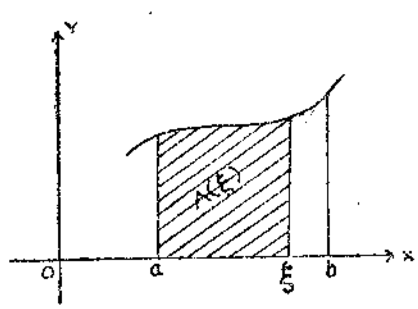

Fig-1

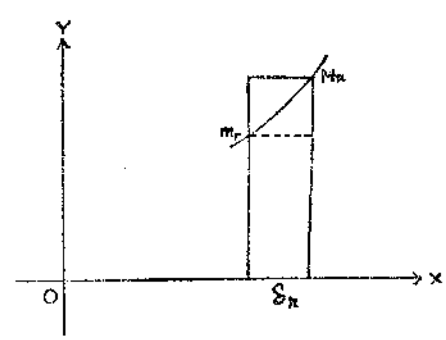

Fig-2

Let M_r be the greatest and m_r the least value assumed by $f(x)$ in the r^{th} interval.

Let $S = \sum_{r=1}^{n} M_r \delta_r$ and $s = \sum_{r=1}^{n} m_r \delta_r$ (1)

From the fig.2, the area under the graph and standing on the subinterval δ_r is less than (equal if the graph is merely $y = M_r$) the area of the rectangle of height M_r standing on the base δ_r.

$\therefore \; A(\xi) \le S$ and $A(\xi) \ge s$

i. e., $s \le A(\xi) \le S$ (2)

Now let us suppose that the graph rises and falls in k definite intervals $(a, \alpha_1), (\alpha_1, \alpha_2)$ .... (α_{k-1}, ξ). Let the curve rises in (a, α_1) falls in (α_1, α_2) ...., falls in (α_{k-1}, ξ).

$$Now \;\; S - s = \sum_{r=1}^{n} (M_r - m_r)\, \delta_r \qquad (3)$$

Let K be the sum of the rises and falls of the graph between a and ξ.

i. e., $K = [f(\alpha_1) - f(a)] + [f(\alpha_1) - f(\alpha_2)] + \ldots + [f(\alpha_{k-1}) - f(\xi)]$.

Let $\in$ be a positive number. Let n be chosen large enough so that δ_r be less than $\in/k$. The contribution to (3) from a to α_1 is less than $[f(\alpha_1) - f(a)] \in/k$ and for α_1 to α_2 is less than $[f(\alpha_1) - f(a)] \in/k$ and so on.

i. e., $S - s < \in$ (4)

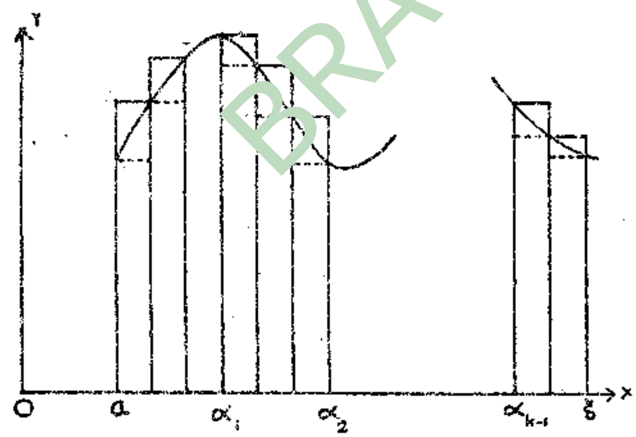

Fig - 3

Therefore, given an arbitrary small $\in$ we can make $S - s$, also $S - A(\xi)$ and $A(\xi) - s$ (from (2)) less than $\in$ by choosing $\delta_r < \in/k$.

If in each δ_r we take any point x_r at random,

then $m_r \le f(x_r) \le M_r$

and so $s \le \sum_{r=1}^{n} f(x_r)\, \delta_r \le S$ (5)

Hence we can make the difference between $A(\xi)$ and $\sum_{r=1}^{n} f(x_r)\ \delta_r$ as small as we please by taking all the δ_r sufficiently small (i.e., by choosing 'n' sufficiently large). We can put this in a rough symbolic fashion in the form

$$A(\xi) = \lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\ \delta_r \qquad (6)$$

Similarly we can prove when $f(x)$ is negative or zero over the whole of the interval (a, ξ), that $\lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\, \delta_r = -$ [The numerical area $A(\xi)$].

Keeping these ideas in mind we can define the definite integral as limit of a sum as follows.

Definition: Let $f(x)$ be defined in the interval (a, b). Divide the interval into a finite number 'n' of subintervals of lengths $\delta_1, \delta_2, \ldots \delta_n$ and let each $\delta_r < \delta$. Let x_r be a point taken at random in the sub-interval δ_r. Then

$$\lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\, \delta_r \qquad (7)$$

(in such a manner $\delta \to 0$)

is called a Definite integral and is denoted by

$$\int_a^b f(x)\, dx \qquad (8)$$

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If the summation (7) does not have a limit, then $f(x)$ has no definite integral. The limit of (7), represents the area under the graph and the ordinates $x = a$, $x = b$.

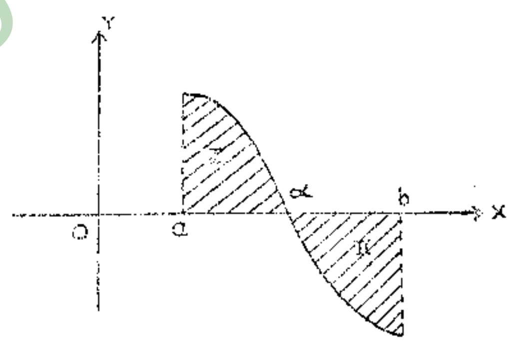

Fig - 4

If the graph of the function $f(x)$ is crossing x – axis at 'α' as shown in fig. 4, then

$$\int_a^b f(x)\, dx = \int_a^{\alpha} f(x)\, dx - \int_{\alpha}^{b} f(x)\, dx \qquad (9)$$

$$= \text{Area I} - \text{Area II}$$

9.4 Analytical approach

Before we take up analytical approach, we give below some definitions.

Bounded Functions

Let the function $f(x)$ be defined in the interval (a, b). If there exists finite numbers L and U such that $L \le f(x) \le U$ for all x in (a, b), then L is called a Lower bound, U an Upper bound of $f(x)$ in (a, b) and $f(x)$ is said to be bounded in (a, b).

Monotonic functions

If a function $f(x)$ satisfies the condition $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$, then f is said to be Monotonic increasing. If it satisfies the condition $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$ it is said to be Monotonic decreasing. Either type of function is said to be Monotonic.

Oscillation of $f(x)$

The oscillation of $f(x)$ in a given interval is the difference between the upper and lower bounds of $f(x)$ in this interval. i.e., $U - L$.

The Riemann integral

Let $f(x)$ be a bounded function defined in the interval (a, b). Let us suppose that the interval (a, b) is divided into n subintervals of lengths $\delta_1, \delta_2 \ldots\ldots, \delta_n$ where n is any finite positive integer. (i.e., $\delta_r = x_r - x_{r-1}$ and $a = x_0, x_1, x_2 \ldots\ldots x_n = b$ are points of division). Let M and m be the upper and lower bounds of $f(x)$ in (a, b) and M_r and m_r those in the closed interval $[x_{r-1}, x_r]$. Consider the sums S and s which we have already defined,

i.e., $S = \sum_{r=1}^{n} M_r \delta_r$ and $s = \sum_{r=1}^{n} m_r \delta_r$.

Corresponding to every mode of subdivision of (a, b) into sub intervels we have $s \leq S$. Over all different modes of sub-division, the sums S have a lower bound J, while the sums 's' have an upper bound I (because $S \geq m(b-a)$ and $s \leq M(b-a)$).

If $I = J$, $f(x)$ is said to be Riemann integrable in (a, b) and the common value of I and J is called the definite integral of $f(x)$ between a and b i.e., $\int_a^b f(x)\,dx$. From the above definition of R–integral, it follows that $\int_a^b f(x)\,dx \not> S$ and $\not< s$ what ever be the mode of sub–division of (a, b).

Note : $I \leq J$

9.5 Necessary and sufficient condition for integrability

Theorem : Given an arbitrary positive number ϵ, there exists a positive δ such that $S - s < \epsilon$, for every mode of subdivision of (a, b) in which each partial interval is less than or equal to δ.

Proof : First we prove that the condition is sufficient.

From the definitions of I and J,

$S \geq J$ and $s \leq I$.

$\therefore\ J - I \leq S - s \leq \epsilon$, i.e., $J = I$.

$\therefore$ The condition is sufficient.

Now we prove that the condition is necessary.

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If $f(x)$ is integrable in (a, b), then $J = I$. Since J is lower bound of upper sum, given an $\in > 0$, there exists a partition 'P_1' of (a, b) such that $J > S_1 - \frac{\in}{2}$, where S_1 is the upper sum of partition P_1. Similarly there exists a partition 'P_2' of (a, b) such that $J < s_2 + \frac{\in}{2}$, where s_2 is lower sum of partition 'P_2'. We know (from the note that follows this theorem) that the upper sum does not increase and lower sum does not decrease when additional points are introduced in a partition. Let 'P' be the partition which contains the points of both P_1 and P_2. Let S and s denote the upper sum and lower sum of partition 'P'. Now

$$S \leq S_1 < J + \frac{\in}{2}$$

$$< s_2 + \frac{\in}{2} + \frac{\in}{2}$$

$$\leq s + \in$$

Hence $S - s \leq \in$

$\therefore$ The condition is necessary.

Note : If $a = x_0, x_1, \ldots\ldots, x_n = b$ is a partition 'P' of (a, b) and ξ is such that $x_{r-1} < \xi < x_r$

Let Q be the partition $a = x_0, x_1, x_{r-1}, \xi, x_r, \ldots, x_n = b$.

Then upper sum S_P for partition P, $S_P = \Sigma M_r \delta_r$.

Let M_{1r}, M_{2r} be the upper bounds of $f(x)$ in the intervals (x_{r-1}, ξ), and (ξ, x_r) and let m_{1r}, m_{2r} be the corresponding lower bounds.

Then if S_P is the upper sum for partition Q, then

$$S_P - S_Q = (M_r - M_{1r})\ \delta_{1r} + (M_r - M_{2r})\ \delta_{2r}.$$

Where δ_{1r} and δ_{2r} stand for $\delta_{1r} = \xi - x_{r-1}$ and $\delta_{2r} = x_r - \xi$. Since the upper bound of a function in a larger set is greater than or equal to the upper bound of $f(x)$ in a smaller set,

$M_r \geq M_{1r}$ and $M_r \geq M_{2r}$.

Hence $S_P - S_Q \geq 0$.

Similarly, we can prove $S_P - S_Q \leq 0$.

Now we prove some important results.

1) **Every continuous function is integrable.**

Let $f(x)$ be continuous in (a, b). Hence it is bounded in (a, b).

To prove the result, we use the following theorem. If $f(x)$ is continuous in a closed interval, then for a given positive number $\in$, the interval can be divided into a finite number of sub-intervals such that $|f(x') - f(x'')| < \in$ where x' and x'' are any two points belonging to the same sub-interval. This is called uniform continuity.

Therefore corresponding to an arbitrary positive number $\in$, there exists a positive number δ, such that the Oscillation of $f(x)$ is less than $\in$ for every mode of division in which the length of each partial interval is less than or equal to δ. Let this mode be defined by the points $x_0 = a, x_1, x_2, \ldots\ldots, x_n = b$.

$$S = \sum_{r=1}^{n} M_r (x_r - x_{r-1}) \text{ and } s = \sum_{r=1}^{n} M_r (x_r - x_{r-1})$$

$$\therefore\ S - s = \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$

But $M_r - m_r < \epsilon$

$$\therefore\ S - s < \epsilon \sum_{r=1}^{n} (x_r - x_{r-1}) < \epsilon\,(b - a).$$

which shows that $f(x)$ is integrable in (a, b)

2) Every bounded monotonic function is integrable.

Let $f(x)$ be a bounded monotonic function in (a, b). Suppose $f(x)$ is monotonic increasing. Let the mode of division be defined by

$$a = x_0, x_1, \ldots\ldots, x_n = b.$$

$$S = \Sigma M_r (x_r - x_{r-1})$$

But $M_r = f(x_r)$ as $f(x)$ is monotonic increasing.

$$\therefore\ S = \Sigma f(x_r)\,(x_r - x_{r-1})$$
$$= f(x_1)(x_1 - a) + f(x_2)(x_2 - x_1) + \ldots + f(b)\,(b - x_{n-1})$$

and similarly,

$$s = f(a)\,(x_1 - a) + \ldots\ldots + f(x_{n-1})\,(b - x_{n-1})$$

If the partial intervals are each less than or equal to δ,

$$S - s = \Sigma [f(x_r) - f(x_{r-1})]\,(x_r - x_{r-1})$$
$$\leq \delta\, \Sigma [\, f(x_r) - f(x_{r-1})\,]$$
$$\leq \delta\, [\, f(b) - f(a)\,]$$

If we choose δ to be $< \frac{\epsilon}{f(b) - f(a)}$ then $S - s < \epsilon$.

$\therefore\ f(x)$ is integrable in (a, b).

3) A bounded function is integrable in (a, b), when all its points of discontinuity can be enclosed in a finite number of intervals whose total length is less than any arbitrary positive number.

Let ϵ be an arbitrary positive number, and let the upper bound of $|f(x)|$ be A in (a, b). By hypothesis all the points of discontinuity of $f(x)$ can be enclosed in a finite number of intervals whose total length is $\not> \frac{\epsilon}{4A}$.

As $S - s = \Sigma (M_r - m_r)(x_r - x_{r-1})$, the contribution to $S - s$ from these intervals enclosing the points of discontinuity is at most $2A \times$ their total length.

i.e., $S - s < 2A\ \frac{\epsilon}{4A} < \frac{\epsilon}{2}$

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and $f(x)$ is continuous in the rest of the remaining closed intervals. Hence $f(x)$ is integrable and there exists a mode of subdivision of this part of (a, b) such that for this part

$$S - s < \frac{\epsilon}{2}$$

$\therefore$ The combined mode gives $S - s < \epsilon$,

$\therefore$ $f(x)$ is integrable in (a, b)

4) If a bounded function is integrable in each of the partial intervals (a, a_1), $(a_1, a_2), \ldots, (a_{n-1}, b)$, it is integrable in the whole interval (a, b)

Since $f(x)$ is integrable in each of the partial intervals, there exists a mode of subdivision for each, such that $S - s$ is less than $\frac{\epsilon}{p}$.

Therefore for the combined mode of division of (a, b), $S - s < \epsilon$

9.6 Properties of definite integral

1) $$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

2) If c is any point in (a, b), then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

3) If $f(x)$ and $g(x)$ are both integrable, then $f(x) + g(x)$ is integrable and

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

4) The product of two integrable functions is integrable.

5) If $f(x) \geq g(x)$ and both are integrable in (a, b), then

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$$

6) $$\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$$

7) A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ and odd function if $f(-x) = -f(x)$ for every x.

$$\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx, \text{ if } f(x) \text{ is even function.}$$

$$= 0, \text{ if } f(x) \text{ is odd function.}$$

9.7 Theorems on definite Integral

First Mean Value Theorem :

If $f(x)$, $\psi(x)$ be two bounded integrable functions in (a, b), $f(x)$ is continuous and $\psi(x)$ preserves the same sign in (a, b), then

$$\int_a^b f(x)\ \psi(x)\ dx = f(\xi) \int_a^b \psi(x)\ dx$$

whether $a \leq \xi \leq b$.

Proof : Let $\psi(x) > 0$ in (a, b). Let M and m be the upper and lower bounds of $f(x)$ in (a, b) i.e.,

$m \leq f(x) \leq M$.

Multiplying by the positive factor $\psi(x)$

$m\ \psi(x) \leq f(x)\ \psi(x) \leq M\ \psi(x)$

Using the properties of integrals, we get

$$m \int_a^b \psi(x)\ dx \leq \int_a^b f(x)\ \psi(x)\ dx \leq M \int_a^b \psi(x)\ dx$$

$$\therefore \int_a^b f(x)\ \psi(x)\ dx = \mu \int_a^b \psi(x)\ dx \text{ where } m \leq \mu \leq M.$$

The above result is true, if $\psi(x)$ is negative also.

Since $f(x)$ is continuous in (a, b), it attains every value between m and M and in particular for some value of x, say ξ, $\mu = f(\xi)$.

$$\therefore \int_a^b f(x)\ \psi(x) dx = f(\xi) \int_a^b \psi(x)\ dx, \quad \text{where } a \leq \xi \leq b.$$

Fundamental theorems of Integral Calculus:

Theorem 2 : If $f(x)$ is bounded and integrable in (a, b) then $F(x) = \int_a^x f(x)\ dx$ is a continuous function of x in (a, b)

Proof : Let $(x + h)$ also lie in the interval (a, b), then

$$F(x+h) - F(x) = \int_a^{x+h} f(x)\ dx - \int_a^x f(x)\ dx$$

$$= \int_x^{x+h} f(x)\ dx = \mu h. \text{ (By first mean value theorem)}$$

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where $m \leq \mu \leq M$, M and m being the upper and lower bounds of $f(x)$ in $(x, x+h)$

$\therefore$ $F(x)$ is continuous function of x in (a, b).

Theorem 3 : If $f(x)$ is continuous in (a, b), then at every point 'x' in (a, b), $F(x)$ has a derivative equal to $f(x)$.

Proof : Since $f(x)$ is continuous in (a, b), we have

$F(x+h) - F(x) = h f(\xi)$ (using the previous theorem)

where either $x \leq \xi \leq x+h$ or $x \geq \xi \geq x+h$ according as $h > 0$ or $h < 0$.

When $h \to 0$, $f(\xi)$ has the limit of $f(x)$

$$\therefore \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

i.e., $F'(x) = f(x)$. Hence the theorem.

If $f(x)$ is continuous in (a, b) and

$$F(x) = \int_a^x f(x)\, dx$$ we know that $\frac{d}{dx} F(x) = f(x)$.

Suppose we can find a continuous function $\phi(x)$, such that $\frac{d}{dx}\phi(x) = f(x)$, then $F(x)$ differs from $\phi(x)$ by a constant, since $\frac{d}{dx}[F(x) - \phi(x)] = 0$.

$\therefore$ $F(x) = \phi(x) + C$, where C is a constant. To determine this constant put $x = a$ in

$$F(x) = \int_a^x f(x)\, dx$$

$$\therefore F(x) = \int_a^a f(x) dx = 0.$$

$$\text{and } F(x) = \phi(a) + C.$$

$$\therefore F(x) = \phi(x) + F(a) - \phi(a)$$

$$= \phi(x) - \phi(a) \quad (\because F(a) = 0)$$

$$\therefore F(x) = \int_a^x f(x)dx = \phi(x) - \phi(a).$$

This will enable you to evaluate the definite integral, provided that you can find an antiderivative $\phi(x)$ of the given function f. However, in practical applications of calculus, it is

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some times necessary to eavulate a definite integral $\int_a^b f(x)\,dx$ for which it is difficult or impossible to find an antiderivative ϕ of f. In such cases the numerical methods of approximation, which you study in the 3rd year can be used to estimate the value of $\int_a^b f(x)\,dx$.

9.8 The evaluation of Definite Integral

The various methods studied in the 8th unit can be used to evaluate definite integrals also. When the variable 'x' in a definite integral $\int_a^b f(x)\,dx$, is changed, we have to change the limits of integration also. The new limits being the values of the new variable which correspond to the values of a and b of x. Now let us consider few examples.

Examples

Ex. 1 : Evalute $\int_a^b \sqrt{a^2 - x^2}\,dx$

Sol. Put $x = a \sin\theta$.

Differentiating both sides, we get

$dx = a \cos \theta\, d\theta$

Since we are changing the varible of integration x, we have to change the limits of integrtion also.

$\therefore$ When $x = a$, $a = a \sin\theta$ $\therefore \sin\theta = 1 \Rightarrow \theta = \pi/2$,

when $x = 0$, $0 = a \sin\theta$ $\therefore \sin\theta = 0 \Rightarrow \theta = 0$.

$\therefore \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$.

$$\therefore \int_0^a \sqrt{a^2 - x^2} = \int_0^{\pi/2} a^2 \cos\theta \cos\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

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$$= \frac{a^2}{2}\left\{\left[\pi/2 + \frac{\sin \pi}{2}\right] - \left(0 + \frac{\sin 0}{2}\right)\right\}$$

$$= \frac{a^2}{2} \cdot \pi/2 = \frac{\pi a^2}{4}.$$

Ex. 2 : Evaluate $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$

Sol. Put $\sin x = t$ $\therefore$ $\cos x\ dx = dt$

when $x = \pi/2,\ t = \sin x = \sin \pi/2 = 1$

$x = 0, t = \sin 0 = 0.$

$$\therefore \int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx = \int_0^1 \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_0^1 = \pi/4$$

Ex.3. Show that $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx = \pi/2 - 1$

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Sol. Put $x = \cos\theta$, $\therefore$ $dx = -\sin\theta\, d\theta$.

$1 - x = 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

$1 + x = 1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

$$\therefore \sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan\left(\frac{\theta}{2}\right)$$

when $x = 1$, $1 = \cos\theta \Rightarrow \theta = 0$

$x = 0, 0 = \cos\theta \Rightarrow \theta = \pi/2$

$$\therefore \int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx = -\int_{\pi/2}^{0} \tan\frac{\theta}{2} \sin\theta\, d\theta.$$

$$= \int_0^{\pi/2} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \left(2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}\right) d\theta.$$

$$= \int_0^{\pi/2} 2 \sin^2 \theta/2 \, d\theta = \int_0^{\pi/2} (1 - \cos \theta) \, d\theta$$

$$= [\theta - \sin \theta]_0^{\pi/2} = \pi/2 - 1.$$

SAQ 1 : Evaluate $\int_0^1 x \sqrt{9 - 5x^2} \, dx$, by using the change of variable.

Ex.4 Evaluate $\int_0^1 x^2 e^{2x} \, dx$

Sol. Using the method of integration by parts, we have

$$\int x^2 e^{2x} \, dx = \frac{x^2 e^{2x}}{2} - \int \frac{2x}{2} e^{2x} \, dx$$

$$\therefore \int_0^1 x^2 e^{2x} \, dx = \left[\frac{x^2 e^{2x}}{2}\right]_0^1 - \int_0^1 x e^{2x} \, dx$$

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$$= \frac{e^2}{2} - \int_0^1 x e^{2x} \, dx$$

and $$\int x e^{2x} \, dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} \, dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$$

$$\therefore \int_0^1 x e^{2x} \, dx = \left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_0^1 = \frac{e^2}{4} + \frac{1}{4}$$

From (1) and (2),

$$\int_0^1 x^2 e^{2x} \, dx = \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4}\right) = \frac{e^2 - 1}{4}$$

SAQ 2 : Evaluate $\int_1^2 x^2 \log x \; dx$.

Ex.5. Prove that $\int_2^5 \sqrt{\frac{5-x}{x-2}}\,dx = 3\pi/2$.

Sol . Let $x = 2\cos^2\theta + 5\sin^2\theta$.

$$\therefore dx = (-4\sin\theta\,.\cos\theta + 10\sin\theta\cos\theta)\,d\theta.$$
$$= 6\sin\theta\cos\theta\,d\theta.$$
$$5-x = 5-2\cos^2\theta-5\sin^2\theta.$$
$$= 5-2(\cos^2\theta+\sin^2\theta)-3\sin^2\theta.$$
$$= 3(1-\sin^2\theta) = 3\cos^2\theta.$$
$$x-2 = 2\cos^2\theta+5\sin^2\theta-2 = 3\sin^2\theta.$$

$$\therefore \sqrt{\frac{5-x}{x-2}} = \sqrt{\frac{3\cos^2 x}{3\sin^2 x}} = \cot\theta.$$

When $x = 5$, $\theta = \pi/2$

$x = 2$, $\theta = 0$

$$\therefore \int_2^5 \sqrt{\frac{5-x}{x-2}}\,dx = 6\int_0^{\pi/2} \cot\theta.\sin\theta\cos\theta\,d\theta$$

-(1) $$= 6\int_0^{\pi/2}\cos^2\theta\,d\theta = 6\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 3\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi}{2}.$$

Ex.6 Evaluate $\int_0^{\pi/2}\sin^n\theta\,d\theta$ and hence find the value of (1) $\int_0^{\pi/2}\sin^7\theta\,d\theta$ and (2) $\int_0^{\pi/2}\sin^4\theta\,d\theta$

(2)

Sol . Let $I_n = \int_0^{\pi/2}\sin^n\theta\,d\theta$

$$= \int_0^{\pi/2}\sin^{n-1}\theta\sin\theta\,d\theta$$

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Integrating by parts, give

$$I_n = \left[\sin^{n-1}\theta \,.\, (-\cos\theta) \right]_0^{\pi/2} - (n-1) \int_0^{\pi/2} \sin^{n-2}\theta \cos\theta (-\cos\theta)\, d\theta$$

$$= 0 + (n-1) \int_0^{\pi/2} \sin^{n-2}\theta \,.\, \cos^2 d\theta$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2}\theta.\ (1 - \sin^2\theta) d\theta,$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2}\theta d\theta - (n-1) \int_0^{\pi/2} \sin^n\theta\, d\theta$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n = \frac{n-1}{n} .\ I_{n-2} .$$

$$\text{and } I_{n-2} = \frac{n-3}{n-2} .\ I_{n-4}, \ldots\ldots$$

$$\therefore I_n = \frac{n-1}{n}\ \frac{n-3}{n-2}\ I_{n-4}$$

$$= \frac{n-1}{n}\ \frac{n-3}{n-2}\ \frac{n-5}{n-4}\ \ldots\ldots\ I_1 \text{ or } I_0 \text{ according as } n \text{ is odd or even.}$$

$$I_1 = \int_0^{\pi/2} \sin\theta\, d\theta = [\cos\theta]_0^{\pi/2} = 1$$

$$I_0 = \int_0^{\pi/2} d\theta = [4]_0^{\pi/2} = \pi/2.$$

When n is odd, $I_n = \frac{n-1}{n}\ \frac{n-3}{n-2}\ \ldots\ldots\ \frac{4}{5}\ \frac{2}{3} . 1.$

When n is even, $I_n = \frac{n-1}{n}\ \frac{n-3}{n-2}\ \ldots\ldots\ \frac{3}{4}\ \frac{1}{2}\ \frac{\pi}{2}$

$$\therefore \int_0^{\pi/2} \sin^7 d\theta = \frac{6}{7}\ \frac{4}{5}\ \frac{2}{3}\ 1 = \frac{16}{35} \quad (\because n = 7 \text{ is odd})$$

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and $\int_0^{\pi/2} \sin^4 d\theta = \frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} = 3\pi/16$ $\quad (\because n = 4 \text{ is even})$

9.9 Summary

If the limit $\lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\,\delta_r$ exists, then the function f is said to be integrable on $[a, b]$ in the sense of Riemann. If f is integrable, then the definite (Riemann) integral of f on interval $[a, b]$ is defined by $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\,\delta_r$.

Essentially, the fundamental theorem of Calculus asserts that the operations of differentiation and integration are inverses of each other. Various methos of integration studied in Unit-8, can be used to evaluate the given definite integral. In addition, when you change the variable from (say) x to (say) $u = g(x)$ in a definite integral of the form $\int_a^b f[g(x)]\, g'(x)dx$, not only must you change the integral just as you did for an indefinite integral, but you must also change the limits of integration so that the integral takes the form $\int_{g(a)}^{g(b)} f(u)\,du$.

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9.10 Sample Examination Questions

I Asnwer the following in detail.

(i) a) Explain the geometrical approach of a definite integral.

b) Find $\int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta$.

(ii) a) Obtain the necessary and sufficient condition for the integrability of a function.

b) Find $\int_0^{\pi} \frac{dx}{3 + 2\cos\theta}$ (Hint : put $t = \tan\theta/2$)

(iii) a) State and prove the first mean value theorem of integral calculus.

b) Show that $\int_0^{\pi/2} \frac{\cos\theta}{6 - 5\sin\theta + \sin^2\theta}\, d\theta = \log\frac{4}{3}$.

(iv) a) State and prove the fundamental theorems of Integral calculus.

b) Evaluate $\int_0^1 \frac{5x^3}{\sqrt{1-x^8}}\,dx$.

II Briefly answer the following.

(i) Find $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\,dx$ (Hint : put $a^2+x^2=t$)

(ii) Show that $\int_0^9 \frac{dx}{1+\sqrt{x}} = 6-2\log 4$ (Hint : put $x=t^2$)

(iii) Find $\int_1^2 x^2 \log x \; dx$.

(iv) Prove that $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \pi/2 - \log 2$

(v) Prove that $\int_{-\pi}^{\pi} x^4 \sin x \; dx = 0$

(vi) Evaluate $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta$.

(vii) Evaluate $\int_0^{\pi/2} \cos^n\theta\, d\theta$ and hence find $\int_0^{\pi/2} \cos^9\theta\, d\theta$.

(viii) If $I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta$, show that $I_n + I_{n-2} = \frac{1}{(n-1)}$ and hence find the value of I_5.

III Answer the following in about 5 lines

(i) Define definite integral of the function $f(x)$.

(ii) Define monotonic function

(iii) State the properties of definite integral

(iv) How do you evaluate a definite integral ?

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Answers

I (ib) $-\frac{1}{3}$; (iib) $\pi/\sqrt{5}$ (iiib) $\log \frac{4}{3}$ (ivb) $\frac{5\pi}{8}$

II (i) $a\left(\sqrt{2}-1\right)$; (iii) $\frac{8}{3}\log 2 - \frac{7}{9}$; (vi) $\frac{\pi}{4}$; (vii) $\frac{5\pi}{32}$;

(viii) $\frac{1}{2}\log 2 - \frac{1}{4}$

9.11 Answers to Self Assessment Questions :

SAQ 1 : Let $u = 9-5x^2$; when $x=0$; $u=9$ and $x=1$; $u=4$.

$$\text{Then, } du = -10x\,dx$$

$$\text{or } x\,dx = -\frac{1}{10}du.$$

$$\therefore \int_0^1 x\sqrt{9-5x^2}\,dx = \int_{u=9}^{4}\sqrt{u}\left(-\frac{1}{10}\right)du.$$

$$= -\frac{1}{10}\int_9^4 \sqrt{u}\,du = \frac{1}{10}\int_4^9 \sqrt{u}\,du$$ [using property 1 of definite Integrals]

$$= \frac{1}{10}\left[\frac{u^{3/2}}{3/2}\right]_4^9$$

$$= \frac{1}{15}\left[9^{3/2}-4^{3/2}\right] = \frac{19}{15}.$$

SAQ 2 : Using the method of integration by parts,

$$\int_1^2 x^2\log x\,dx = \left[\log x\cdot\frac{x^3}{3}\right]_1^2 - \int_1^2 \frac{x^3}{3}\cdot\frac{1}{x}\,dx.$$

$$= \left[\log x\cdot\frac{x^3}{3}\right]_1^2 - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{8}{3}\cdot\log 2 - 0 - \frac{1}{3}\cdot\frac{8}{3} + \frac{1}{9}$$

$$= \frac{8}{3}\log 2 - \frac{7}{9}.$$

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Unit - 10 : Double and Triple Integrals

10.0 Contents

10.1 Aims and Objectives

After going through this unit, you must be able to evaluate the double and triple integrals and change the order of integration wherever necessary.

10.2 Introduction

When we were dealing with a function of a single variable it was found to be advantageous to introduce the concept of a definite integral. The motivation for this was evaluating the area under a curve. Subsequently, this technique was exploited to evaluate volumes of revolution and rectification of curves. It now becomes clear that if we could define the definite integral of a function of two or more variables then we could directly evaluate the areas and volumes without converting the given function of two or more variables into a function of single variable - which at times will be very combersome. In applications to physical problems the double and triple integrals will be of very great use and help solving the problems with much ease.

In units 8 and 9 you learnt that integration may be considered either as the inverse of differentiation or as a process of summation. We evaluated $\int_a^b f(x)\, dx$ using the result,

$$\int_a^b f(x)\, dx = F(b) - F(a), \text{ where } \frac{d}{dx} F(x) = f(x).$$

In a similar manner, we can define Double and Triple integrals.

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10.3 Double Integrals

Let $f(x, y)$ be a continuous function of x and y within a region R, then the form of a double integral is $\int\int_R f(x, y)\, dx\, dy$, where R denotes the region over which the integration is performed. Now let us see about the interpretation of a double integral.

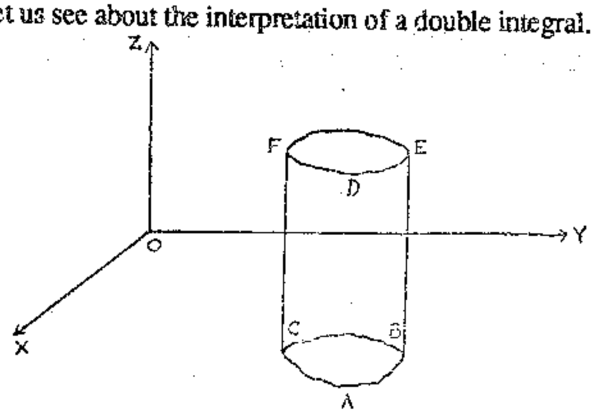

Fig-1.

Let ABC be a closed curve in the XOY plane. From the points inside and on the boundary of the closed curve ABC, vertical lines parallel to OZ have been drawn which intersect a given surface $z = f(x, y)$ in the curve DEF. Any line parallel to OZ meets the surface in only one point since we assumed that $f(x, y)$ is single valued. Now our intention is to evaluate the volume of the cylinder $ABCDEF$.

In the plane $z = 0$, let the curve ABC be wholly enclosed within a rectangle defined by $x = a$, $x = b$, $y = c$ and $y = d$, where $a < b$, $c < d$. Now divide the ractangle into mn small rectangles by means of the lines $x = x_r$, $r = 0, 1, 2 \dots m$ and $y = y_s$, $s = 0, 1, 2, \dots n$.

Let $x_0 = a < x_1 < x_2 \dots < x_m = b$

and $y_0 = c < y_1 < y_2 \dots < y_n = d$.

In the small rectangle bounded by the lines $x = x_{r-1}$, $x = x_r$; $y = y_{s-1}$, $y = y_s$, consider a point (ξ_{rs}, η_{rs}). Let $F(x, y)$ be a function such that

$F(x, y) = f(x, y)$; for all points inside and on the curve ABC.

$= 0$, for all points of the rectangle outside the curve ABC.

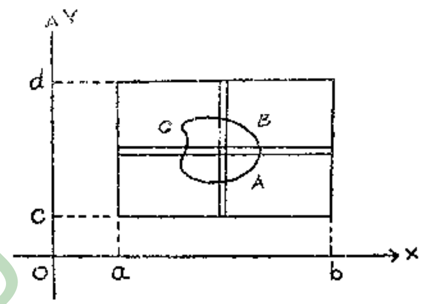

Fig-2.

The volume of cuboid on the small rectangle as base and of height $F(\xi_{rs}, \eta_{rs})$ is given by

$$F(\xi_{rs}, \eta_{rs})(x_r - x_{r-1})(y_s - y_{s-1})$$

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The sum of all such expressions over the small rectangles will give us approximately the volume of the solid figure, which we want to measure. Let m and n tend to infinity in such a way that the area of each small rectangle tends to zero, then

$$\lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{\substack{r=1,m \\ s=1,n}} F(\xi_{rs}, \eta_{rs})(x_r - x_{r-1})(y_s - y_{s-1})$$

becomes nearly equal to the required volume. This is expressed as a double integral of the function $f(x, y)$, over the area enclosed by ABC and is written as $\int\int_R f(x, y)\, dx\, dy$, where R denotes the area enclosed by the curve ABC.

Therefore, the double integral measures the volume of the solid figure bounded by the plane $z = 0$, the surface $z = f(x, y)$ and the cylinder generated by lines parallel to the z – axis passing through the curve ABC in the plane $z = 0$. The area over which we integrate in the plane $z = 0$, i.e., the area enclosed by the curve ABC, is called the field of integration.

If $f(x, y) = 1$, the double integral $\int\int_R dx\, dy$, measures the area R enclosed by the curve ABC

Now we will show that the integration can be performed in two stages, each stage being an ordinary integration.

Let the curve ABC be enclosed by the smallest rectangle given by $x = a, x = b, y = c$ and $y = d$. We divide the rectangle into small rectangles by the type of lines $x = x_{r-1}, x = x_r, y = y_{s-1}, y = y_s$. Now form the sum $\Sigma F(\xi_{rs}, \eta_{rs})(x_r - x_{r-1})(y_s - y_{s-1})$. We perform the summation in a certain order. Let r be fixed and we obtain the sum of all cuboids lying between the planes $x = x_{r-1}, x = x_r$ by summing over s from $s = 1$ to $s = n$.

Fig - 3

Therefore for sufficiently large values of m and n, the volume of the part of the solid figure lying between the planes $x = x_{r-1}$, $x = x_r$ is approximately given by

$$(x_r - x_{r-1}) \lim_{n\to\infty} \sum_{s=1}^{n} F(x, \eta_{rs})(y_s - y_{s-1})$$

or $(x_r - x_{r-1}) \displaystyle\int_c^d F(x_r, y)\, dy$. But from the definition of $F(x, y)$, it is equal to

$$(x_r - x_{r-1}) \int_{MP}^{MP'} f(x_r, y)\, dy \quad - (A),$$ in this integration x is kept constant.

Here we assumed that any line parallel to x or y axes would meet the curve ABC in atmost two points. If this is not so, we can divide the area into portions for which the above condition is satisfied and then perform the integration for each of these portions. Let the curves GDE and GFE, the lower and upper portions of the curve ABC have equations $y = \phi_1(x), y = \phi_2(x)$.

$\therefore\ MP = \phi_1(x_r),\ MP' = \phi_2(x_r)$

$\therefore \displaystyle\int_{MP}^{MP'} f(x_r, y)\ dy$ is a function of x_r and let it be equal to $\phi(x_r)$

$\therefore$ (A) takes the form $\displaystyle\sum_{r=1}^{m} \phi(x_r)(x_r - x_{r-1})$

Now $\displaystyle\lim_{m\to\infty} \sum_{r=1}^{m} \phi(x_r)\ (x_r - x_{r-1})$

$$= \int_a^b \phi(x)\, dx, \text{ where } \phi(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy$$

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since as $m \to \infty$, it is assumed that each $x_r - x_{r-1}$ tends to zero. Hence the required volume

$$= \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\ dy \right] dx, \text{ where the inner integral is kept constant during integration}$$

The above double integral can be written as $\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy$ or $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dx\ dy$

This type of integral is called *repeated integral.*

Similarly, if we keep s fixed and by summing first over r from $r = 1$ to $r = m$, the sum of the volumes of the cuboids lying between the planes $y = y_{s-1}$ and $y = y_s$, then as before, summing with respect to s, we obtain

$$\int_c^d \left[\int_{QR}^{QR'} f(x, y)\ dx \right] dy.$$

If the equations of the curves FGD, FED are given by $x = \Psi_1(y)$, $x = \psi_2(y)$ respectively,

then the volume is given by $\int_c^d dy \int_{\Psi_1(y)}^{\Psi_2(y)} f(x, y)\, dx$

In the inner integral y is kept constant. Since the volume of the solid figure has been expressed as repeated integrals the two repeated integrals must be equal and is equal to the double integral, i.e.,

$$\int\int_R f(x, y))\ dx\ dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dx\ dy = \int_c^d \int_{\Psi_1(y)}^{\Psi_2(y)} f(x, y)\ dy\ dx$$

If the function $f(x, y)$ has discontinuities within ABC, it may happen that the two integrals are not the same.

The volume of a column which stands on the base of area $dx\, dy$ and whose height is z is written as $z\, dx\, dy$ and by integrating over the area in the plane $z = 0$, we get the required volume of the solid figure as $V = \int\int z\ dx\ dy.$

In some cases the polar element of area is more convenient for the purpose of integration and hence volume is written as

$$V = \int\int z\ r\ d\theta\, dr, \text{ where the integration is performed over the area in plane } z = 0.$$

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With the substitution $x = r\cos\theta$, $y = r\sin\theta$, we can transform the cartesian element of area i.e., $dx\,dy$ into $r\,d\theta.\,dr$.

Note 1 : For constant limits, the order of integration is immaterial.

Note 2 : The integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\; dy\; dx$, is the integral over the region bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ for the values of x between a and b. For, to change the order of integration, we have to sketch the region of integration, and from the sketch the limits of x and y should be determined.

Examples

Ex. 1 : Evaluate $\int\int xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Sol : If we keep x as constant, y varies from 0 to $\sqrt{a^2 - x^2}$. To cover the whole area x varies from 0 to a.

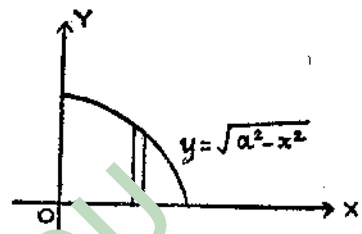

Fig - 4

$$\therefore \int\int xy\,dx\,d) = \int_0^a \int_0^{\sqrt{a^2 - x^2}} xy\,dy\,dx$$

$$= \int_0^a \left[\frac{xy^2}{2}\right]_0^{\sqrt{a^2 - x^2}} dx$$

$$= \int_0^a x\left(\frac{a^2 - x^2}{2}\right) dx \quad = \left[\frac{a^2 x^2}{4} - \frac{x^4}{8}\right]_0^a = \frac{a^4}{8}$$

Ex. 2 : Evaluate $\int_0^3 \int_1^2 xy(x + y)\,dy\,dx$

Sol : $\int_0^3 \int_1^2 xy(x + y)\,dy\,dx = \int_0^3 \left[\int_1^2 \left(x^2 y + xy^2\right) dy\right] dx$

$$= \int_0^3 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3}\right]_1^2 dx$$

$$= \int_0^3 \left(\frac{3}{2}x^2 + \frac{7}{3}x\right) dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^3}{3} + \frac{7}{3}\,\frac{x^2}{2}\right]_0^3 = 24$$

Ex. 3 : Evaluate $\int\int (x^2 + y^2)\, dx\ dy$ over the region for which x, y are each ≥ 0 and $x + y \leq 1$

Sol : The region is the triangle formed by the lines

$x = 0,\ y = 0$ and $x + y = 1.$

x varies from 0 to 1,

y varies from 0 to $1 - x$.

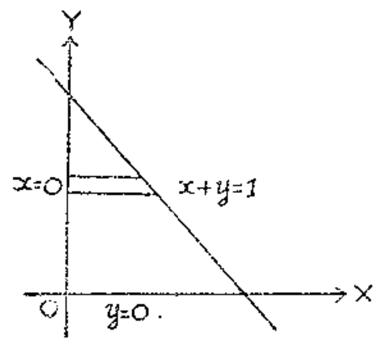

Fig - 5

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$$\therefore \quad \int\int \left(x^2 + y^2\right) dx\ dy = \int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) dy\ dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2 (1 - x) + \frac{(1 - x)^3}{3}\right] dx = \frac{1}{6}$$

SAQ 1 : Evaluate $\iint_R x \cos xy\ dx\ dy$, where R is the region $1 \leq x \leq 2$ and $\pi/2 \leq y \leq 2\pi/x$

10.4 Triple Integrals

The process of double integration can be easily extended to that of triple integration.

The form of triple integral is $\iiint_V f(x, y, z)\, dx\, dy\, dz$

In this case the function to be integrated is $f(x, y, z)$, a function of three variable x, y, z and the field of integration is now a closed volume V in three dimensional space. We now enclose this volume in a parallelopiped whose faces have the equations $x = a,\ b;\ y = c,\ d;\ z = e, f.$ This parallelopiped is divided into lmn small parallelopipeds by the planes $x = x_r,\ y = y_s,\ z = z_r$ where

$r = 0, 1, \ldots, l,\ s = 0, 1, 2 \ldots m,\ t = 0, 1, 2, \ldots n,\ x_0 = a,\ x_l = b,\ y_0 = c,\ y_m = d,\ z_0 = e,\ z_n = f$. In each small volume we take any point $(\xi_{rst}, \eta_{rst}, \zeta_{rst})$ being the typical one, we form the sum

$$\Sigma F(\xi_{rst}, \eta_{rst}, \zeta_{rst})\ (x_r - x_{r-1})\ (y_s - y_{s-1})\ (z_t - z_{t-1})$$

summed over all the lmn small volumes and where $F(x, y, z)$ is a function which is equal to $f(x, y, z)$ for points inside and on V and is equal to zero outside V. The limit of this sum when l, m, n tend to infinity so that each small volume tends to zero is called the triple integral of $f(x, y, z)$ throughout the given volume. That is

$$\lim_{\substack{l \to \infty \\ m \to \infty \\ n \to \infty}} \sum_{\substack{r = 0, l \\ s = 0, m \\ t = 0, n}} F(\xi_{rst}, \eta_{rst}, \zeta_{rst})\ (x_r - x_{r-1})\ (y_s - y_{s-1})\ (z_t - z_{t-1})$$

$$= \iiint_V f(x, y, z)\, dx\, dy\, dz.$$

If $f(x, y, z) = 1$, the triple integral measures the volume of the space V.

As in the case of the double integral, this triple integral can be expressed as a repeated integral as

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz.$$

The integration is performed in the order z, y, x.

Now let us consider the following examples

Examples

Ex. 1 : Evaluate $\iiint \dfrac{dx\, dy\, dz}{(x + y + z + 1)^3}$ throughout the volume bounded by the planes $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

Sol : First we integrate with respect to z and the limits of z are $z = 0$ and $z = 1 - x - y$, since any line parallel to the z – axsis through the point $(x, y, 0)$ meets the plane $x + y + z = 1$ in the point $(x, y, 1 - x - y)$. The projection of the plane $x + y + z = 1$ on the plane $z = 0$ is $x + y = 1$. Therefore, any line through the point $(x, 0)$ and parallel to the y - axis in the xy plane meets the line $x + y = 1$ in the point $(x, 1 - x)$, hence the limits for y are 0 and $1 - x$. The line $x + y = 1$ meets the x - axis in the point $(1, 0)$. $\therefore$ The limits of x are 0 and 1.

$$\therefore \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1 + x + y + z)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[-\frac{1}{2(1 + x + y + z)^2}\right]_0^{1-x-y}$$

$$= \int_0^1 dx \int_0^{1-x} \left[\frac{1}{2(x+y+1)^2} - \frac{1}{2^3}\right] dy$$

$$= \int_0^1 dx \left[-\frac{1}{2(x+y+1)} - \frac{1}{2^3} y\right]_0^{1-x}$$

$$= \int_0^1 \left[\frac{1}{2(x+1)} - \frac{1}{4} - \frac{1-x}{8}\right] dx$$

$$= \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

Ex. 2 : Evaluate $\iiint xyz\ dx\ dy\ dz$ taken through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

Sol : In the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$, z varies from 0 to $\sqrt{a^2 - x^2 - y^2}$, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a. Hence the required integral is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\ dz\ dy\ dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{x\ y\ z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\ dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{xy\ (a^2 - x^2 - y^2)}{2}\ dy\ dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} [x\ y\ a^2 - x^3 y - xy^3]\,dy\ dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{x\ y^2\ a^2}{2} - \frac{x^3\ y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} x$$

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$$= \frac{1}{2}\int_0^a \left[\frac{x\,(a^2 - x^2)a^2 - x^3\,(a^2 - x^2)}{2} - \frac{x\,(a^2 - x^2)^2}{4} \right] dx = \frac{a^6}{48}$$

SAQ 2: Evaluate $\int\int_S\int$ where S is the solid bounded above by the plane $z = 2 - x - y$, below by the plane $z = 0$ and laterally by the region $R : 0 \le x \le 1, 0 \le y \le 1 - x$.

10.5 Change of order of integration

In a double integral when integration is performed first with respect to y, x is kept constant and the limits of y are expressed as a function of x. In this case the region of integration is subdivided by means of strips parallel to y- axis. When the order is changed integration is performed first with respect to x keeping y constant and the limits of x are to be expressed as a function of y. In this case the region of integration is subdivided by means of strips parallel to x-axis. For expressing the limits of x as a function of y it is necessary to make a sketch of the field of integration. In the process of the change of order, quite often it is seen that strips change their character at some points and the double integral is expressed as the sum of two or more double integrals. Some times the integral becomes easier with the change in the order of integration.

Note: If the field of integration is a rectangle with sides $x = a, x = b, y = c, y = d$, then the limits of integration for x and y in the double integral remain the same when the order of integration is changed i.e.,

$$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dy\, dx.$$

If $f(x, y)$ can be expressed as the product of a function of x and a function of y, the double integral is simply the product of two difinite integrals, i.e.,

$$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \phi(x)\, dx \int_c^d \psi(y)\, dy$$

if $f(x, y) = \phi(x) \,.\, \psi(y)$

Examples

Ex 1. By changing the order of integration evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$

Sol: Let $I = \int_0^\infty dx \int_x^\infty \frac{e^{-y}}{y}\, dy$

Integrate w.r.t. y from x to ∞ and then w.r.t. x from 0 to ∞. Let OA be the stright

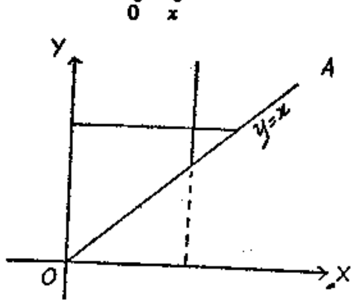

Fig-6

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line $y = x$. Region of integration is R, above OA. Reverse the order of integration. Keep y constant, x varies for 0 and y. Then allow y to vary from 0 to ∞ to cover R.

$$\text{Hence } I = \int_0^\infty e^{-y}\frac{dy}{y}\int_0^y dx = \int_0^\infty \frac{e^{-y}}{y}dy\,[x]_0^y = \int_0^\infty e^{-y}\,dy = 1.$$

Ex 2. Change the order of integration in $\int_0^a \int_0^{\frac{b}{b+x}} f(x, y)\, dx\, dy$

Sol. The region of integration is given by

$$y = 0, y = \frac{b}{b+x}, x = 0\ x = a.$$

The equation $y(b + x) = b$ represents an hyperbola. Hence the approximate shape of the region of integration is given in the Fig.7. The region of integration is in the shape $OADB$. When the strips are taken parallel to x- axis, they change their character at the point D. Hence the space $OADB$ is to be considered as the sum of two spaces, i.e., $OADC$ and CDB. For space $OADC$ we must integrate from $x = 0$ to $x = a$, and then from

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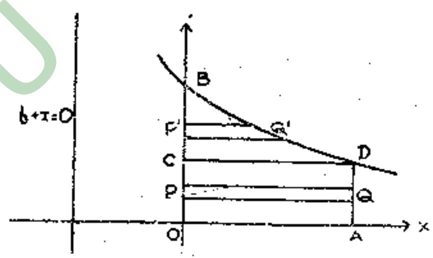

Fig-7.

$y = 0$ to $y = \frac{b}{b+a}$. For the space CDB we must integrate from $x = 0$ to $x = \frac{b(1-y)}{y}$ and then from $y = \frac{b}{b+a}$ to $y = 1$. Therefore the transformed integral is

$$\int_0^{\frac{b}{b+x}}\int_0^a f(x, y)\, dy\, dx + \int_{\frac{b}{b+a}}^1 \int_0^{\frac{b(1-y)}{y}} f(x, y)\, dy\, dx.$$

10.6 Summary

Double and Triple integrals were defined in a similar manner as that of a single integral. The geometrical meanings of Double and Triple integrals are given. Some times the evaluation of double or triple integrals becomes easier with the change in the order of integration.

10.7 Sample Examination Questions

I. *Answer the following questions in detail*

 (i) a) Interpret the double integral geometrically

b) Evaluate $\int_0^a \int_0^o (x^2 + y^2)\, dx\, dy$.

(ii) a) Explain the process of triple integratior

b) Evaluate $\iiint (x + y + z)^2)dx\, dy\, dz$ over the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

II. Briefly answer the following

(i) Find the value of $\int_1^2 \int_1^x xy^2\, dy\, dx$

(ii) Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \sin\theta\, dr\, d\theta$

(iii) Find the volume of the tetrahedron bounded by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes

(iv) Evaluate $\int_0^{\pi a/2} \int_0^{\pi b/2} \sin\left(\frac{x}{a} + \frac{y}{b}\right) dx\, dy$

(v) Integrate $(x^2 + y^2)$ over the circle $x^2 + y^2 = a^2$

(vi) Show that the double integral of $x^3 + y^3$ over the triangle bounded by $y = 0, y = 2x$ and $x = a$ is equal to $\frac{6}{5} a^5$.

(vii) Evaluate $\iint (x + y)^2\, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(viii) When the field of integration in the triangle given by $y = 0, y = x$ and $x = 1$ show that

$$\iint \sqrt{(4x^2 - y^2)}\, dx\, dy = \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

(ix) Change the order of integration in the integral

$$\int_0^a \int_{x^2/a}^{2a-x} f(x, y)\, dx\, dy$$ and evaluate it.

(x) By means of substitution $x + y + z = U, y + z = UV, z = UVW$ or otherwise show that

$$\iiint \sqrt{\frac{1-x-y-z}{xyz}}\, dx\, dy\, dz$$ taken over the volume bounded by

$x = 0, y = 0, z = 0$ and $x + y + z = 1$ is $\pi^2/4$.

III Answer the following in about 5 lines.

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(i) Define field of integration

(ii) Comment on the change of integration.

Answers

I (ib) $\frac{1}{3}\, ab\,(a^2+b^2)$

II (i) $\frac{9}{4}$ (ii) $\frac{4\pi a^3}{3}$ (iii) $\frac{abc}{6}$ (iv) xab (v) $\frac{\pi a^4}{2}$

(vii) $\frac{1}{4}\,\pi ab\,(a^2+b^2)$ (ix) $\int_0^a \int_0^{\sqrt{ay}} f(x,y)\,dy\,dx + \int_a^{2a} \int_0^{2a-y} f(x,y)\,dy\,dx$

10.8 Solutions to SAQ's

SAQ 1: $$\iint_R x\cos xy\;dx\,dy = \int_{x=1}^{x=2}\left(\int_{y=\pi/2}^{2\pi/x} x\cos xy\;dy\right)dx.$$

$$= \int_{1}^{2} x\left[\frac{\sin xy}{x}\right]_{y=\pi/2}^{y=2\pi/x} dx.$$

$$= \int_1^2 \left(\sin 2\pi - \sin\frac{\pi}{2}x\right)dx.$$

$$= \int_1^2 (-\sin \pi x/2)\,dx = \frac{2}{\pi}\left[\cos \pi x/2\right]_1^2$$

$$= \frac{2}{\pi}\cos\pi - \frac{2}{\pi}\cos\ \pi/2 = -\frac{2}{\pi}$$

SAQ 2: $$\iiint_S (x+y+z)\,dx\,dy\,dz = \iint_R\left[\int_{z=0}^{z=2-x-y}(x+y+z)dz\right]dx\,dy$$

$$= \iint_R \left[\left(xz + yz + \frac{z^2}{2}\right)_0^{z=2-x-y}\right]$$

$$= \iint_R \left[x\,(2-x-y) + y\,(2-x-y) + \frac{(2-x-y)^2}{2}\right]dx\,dy$$

$$= \iint_R \left(2 - \frac{x^2}{2} - xy - \frac{y^2}{2}\right)dx\,dy$$

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$$= \int_{x=0}^{1} \left[\int_{y=0}^{1-x} \left(2 - \frac{x^2}{2} - xy - \frac{y^2}{2} \right) dy \right] dx$$

$$= \int_{x=0}^{1} \left[\left(2y - \frac{x^2 y}{2} - \frac{xy^2}{2} - \frac{y^3}{6} \right)_{y=0}^{y=1-x} \right] dx$$

$$= \int_{x=0}^{1} \left(\frac{11}{6} - 2x + \frac{x^3}{6} \right) dx$$

$$= \left(\frac{11}{6} x - x^2 - \frac{x^4}{24} \right)_0^1 = \frac{7}{8}$$

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Block -4: APPLICATIONS OF INTEGRATION

Introduction

In Block-3 we discussed integration. In many branches of mathematics we notice the applications of integration. In this Block, our intension is to acquaint you with some applications of integration.

In units 11,12 and 13 we discuss geometrical applications and in unit 14 few applications to Mechanics. Unit 11 deals with the evaluation of areas and volumes. In unit 12 we try to get the expressions for the area of surface of revoluation in cartesian and polar coordinates and in unit 13 we discuss the process of determination of the lengths of arcs of plane curves whose equations are known.

Finally, we take up the applications of integration to Mechanics, in particular to study to concepts of Centre of Gravity and Moment of Intetia in unit 14.

Unit -11 : Evaluation of Areas and Volumes

11.0 Contents

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11.1 Aims and Objectives

After goingthrough this unit you must able to evaluate the areas and volumes of given figures and bodies using the methods of integration.

11.2 Introduction

One of the most important classes of problems in the 17th century was concerned with the determination of areas and volumes bounded by curves and surfaces. Elementary mathematics can deal with evaluation of areas of simple figures like square, rectangle or a regular polygon. But, even a simple looking geometric figures like circle can not be dealt with easily. From Greek times until 17th century, very little progress was made in finding areas and volumes bounded in whole or in part by curved figures. With the advent of calculus the idea of definite integral has been exploited to evaluate the area under a curve bounded by either x or y - axis.

To determine volumes of solids in the form of cubes and parallelopipeds is very easy with the use of elementary mathematics. Even with the advent of caluclus we can not easily find volumes all solids. We can find, however, volumes of solids that are generated by revolving a curve around an axis. Such solids are called solids of revolution. For example, a sphere is generated by revolving a semicircle around its diameter.

11.3 Area

In Block-3, Unit-9, we have seen that $\int_a^b f(x)\ dx$ represents the area bounded by a curve $y = f(x)$, the x-axis and two ordinates $x = a$ and $x = b$.

Similarly $\int_c^d f(y)\ dy$ represents the area bounded by a curve $x = f(y)$, the y-axis and the two abscissae $y = c$ and $y = d$.

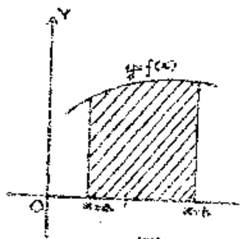

Fig-1.

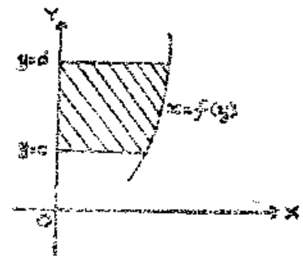

Fig-2.

If $f(x)$ is partly +ve and partly –ve in $[a, b]$,then the curve crosses the x-axis at some point 'c'.

$$\therefore \text{ The required area of the curve} = \int_a^c f(x)\ dx - \int_c^b f(x)\ dx .$$

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The area of the region bounded by the lines $x = a$, $x = b$ and two curves $y = f(x)$ and $y = g(x)$ is

$$\int_a^b [f(x) - g(x)]\ dx.$$

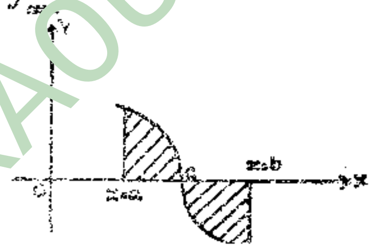

Fig-3.

If the two curves $y = f(x)$ and $y = g(x)$ intersect at some point, then first solve them and find the point of intersection. These points are limits of integration.

The process of determining the area of a plane region is known as quadrature.

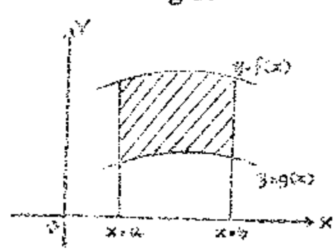

Fig-4.

Examples

Ex.1 Find the area of the circle $x^2 + y^2 = a^2$

Sol. The circle $x^2 + y^2 = a^2$ is symmetrical about both the coordinate axes, so that the two axes divide into four equal parts.

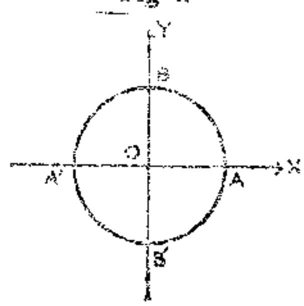

Fig-5.

We determine the area of the portion OAB, which is in the first quadrant. Multiplying by 4, we get the required area of the circle.

Area of OAB is the area formed by the curve AB, x - axis and the ordinates $x = 0$ and $x = a$.

$x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$

$$\therefore \text{Area of } OAB = \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \cdot \sin^{-1}\frac{x}{a}\right]_0^a = \frac{\pi a^2}{4}$$

$\therefore$ Area of the circle $= 4 \times \frac{\pi a^2}{4} = \pi a^2$. Sq. units.

Ex2. Prove that the area of the curve $a^4 y^2 = x^5 (2a - x)$ is $\frac{5a^2\pi}{4}$.

Sol. Tracing the curve, we note that the curve consists of a loop lying between the lines $x = 0$ and $x = 2a$ and is symmetrical about the x-axis.

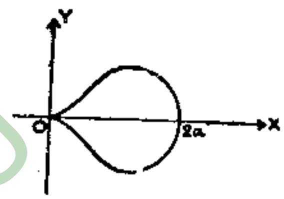

Fig-6.

$a^4 y^2 = x^5 (2a - x)$

$\therefore y = \frac{1}{a^2} x^{5/2} \sqrt{2a - x}$

The area of the curve $= 2\int_0^a \frac{1}{a^2} x^{5/2} \sqrt{2a - x}\, dx$

$= \frac{5a^2\pi}{4}$ (putting $x = 2a\sin^2\theta$)

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Ex3. Find the area bounded by the curve $y^2 = x$ and the line $x = 4$.

Sol. The curve is symmetrical about the x-axis. From the equation, the curve passes through (0,0).

The points of intersection of $y^2 = x$ and $x = 4$ are (4, 2) (4, –2).

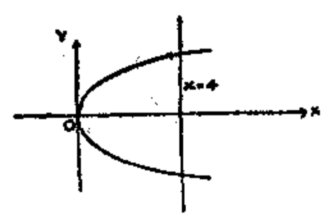

Fig-7.

$$\therefore \text{The required area} = \int_0^4 \sqrt{x}\, dx - \int_0^4 -\sqrt{x}\, dx .$$

$$= 2\int_0^4 \sqrt{x}\, dx = \frac{32}{3} \text{ Sq. units.}$$

Ex.4 : Find the area, lying above x-axis, and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

Sol. Solving $x^2 + y^2 = 2ax$ and $y^2 = ax$, we get $(0, 0)$, (a, a) and $(a, -a)$ as points of intersection.

Area of $OAPB$ = Area of $OBPCO$ – Area of $OAPCO$

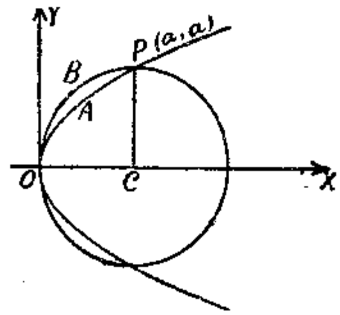

Fig-8.

$$\text{Area of } OBPCO = \int_0^a \sqrt{(2ax - x^2)}\,dx \quad -\text{(A)}$$

$$\text{Area of } OAPCO = \int_0^a \sqrt{(ax)}\,dx = \frac{2a^2}{3} \quad -\text{(B)}$$

Now consider (A),

$$\int_0^a \sqrt{(2ax - x^2)}\,dx = \int_0^a \sqrt{(a^2 - (a-x)^2)}\,dx$$

put $a - x = a\sin\theta$ $\quad\therefore -dx = a\cos\theta\,d\theta.$

$$\therefore \int_0^a \sqrt{(2ax - x^2)}\,dx = \int_{\pi/2}^0 a\cos\theta\,(-a\cos\theta)\,d\theta.$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\,d\theta = \frac{\pi a^2}{4}.$$

$$\therefore \text{Required area} = \frac{\pi a^2}{4} - \frac{2a^2}{3} = a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

Ex.5: Find the area between the parabola $y^2 = 4x$ and $y = 2x - 4$.

Sol. Solving $y^2 = 4x$ and $y = 2x - 4$, the points of intersection are $B(4, 4)$ and $A(1, -2)$. Trace the curves. Draw BP and AQ perpendiculars to y-axis.

$\therefore$ The required area = Area of $AQPB$ – (Area of BPO + Area of AQO)

From $y^2 = 4x \Rightarrow x = \dfrac{y^2}{4}$

$y = 2x - 4 \Rightarrow x = \dfrac{y+4}{2}$

$$\therefore \text{Required area} = \int_{-2}^{4}\left(\frac{y+4}{2}\right)dy - \int_{-2}^{4}\frac{y^2}{4}\,dy$$

$= 9$ Sq. units.

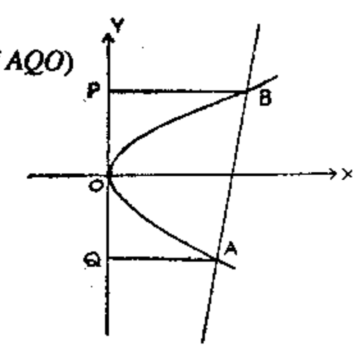

Fig – 9

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11.4 Volumes

Let AB be any arc of a curve and let PQ be a straight line which does not intersect the curve. Draw AC and BD perpendiculars to the line PQ.

A solid will be obtained if the region $ACDBA$ revolves about the line PQ. This solid is obtained by the revolution of the arc AB about the line PQ. The line PQ about which the curve rotates is called the axis of revolution. Our aim is to get expressions for the volume of this solid. First we consider the cases where x-axis is the axis of revolution and then we consider the cases when any line is the axis of revolution.

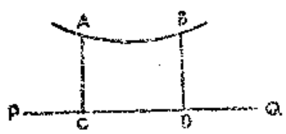

Fig-10.

Now we show that the volume obtained by revolving about x-axis, the arc of the curve $y = f(x)$, intercepted between the points whose abscissae are a, b is $\int_a^b \pi y^2 \, dx$

i.e., $\int_a^b \pi [f(x)]^2 \, dx$, assuming that the arc does not cut x - axis.

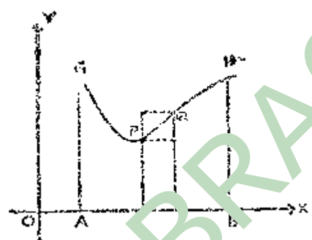

Fig-11(a)

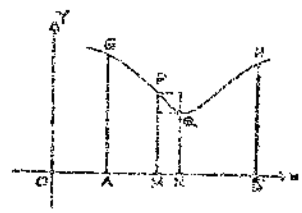

Fig-11(b).

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Let G, H be points on the curve $y = f(x)$ with the abscissae a and b. Let $P(x, y)$ be any variable point on the curve. Let V be the volume of the solid obtained by revolving the arc GP about x-axis. Therefore, V is a function of x. Consider another point $Q(x + \delta x, y + \delta y)$ on the curve so near to P that as a point moves on the curve from P to Q, its ordinate either constantly increases (Fig.11 (a)) or constantly decreases (Fig. 11(b)). Complete the rectangles $MPNQ$.

Now δV, which is the volume obtained by revolving the arc PQ, lies between the volumes of the two discs obtained by revolving the two rectangles $MPNQ$ about x-axis.

From Fig. 11(a), $\pi y^2 \delta x < \delta V < \pi (y + \delta y)^2 \delta x$.

i.e., $\pi y^2 < \dfrac{\delta V}{\delta x} < \pi (y + \delta y)^2$.

Considering the limit as $\delta x \to 0$, we get $\dfrac{dV}{dx} = \pi y^2$.

From Fig.11(b), $\pi (y + \delta y)^2 \delta x < \delta V < \pi y^2 \delta x$.

i.e., $\pi (y + \delta y)^2 < \dfrac{\delta V}{\delta x} < \pi y^2$.

$\therefore \dfrac{dV}{dx} = \pi y^2$.

$$\therefore \int_a^b \pi y^2 dx = \int_a^b \frac{dV}{dx} dx = [V]_a^b = V(b) - V(a).$$

= the volume of the solid obtained by revolving the arc GH about x-axis.

Similarly, we can show that volume obtained on revolving about y-axis, the arc of the curve $x = f(y)$ intercepted between the points whose ordinates are a and b is $\int_a^b \pi x^2 dy$, assuming that the arc does not cut y-axis.

11.5 Any axis of revolution

Let 'y' be the length of the perpendicular PM of any point 'P' on the curve from the axis of revolution, M be the foot of the perpendicular, x denotes the distance of the foot of the perpendicular M from a fixed point 'O' on the axis, a and b are the distances from the fixed point 'O' of the feet A, B of the perpendiculars from the extreme ends G, H of the given arc.

From this we deduce (using previous result) that the volume obtained by revolving the arc GH about the line AB is

$$\int_{OA}^{OB} \pi (MP)^2 \, d(OM)$$

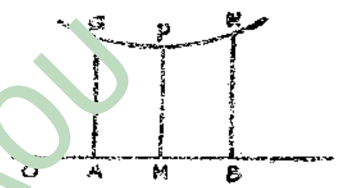

Fig-12

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Examples

Ex.1 : Find the volume of the solid obtained by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x – axis.

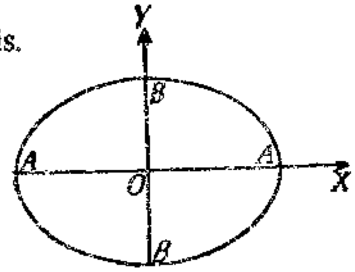

Fig - 13

Sol : The solid obtained by revolving the arc ABA' about x – axis is same as the solid obtained by revolving the whole ellipse and the volume of this solid is double the volume of the solid obtained by revolving the arc AB.

$$\therefore \text{ The required volume } = 2 \int_0^a \pi y^2 dx = 2\pi \frac{b^2}{a^2} \int_0^a (a^2 - x^2)\, dx = \frac{4}{3} \pi\, ab^2.$$

Ex. 2 : Find the volume of the solid obtained by revolving the cardioide $r = a\ (1 + \cos\theta)$ about the initial line.

Sol : The required volume $= \pi \int_0^{2a} y^2 dx$

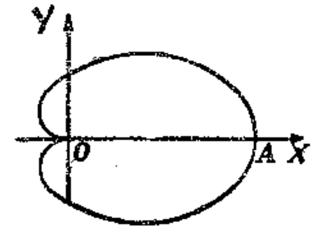

Fig - 14

Now change the variable x to θ,

$x = r \cos\theta$

$= a(1 + \cos\theta)\cos\theta$

$$y = r\sin\theta = a(1+\cos\theta)\sin\theta.$$

$$\therefore\ dx = -a(\sin\theta + 2\sin\theta.\cos\theta)\,d\theta.$$

$$\text{When } x = 0,\ \theta = \pi$$

$$x = 2a,\ \theta = 0.$$

$$\therefore \text{ Required volume} = \pi \int_0^{2a} y^2\,dx$$

$$= -\pi \int_\pi^0 a^2 \sin 2\theta.\ (1+\cos\theta)^2\, a\sin\theta.\ (1+2\cos\theta)\,d\theta$$

$$= 128\,\pi\, a^2 \int_0^\pi \sin^3\frac{\theta}{2}.\ \cos^9\frac{\theta}{2}\,d\theta$$

$$-32\,\pi\, a^3 \int_0^\pi \sin^3\frac{\theta}{2}\cos^7\frac{\theta}{2}\,d\theta$$

$$\text{Let } \phi = \frac{\theta}{2}$$

$$= 256\,\pi\, a^3 \int_0^{\pi/2} \sin^3\phi\ \cos^9\phi\,d\phi$$

$$-64\,\pi\, a^3 \int_0^{\pi/2} \sin^3\phi\ \cos^7\phi\ d\phi.$$

$$= \frac{8\,\pi\, a^3}{3} \text{ (using the reduction formula)}$$

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11.6 Summary

(i) The area bounded by the curve $y = f(x)$, x – axis and the two ordinates $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.

(ii) The area bounded by $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, is given by $\int_a^b [f(x) - g(x)]\,dx$

(iii) The volume obtained by revolving the curve $y = f(x)$ about x –axis, intercepted between the points whose abscissae are a, b is given by $\int_a^b \pi\, y^2 dx$

11.7 Sample examination Questions

I *Answer the following in detail.*

(i) a) Explain the process 'Quadrature'. What is the area of the region bounded by the lines $x = a$, $x = b$ and two curves $y = f(x)$ and $y = g(x)$?

b) Find the area enclosed by the curves $x^2 = 4ay$ and $x^2 + 4a^2 = \frac{8a^3}{y}$

(ii) a) Show that the volume obtained by revolving the curve $y = f(x)$, interecepted between the points whose abscissae are a and b, about x - axis is $\int_a^b \pi\, y^2 dx$

b) Find the volume generated by the portion of the arc $y = \sqrt{1 + x^2}$, lying between $x = 0$ and $x = 4$ as it revolves about the x – axis.

II. *Briefly answer the following.*

(i) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(ii) Find the area between the curves $y = \sin x$ in the interval $(0, 2\pi)$.

(iii) Find the area between the curves $y = \sqrt{x}$ and $y = x$

(iv) Find the area enclosed between one arc of cycloid $x = a\,(\theta - \sin\theta)$; $y = a\ (1 - \cos\theta)$ and its base.

(v) Find the area enclosed by the curve $xy^2 = 4\,(2 - x)$, and y – axis.

(vi) Find the volume of the solid obtained by revolving one arc of the cycloid $x = a\,(\theta + \sin\theta)$, $y = a\,(1 + \cos\theta)$ about x - axis.

(vii) The area enclosed by the parabolas $x^2 = 4ay$ and $x^2 = 4a\ (2a - y)$ revolves about the line $y = a$, find the volume of the solid so generated.

(viii) Show that the volume obtained by revolving about $x = \frac{a}{2}$ of the area enclosed between the curves $xy^2 = a^2\,(a - x)$, $(a - x)\,y^2 = a^2x$ is $\frac{\pi a^3\,(4 - \pi)}{4}$.

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Answers

I (ib) $\frac{2}{3}\,(3\pi - 2)\,a^2$ (iib) $76\ \pi/3$.

II (i) $\pi\ ab$ (ii) 4 (iii) $\frac{1}{3}$ (iv) $3\,a^2\pi$. (v) 4π (vi) $5\pi^2\,a^3$ (vii) $\frac{76\pi}{3}$ (viii) $\frac{32\pi a^3}{15}$

Unit -12 : Areas of Surfaces of Revolution in Cartesian and Polar Coordinates

12.0 Contents

12.1 Aims and Objectives.

After going through this unit you will be able to evaluate the area of the surface of revolution in cartesian and polar coordinates.

12.2 Introduction

In Unit 11, we discussed about the volume of a solid of revolution when (1) the axis of revolution is x - axis (2) any axis of revolution. A solid is said to be obtained by the revolution of the arc about any line. We got expressions for the volume of a solid of revolution. The area of the surface of a solid is said to be Area of surface of revolution. Now our intention is to get the expressions for the Area of surface of revolution.

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12.3 Area of the surface of the frustum of a cone

We know the area of the surface of a right circular cone, the radius of whose circular base is 'r', and whose slant height is 'ρ' is $\pi r \rho$.

If we tear a right circular cone along one of its generators, we get a circular sector whose radius OA is equal to the slant height and whose arc is equal to the circumference of the cicular base of the cone. The area of this sector = The surface of the cone.

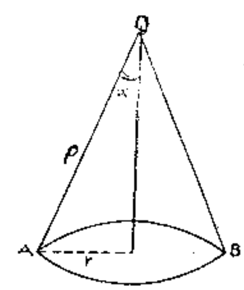

Fig. 1

Let 'α' be the semi - vertical angle of the cone.

$$\therefore \ \sin\alpha = \frac{r}{\rho}$$

$\therefore$ Area of the surface of the cone $= \pi \rho \sin\alpha \ \rho = \pi \rho^2 \sin\alpha$.

Consider the frustum $CABD$ of a cone OAB. Let $O''A = r_1, O'C = r_2$ and slant height of the frustum $= CA = \rho_1$.

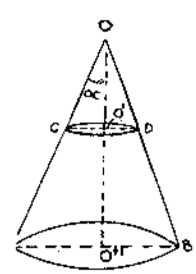

Fig - 2

$\therefore$ The area of the surface of this frustum

$= \pi \ (AO^2 - CO^2) \ \sin\alpha$

$= \pi \ (AO - CO)\,(AO + CO) \ \sin\alpha$

$= \pi \ (AC) \ (AO \sin\alpha + CO \sin\alpha)$

$= \pi\rho \ (AO'' + CO') = \pi\rho \ (r_1 + r_2)$

$= \pi$ x Slant height x Sum of radii of the two bases.

12.4 Surface of revolution

Now we show that the area of the surface of the solid obtained by revolving about x – axis; the arc of the curve $y = f(x)$ intercepted between the points whose abscissae are a, b is

$$\int_a^b 2\pi y \frac{dS}{dx}\, dx = 2\pi \int_a^b f(x) \sqrt{\left(1 + \{f'(x)\}^2\right)}\, dx.$$

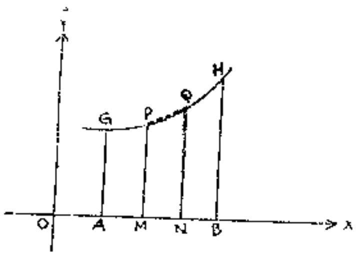

Fig - 3

Let $G(a, f(a))$ be a fixed point and $P(x, y)$, a variable point on the arc. Let H be the point $[b, f(b)]$. Let σ be the area of the surface of the solid obtained by revolving the arc GP about x-axis so that 'σ' is a function of x. Let $Q(x + \delta x, y + \delta y)$ be the point on the curve lying near P. $\delta\sigma$ is the area of the surface of the solid obtained by revolving the arc PQ.

Let arc $GP = S$ and arc $PQ = \delta S$.

By revolving the chord PQ about x - axis, we get a frustum of the cone whose slant hight is PQ and the radii of whose circular ends are MP, NQ.

$\therefore$ The area of the surface of this frustum $= \delta\Sigma$.

$$= \pi. PQ. (PM + QN)$$

$$= \pi. PQ. (y + y + \delta y)$$

We take it as an axiom that

$$\text{Lim} \frac{\delta\Sigma}{\delta x} = \text{Lim} \frac{\delta\sigma}{\delta x}. \text{ i.e., } \frac{d\Sigma}{dx} = \frac{d\sigma}{dx}$$

$$\therefore \frac{\delta\Sigma}{\delta x} = \pi \ (y + y + \delta y) \frac{PQ}{\delta x}$$

$$= \pi (2y + \delta y). \frac{\text{Chord } PQ}{\text{arc } PQ}. \frac{\text{arc } PQ}{\delta x}$$

$$\therefore \text{Lim} \frac{\delta\Sigma}{\delta x} = \frac{d\Sigma}{dx} = 2\pi y. 1. \frac{dS}{dx} = 2\pi y \frac{dS}{dx}$$

$$\text{and } \frac{d\sigma}{dx} = 2\pi y \frac{dS}{dx}.$$

$$\therefore \int_a^b 2\pi y \frac{dS}{dx}\, dx = \int_a^b \frac{\delta\sigma}{\delta x}\, dx = [\sigma]_a^b$$

$=$ Area of the surface obtained by revolving the arc GH.

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12.5 Area of surface of revolution in polar coordinates

If we take 'S' as the independent variable, then the surface area $= \int 2\pi y\, dS$. In the case of the polar curve $r = f(\theta)$, where θ is the independent variable, the surface area is equal to

$$\int 2\pi y \frac{dS}{d\theta}\, d\theta.$$

In the case of the curve $x = f(t)$ and $y = \phi(t)$, where 't' is the parameter, surface area is equal to $\int 2\pi y \frac{dS}{dt}\, dt.$

12.6 Theorem of Pappus on surface of revolution

If a closed plane curve revolves about a straight line in its plane (the straight line not intersecting the curve), then the surface of the solid of revolution thus formed is obtained on multiplying the length of the curve with that of the path described by the centroid of the curve.

Proof : Take x - axis as the axis of revolution. If 'ρ' denotes the length of the arc of the curve and if y is a function of S, then the surface S of the solid of revolution is given by

$$S = \int_0^{\rho} 2\pi y\, dS \quad ..(1)$$

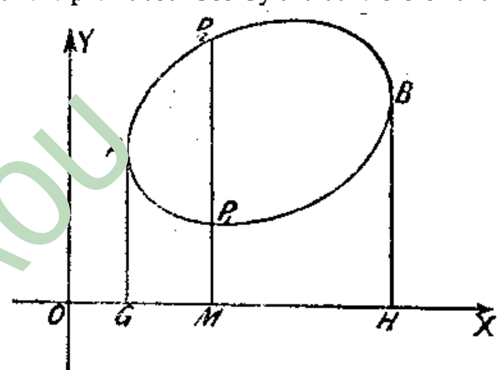

Fig - 4

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The centroid $\overline{y}$, of the curve is given by $\overline{y} = \int_0^{\rho} y\, dS/\rho$ (2)

From (1) and (2), $S = 2\pi \overline{y} \rho$, where $2\pi \overline{y}$ is the length of the path described by the centroid.

Hence the result.

Examples

Ex. 1 : Find the area of the whole surface of a sphere of radius 'a'.

Sol : Let the sphere be formed by the rotation of the circle $x^2 + y^2 = a^2$ about the axis of x. Let 'θ' be the inclination of the radius OP to the axis of x, then the coordinates of P are $(a\cos\theta, a\sin\theta)$ and the length of the arc 'S' from A to P is $a\theta$.

The whole surface is twice the surface generated by the rotation of AB.

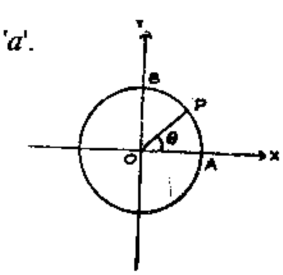

Fig - 5

$$= 2\int_0^{\pi/2} 2\pi y \frac{dS}{d\theta} . d\theta.$$

$$= 4\pi \int_0^{\pi/2} a \sin\theta.\, a\, d\theta = 4\,\pi a^2 \int_0^{\pi/2} \sin\theta\, d\theta$$

$$= 4\,\pi a^2.$$

Ex. 2 : Find the surface of the solid formed by revolving the cardioide $r = a\,(1 + \cos\theta)$ about the initial line.

Sol : The required surface $= \int_0^{\pi} 2\,\pi\, y\, \frac{dS}{d\theta}.\, d\theta.$

$$\text{Now } y = r \sin\theta$$

$$= a\,(1 + \cos\theta)\sin\theta.$$

$$\text{and } \frac{dS}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2\,(1 + \cos\theta)^2 + a^2 \sin^2\theta} = 2a \cos\frac{\theta}{2}.$$

$$\therefore \text{ The required surface} = \int_0^{\pi} 2\,\pi\, a\, \sin\theta\,(1 + \cos\theta)\, 2a.\cos\frac{\theta}{2}.\, d\theta$$

$$= 4\,\pi a^2 \int_0^{\pi} \sin\theta.\,(1 + \cos\theta)\cos\frac{\theta}{2}\, d\theta$$

$$= 4\,\pi a^2 \int_0^{\pi} 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}.\, 2\cos^2\frac{\theta}{2}.\cos\frac{\theta}{2}\, d\theta$$

$$\text{Let } \phi = \frac{\theta}{2}$$

$$= 32\,\pi a^2 \int_0^{\pi/2} \sin\phi.\cos 4\,\phi\, d\phi$$

$$= \frac{32}{5}\,\pi\, a^2.$$

Ex. 3 : Evaluate the surface area of the solid generated by revolving the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about the line $y = 0$.

Sol : We know that

$$\frac{dS}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{a^2\,(1 + \cos\theta)^2 + a^2 \sin^2\theta}$$

$$= 2\,a \sin\frac{\theta}{2}.$$

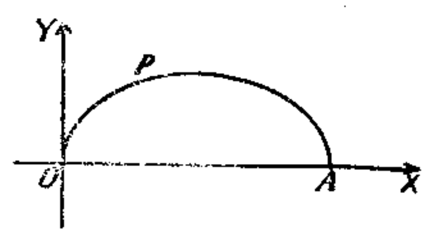

Fig - 6

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For the part OPA of the cycloid $x = a(\theta - \text{Sin}\,\theta)$, θ varies from 0 to 2π.

$$\therefore \quad \text{The required surface} = \int_0^{2\pi} 2\pi y \cdot \frac{dS}{d\theta} \cdot d\theta$$

$$= \int_0^{2\pi} 2\pi a\,(1 - \cos\theta)\, 2a.\sin\frac{\theta}{2} \cdot d\theta$$

$$= 8\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\, d\theta$$

$$\text{Let } \phi = \frac{\theta}{2}$$

$$= 16a^2 \int_0^{\pi} \sin^2\phi\, d\phi = \frac{64\pi a^2}{3}.$$

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12.7 Summary

The expression for the area of the surface of revolution in polar and cartesian coordinates have been obtained. The theorem of pappus on surface of revolution is proved.

12.8 Sample Examination Questions

I *Answer the following questions in detail.*

(i) a. Obtain the expression for the area of the surface of the frustum of a cone.

b. Find the surface of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about x – axis.

(ii)a. State and prove Pappus theorem on surface of revolution.

b. Find the surface of the solid generated by revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ about the axis of x.

II. *Briefly answer the following.*

(i) Find the surface of the solid obtained by revolving the cardioide $r = a(1 - \cos\theta)$ about the initial line.

(ii) Prove that the surface of the solid obtained by revolving the ellipse $e^2x^2 + a^2y^2 = a^2b^2$ about the axis of x is

$$2\pi ab\left[\sqrt{(1 - e^2)} + \frac{1}{e}\sin^{-1} e\right]$$

where 'e' is the eccentricity of the ellipse.

(iii) Show that the surface of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about x – axis is $\pi^2\left[\sqrt{2} + \log\,(1 + \sqrt{2})\right]$.

(iv) Prove that the surface and the volume of the solid generated by the revolution about the x - axis, of the loop of the curve $x = t^2,\ y = t - \frac{t^3}{3}$ are respectively 3π and $\frac{3}{4}\pi$.

(v) Find the surface area of the solid formed by the revolution about the initial line of the curve $r^2 = a^2 \cos\theta + b^2 \sin\theta\ (a^2 > b^2)$.

(vi) Find the area of the surface of the solid formed by the rotation about the axis of x of the parabola $y^2 = 16x$ from $x = 5$ to $x = 12$.

(vii) The arc of the catenary $y = c \cosh\left(\frac{x}{c}\right)$ between $x = -c$ and $x = c$ revolves about the axis of x, find the area of the surface generated.

(viii) The part of the curve $y = e^x$ from $x = 0$ to $x = -\infty$, rotates about the x – axis. Find the area of the surface described.

Answers :

I (ib) $\frac{8a^2\pi(2\sqrt{2}-1)}{3}$ (ii b) $\frac{12\pi a^2}{5}$

II (i) $\frac{32\pi a^2}{5}$

(v) $\pi\left[a^2 + \frac{b^4}{\sqrt{a^4-b^4}} \log \frac{a^2+\sqrt{a^4-b^4}}{b^2}\right]$

(vi) 620 (vii) $\pi c^2 (2 + \sinh 2)$ (viii) $\pi\left[\sqrt{2} + \log\left(\sqrt{2}+1\right)\right]$

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Unit – 13: Rectification

13.0 Contents

13.1 Aims and Objectives

After going through this unit you must be able to determine the length of arc of plane curves whose equations are given.

13.2 Introduction

The notion of length presents no fundamental conceptual difficulties. For measuring the length of a line segment, we first determine the unit length and measure how many unit lengths and fraction of unit lengths the given line segment has. Even here it is not easy since the resulting number may be an irrational number.

The problem of determining the length of a curve can not be approached this way. The reason is that curves change shape from one place to another and hence it is difficult to find a unit length whose shape will fit along the curve. We once again use definite integral to determine the length of a curve.

The process of determination of the lengths of arcs of plane curves whose equations are given in the cartesian, parametric or polar form is known as rectification.

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We have the following formulae which we will use now

1. $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

2. $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$

3. $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

4. $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

13.3 Cartesian equations $y = f(x)$

We prove that the length of the arc of the curve $y = f(x)$ included between two points whose abscissae are a and b is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Let A, B be the two points with abscissae a, b on the curve $y = f(x)$. Let 'S' denotes the length of the arc of the curve included between a fixed point A and a variable point P whose abscissae is x, so that it is a function of x.

We have $\frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\therefore \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \frac{dS}{dx} dx = [S]_a^b$$

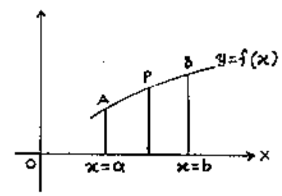

Fig. 1

= Value of S for $x = b$ – value of S for $x = a$

= Arc AB

Hence the result.

13.3.1 Cartesian equation $x = f(y)$

The length of the arc of the curve $x = f(y)$, included between two points whose ordinates are c, d is

$$= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad = \int_c^d \sqrt{1 + (f'(y))^2}\, dy$$

13.3.2 Parametric cartesian equations $x = f(t), y = \phi(t)$

The length of the arc of the curve $x = f(t), y = \phi(t)$ included between two points whose parametric values are α, β is

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_\alpha^\beta \sqrt{f'^2(t) + \phi'^2(t)}\, dt.$$

13.3.3 Polar equations $r = f(\theta)$

The length of the arc of the curve $r = f(\theta)$ included between two points whose vectorial angles are α and β is

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

Now let us consider few examples based on the above results.

Examples :

Ex. 1: Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum.

Sol. The abscissae of the extremity L of the latus rectum is $2a$.

From $x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$

$$\therefore \frac{dy}{dx} = \frac{x}{2a}$$

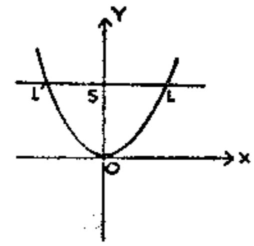

Fig.2

$$\therefore \text{ The required length } = \int_0^{2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{1}{2a} \int_0^{a} \sqrt{\left(4a^2 + x^2\right)}\, dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{x^2 + 4a^2}}{2} + 2a^2 \sinh^{-1}\frac{x}{2a}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[2\sqrt{2}\, a^2 + 2a^2 \sinh^{-1} 1\right]$$

$$= a\left[\sqrt{2} + \sinh^{-1} 1\right]$$

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Ex. 2: Rectify the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

Sol.: The parameter θ increases from $-\pi$ to π as a point moves from one end A' to other end A of one arc. $\theta = 0$ for the vertex 'O'. Since the arc is symmetrical about OY arc AOA′ = 2 arc OA.

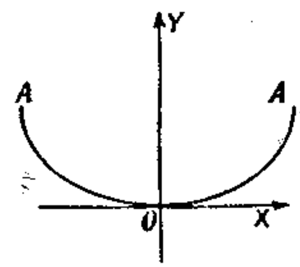

Fig. 3

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \quad \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{2a^2(1 + \cos\theta)}$$

$$= \sqrt{4a^2 \cos^2\frac{\theta}{2}}$$

$$= 2a \cos\frac{\theta}{2}$$

$$\therefore \text{ The length of the arc } = 2\int_0^{\pi} 2a \cos\frac{\theta}{2}.\, d\theta$$

$$= 2\left[4a \sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

Ex. 3: Find the perimeter of the cardioide $r = a(1 - \cos\theta)$

Sol: The curve is symmetrical about the initial line, and, therefore its perimeter is double the length of the arc of the curve lying above the same.

$$\frac{dr}{d\theta} = a\sin\theta$$

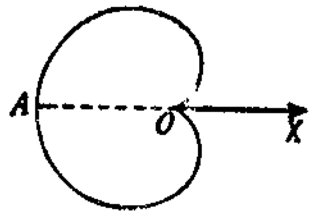

Fig. 4

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{2a^2(1-\cos\theta)} = 2a\sin\frac{\theta}{2}.$$

$$\therefore \text{The required length} = 2\int_0^{\pi} 2a\sin\frac{\theta}{2}.\,d\theta = 8a.$$

13.4 Intrinsic equations of a curve

If 'S' denotes the length of the arc of a curve measured from some fixed point A to a variable point P, and ψ denotes the angle between the tangents at A and P, then a relation between 'S' and ψ is called an **Intrinsic equation** of the curve. 'S' and ψ are called the intrinsic coordinates.

S and ψ for any point depend only upon the form of the curve and not on its position in the plane, so that they are inherently associated with the curve. That is why 'S' and 'ψ' are called intrinsic. Intrinsic coordinates of any point on a curve will not change if the curve changes its position in the plane. This is not the case for the ordinary cartesian or polar coordinates.

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13.4.1 Derivation of Intrinsic equations from cartesian equations

Let $y = f(x)$ be the cartesian equation of a given curve. We suppose that the abscissa of the fixed point A is 'a' and the tangent at A is parallel to x–axis. Let $P(x, y)$ be any variable point on the curve. Let arc $AP = S$. Let the tangent at P make angle ψ with x–axis.

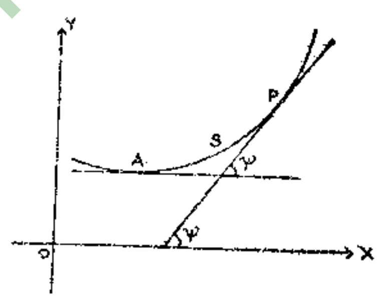


Fig. 5

$$\text{We have } S = \int_a^x \sqrt{\left[1 + (f'(x))^2\right]}\,dx \quad (1)$$

$$\tan\psi = f'(x) \quad (2)$$

Eliminating x between (1) and (2), we obtain the required intrinsic equation.

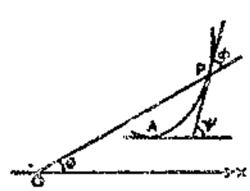
13.4.2 Derivation of intrinsic equations from polar equations

Let $r = f(\theta)$ be the polar equation of a given curve. Let the vectorial angle of the fixed point 'A' be 'α'. We suppose that the tangent at A is parallel to the initial line. We take any point P on the curve whose vectorial angle is 'θ'.

Let arc $AP = S$.

$$\therefore \tan\phi = r\frac{d\theta}{dr}$$

$$= f(\theta)/f'(\theta) \quad ...(1)$$



ϕ being the angle between the radius vector and the tangent.

$$\text{Also } \psi = \theta + \phi \quad ...(2)$$

Fig. 6

$$\therefore\ S = \int_{\alpha}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\alpha}^{\theta} \sqrt{f^2(\theta) + f'(\theta)^2}\, d\theta \quad \text{... (3)}$$

Eliminating θ and φ from (1), (2) and (3), we obtain the required intrinsic equation.

Examples :

Ex. 1: Obtain the intrinsic equation of the catenary $y = a\cosh\left(\frac{x}{a}\right)$, taking the vertex $(0, a)$ as the fixed point.

Sol.: The tangent at $A\ (0, a)$ is parallel to x–axis.

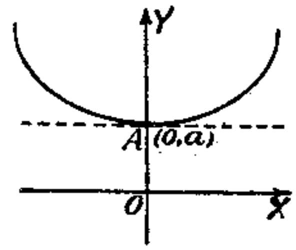


Fig. 7

$$\text{We have } S = \int_{0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{0}^{x} \cosh\left(\frac{x}{a}\right) dx = a \sinh\left(\frac{x}{a}\right) \quad \text{.... (1)}$$

$$\text{Also } \tan\psi = \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right) \quad \text{.... (2)}$$

∴ From (1) and (2), we obtain $S = a\tan\psi$ which is the required intrinsic equation.

Ex. 2: Obtain the intrinsic equation of the cycloid $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$, the fixed point being the origin.

Sol.: Here x–axis is the tangent at the fixed point 'O'. Let $P(\theta)$ be variable point on the cycloid (refer Fig.3). We have

$$\frac{dx}{d\theta} = a\,(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \tan\psi = \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\therefore \psi = \frac{\theta}{2} \quad \text{.... (1)}$$

$$S = \int_{0}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2a \int_{0}^{\theta} \cos\frac{\theta}{2}\, d\theta = 4\,a\sin\frac{\theta}{2} \quad \text{.... (2)}$$

∴ From (1) and (2), $S = 4a\sin\psi$.

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Ex. 3: Obtain the intrinsic equation of cardioid $r = a(1 - \cos\theta)$ taking pole as the fixed point.

Sol.: Initial line is the tangent at the pole. (refer Fig. 4)

$$\frac{dr}{d\theta} = a \sin\theta.$$

$$\therefore \tan\phi = r\frac{d\theta}{dr}$$

$$= \frac{1 - \cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\phi = \frac{\theta}{2}$$

$$\text{and } \psi = \phi + \theta = \frac{3\theta}{2} \qquad \dots (1)$$

$$S = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 2a\int_0^\theta \sin\frac{\theta}{2}.\, d\theta.$$

$$= 4a\left[1 - \cos\frac{\theta}{2}\right] \qquad \dots (2)$$

$\therefore$ From (1) and (2), $S = 4a\left[1 - \cos\frac{\psi}{3}\right]$

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13.5 Summary

(i) The length of the arc of the curve $y = f(x)$ included between a and b is given by

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

(ii) The length of the arc of curve whose equations are given by the parametric equations $x = f(t)$; $y = \phi(t)$ included between α and β is given by $\int_\alpha^\beta \sqrt{(f'(t))^2 + (\phi'(t))^2}\, dt$

(iii) If the equation of the curve is in the polar form $r = f(\theta)$;

then the length $= \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$

(iv) The intrinsic equations of a curve $y = f(x)$ are given by $S = \int^x \sqrt{1 + f'^2}\, dx$ and $\tan\psi = f'(x)$.

13.6 Sample Examination Questions

I. *Answer the following questions in detail.*

i) a. Obtain the length of a curve included between two points in various coordinate systems.

b. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/3$

ii) a Obtain the intrinsic equation of a curve in various coordinate systems.

b. Show that the intrinsic equation of the cardioid $r = a\,(1 + \cos\theta)$ is $s = 4a \sin \psi/3$, taking $\theta = 0$ as the fixed point.

II *Briefly answer the following.*

i) Find the length of the arc of the parabola $y^2 = 4\,ax$ cut off by its latus rectum.

ii) Find the length of the arc of the curve $x = t^2 \cos t$, $y = t^2 \sin t$, from the origin to the point 't'.

iii) Find the length of the curve $y = \log\dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

iv) Show that the length of the arc of the curve given by $x = a\,(3 \sin\theta - \sin^3\theta), y = a \cos^3\theta$ measured from $(0, a)$ to any point (x, y) is $\frac{3}{2} a\,(\theta + \sin\theta \,.\, \cos\theta)$

v) Find the intrinsic equation of the parabola $y^2 = 4\,ax$, by taking origin as the fixed point.

vi) Show that the intrinsic equation of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ taking $(a, 0)$ as the fixed point is $= \frac{3}{2} a \sin^2\psi$

vii) Show that the intrinsic equation of the curve $x = e^{\theta} \sin\theta$. $y = e^{\theta}\cos\theta$ is $e\,c^{(-\pi/4)} + \sqrt{2} = \sqrt{2}\,(\cosh\psi - \sin\psi)$ where $\theta = \pi/4$ is a fixed point.

viii) Find the intrinsic equation of the curve $y = \log x$, taking the point $(1, 0)$ as the fixed point.

III Answer the following in about 5 lines.

i) What is Rectification ?

ii) What is an intrinsic equation of a curve ?

iii) Obtain the intrinsic equation of a curve $y = f(x)$.

Answers

I. (ib) $\log(2 + \sqrt{3})$

II. (i) $2\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right] a$; (ii) $\frac{1}{3}\left[(4 + t^2)^{3/2} - 8\right]$

(iii) $\log\left[e + e^{-1}\right]$, (v) $s = a\,[\tan\psi \sec\psi + \log(\tan\psi + \sec\psi)]$

viii) $s = \dfrac{\sqrt{2}}{\cos\psi - \sin\psi} - \sqrt{2} + \log(\sqrt{2} + 1) - \log\dfrac{\sqrt{2} + \cos\psi - \sin\psi}{\cos\psi + \sin\psi}$

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Unit – 14 : Centre of Gravity and Moment of Inertia

14.0 Contents

14.1 Aims and Objectives

After going through this unit, you must be able to : (i) Evaluate the Centre of Gravity of a given body, (ii) Evaluate the Moment of Inertia of a given body with respect to a given axis.

14.2 Introduction

In units 11, 12 and 13 we discussed the geometrical applications of integral calculus, i.e., determination of lengths, areas and volumes. In this unit we study the applications of integral calculus to mechanics, in particular, we will take up the concepts of centre of grarvity (*C.G.*) and moment of inertia (*M. I.*).

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14.3 Centre of Gravity

14.3.1 Discrete Distribution of matter

For a discrete distribution, consider a system of n particles with weights $w_1, w_2, \ldots, w_r, \ldots, w_n$ on a plane with coordinates

$(x_1, y_1), (x_2, y_2), \ldots, (x_r, y_r), \ldots, (x_n, y_n)$

Then the *C. G.* $(\overline{x}, \overline{y})$ of the system is given by

$$\overline{x} = \frac{\sum_{r=1}^{n} w_r x_r}{\sum_{r=1}^{n} w_r}, \quad \overline{y} = \frac{\sum_{r=1}^{n} w_r y_r}{\sum_{r=1}^{n} w_r}$$

If gravity at each of the points is the same and is denoted by g so that $w_r = m_r g$, where m_r denoting the mass of the particle with weight w_r.

$$\overline{x} = \frac{\sum_{r=1}^{n} m_r x_r}{\sum_{r=1}^{n} m_r}, \quad \overline{y} = \frac{\sum_{r=1}^{n} m_r y_r}{\sum_{r=1}^{n} m_r}$$

14.3.2 Continuous distribution of matter

C.G of a continuous distribution of matter will be obtained through the limiting process from that of a discrete distribution. We replace the given continuous distribution by appropriately selected finite system of particles. The *C.G* of former one will be obtained through a limiting process from that of the later.

14.4 C.G. of a uniform plane curve

Consider a portion of curve $y = f(x)$ intercepted between points A and B with abscissae a and b. Imagine the curve as a thin rigid wire. Draw AL and BM perpendiculars to x – axis. Divide the interval (a, b) into n equal parts each of length 'h' so that on LM we obtain the following points of division with abscissae given by a, $a + h, .., a + rh, .., a + nh$. Let us denote them by

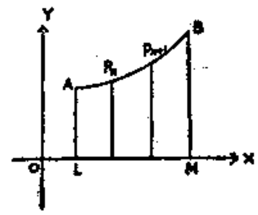

Fig - 1

$x_0, x_1, ..., x_r, ..., x_n$. Let the ordinates at these points meet the arc AB at points $P_0 = A$, $P_1, P_2, ..., P_r, ... P_n = B$.

$\therefore$ $y_r = f(x_r) = f(a + rh)$. Let arc $AP_r = S_r$, ρ be the density i. e., mass per unit length. We replace the arc $P_r P_{r+1}$ by a particle of equal mass $\rho\,(S_{r+1} - S_r)$ at its initial point P_r. Thus we obtain a system of n particles with masses $\rho\,(S_1 - S_0), ..., \rho\,(S_{r+1} - S_r), ..., \rho\,(S_n - S_{n-1})$ at points with coordinates (x_0, y_0) ...., (x_r, y_r),...., (x_{n-1}, y_{n-1}). The *C.G.* of this system is given by

$$\left[\frac{\sum_{r=0}^{n-1} \rho\,(S_{r+1} - S_r)\,x_r}{\sum_{r=0}^{n-1} \rho\,(S_{r+1} - S_r)},\ \frac{\sum_{r=0}^{n-1} \rho\,(S_{r+1} - S_r) y_r}{\sum_{r=0}^{n-1} \rho\,(S_{r+1} - S_r)}\right]$$

$$\sum_{r=0}^{n-1} (S_{r+1} - S_r) = \text{length of the complete arc} = l$$

Write $(S_{r+1} - S_r)\,x_r$ as

$$(S_{r+1} - S_r)\ x_r = x_r \left(\frac{S_{r+1} - S_r}{x_{r+1} - x_r}\right) (x_{r+1} - x_r)$$

$$= x_r\ \frac{S_{r+1} - S_r}{x_{r+1} - x_r}\ .\ h$$

In the limit, we have

$$\lim_{x_{r+1} \to x_r} \frac{S_{r+1} - S_r}{x_{r+1} - x_r} = \left(\frac{dS}{dx}\right)_{x\,=\,x_r}$$

Thus as an approximation, we write $(S_{r+1} - S_r)\,x_r = h\,x_r \left(\frac{dS}{dx}\right)_{x\,=x_r}$

$\therefore$ *C. G.* is given by

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$$\left[\frac{h \sum_{r=0}^{n-1} x_r \left(\frac{dS}{dx}\right)_{x=x_r}}{l} , \frac{h \sum_{r=0}^{n-1} y_r \left(\frac{dS}{dy}\right)_{y=y_r}}{l} \right]$$

By the fundamental theorem of integral calculus, we see that these expressions tend to the limits,

$$\left[\frac{\int_a^b x \frac{dS}{dx} dx}{l} , \frac{\int_a^b y \frac{dS}{dy} dx}{l} \right]$$

14.5 C.G. of uniform plane area

Consider a plane area bounded by a curve $y = f(x)$, x – axis and the two abscissae $x = a$, $x = b$. Divide the segment LM into n equal segments of length 'h' by points with abscissae $a, a+h, \ldots, a+rh, \ldots, a+nh = b$, which we denote as $x_0, x_1 \ldots, x_r, \ldots, x_n$ and erect ordinates at these points $A = P_0, P_1, \ldots, P_r, \ldots, P_n = B$ respectively .

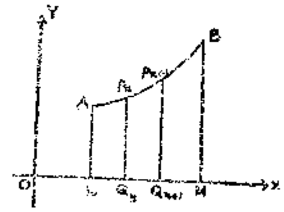

Fig - 2

We write $y_r = f(x_r) = f(x_r) = f(a + rh)$, and replace the area below $P_r P_{r+1}$ by the rectangle $P_r Q_{r+1}$ with area hy_r and $C.G.$ at point

$$\left[\frac{1}{2}(x_r + x_{r+1}) , \frac{1}{2} y_r \right]$$

Again we replace this rectangle by a particle of mass $\rho h y_r$ at point $\left[\frac{1}{2}(x_r + x_{r+1}) , \frac{1}{2} y_r \right]$ so that we obtain n particles with. C.G.

$$\left[\frac{\Sigma \rho h y_r \frac{1}{2}(x_r + x_{r+1})}{\Sigma \rho h y_r} , \frac{\Sigma \rho h y_r \frac{1}{2} y_r}{\Sigma \rho h y_r} \right] .$$

When $n \to \infty$ and $h \to 0$ $\text{Lim } \Sigma hy_r = \int_b^a y \, dx$

$$\text{Lim } \Sigma hy_r \cdot \frac{1}{2}(x_r + x_{r+1}) = \int_a^b yx \, dx$$

$$\text{Lim } \Sigma \frac{1}{2} hy_r^2 = \frac{1}{2} \int_a^b y^2 \, dx$$

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Thus, we obtain a point with coordinates $\left[\dfrac{\int_a^b xy\,dx}{\int_a^b y\,dx}, \dfrac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}\right]$

which we take as the $C.G.$ of the given area or the given lamina.

Similarly the $C.G.$ of a plane area bounded by a curve $x = f(y)$, y – axis and the two ordinates $y = c, y = d$ is

$$\left[\frac{\frac{1}{2}\int_c^d x^2 dy}{\int_c^d x\,dy}, \frac{\int_c^d xy\,dy}{\int_c^d x\,dy}\right]$$

14.6 C.G. of a sectorial area

Consider a sectorial area bounded by a curve $r = f(\theta)$ and two radii vectors $\theta = \alpha$, $\theta = \beta$.

We divide the angle AOB into 'n' equal parts each part being equal to 'h'. Let OP_r, OP_{r+1} be two lines through 'O' making angles $\alpha + hr$, $\alpha + h(r+1)$ respectively with OX.

$OP_r = f(\alpha + hr)$. We replace the area bounded by the arc $P_r P_{r+1}$ by that of traingle $OP_r P_{r+1}$. The $C.G$ of the triangle $OP_r P_{r+1}$ is a point on the median through 'O'. We replace the same by the point on OP_r at a distance $\frac{2}{3} OP_r$ from 'O'. Finally, we replace the area bounded by the arc $P_r P_{r+1}$ by a particle of mass equal to that of the triangle at a point on OP_r at a distance equal to $\frac{2}{3} OP_r$ from O.

Thus we have a mass $\frac{1}{2} OP_r . OP_{r+1} \sin h$ at the point with co-ordinates $\frac{2}{3} OP_r \cos(\alpha + hr)$, $\frac{2}{3} OP_r \sin(\alpha + hr)$.

i.e., at the point with coordinates $\frac{2}{3} f(\alpha + hr)\cos(\alpha + hr), \frac{2}{3} f(\alpha + hr)\sin(\alpha + hr)$

The abscissa of the $C.G$ of the system of 'n' particles is

$$= \frac{\Sigma\, \rho\, \frac{1}{2} f(\alpha + hr) f'(\alpha + h(r+1)) \sin h . \frac{2}{3} f(\alpha + hr)\cos(\alpha + hr)}{\Sigma\, \rho\, \frac{1}{2} . f(\alpha + hr) f(\alpha + (r+1)h) \sin h}$$

i.e., $$\frac{\frac{2}{3}\Sigma f^3(\alpha + hr)\cos(\alpha + hr)\, h}{\Sigma f^2(\alpha + hr)\, h}$$

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which, in the limiting case becomes

$$\frac{\frac{2}{3}\int_{\alpha}^{\beta} f^{3}(\theta)\cos\theta \, . \, d\theta}{\int_{\alpha}^{\beta} f^{2}(\theta)\, d\theta} = \frac{\frac{2}{3}\int_{\alpha}^{\beta} r^{3}\cos\theta \, . \, d\theta}{\int_{\alpha}^{\beta} r^{2}\, d\theta}$$

Similarly the ordinate of $C.G.$ is

$$\frac{\frac{2}{3}\int_{\alpha}^{\beta} r^{3}\sin\theta \, . \, d\theta}{\int_{\alpha}^{\beta} r^{2}\, d\theta}$$

14.7 C.G. of a volume of Revolution

Let a curve $y = f(x)$ intercepted between two points with abscissae a and b revolve about x–axis. We have to find the $C.G.$ of the solid thus obtained. By symmetry we note that the $C.G.$ lies on x–axis.

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Divide LM into n equal parts each of length 'h' and thus obtain the points with abscissae a, $a+h$, ..., $a+rh$, ..., $a+nh=b$.

Let us denote them as $x_0, x_1, ...,x_n$. Let the ordinates through these points meet the curve at points $P_0 = A, P_1, ..., P_r, ..., P_n = B$.

The ordinate of $P_r = f(a+rh) = y_r$ (say). Consider the part of the volume obtained on revolving the rectangle $P_r Q_{r+1}$ about x–axis.

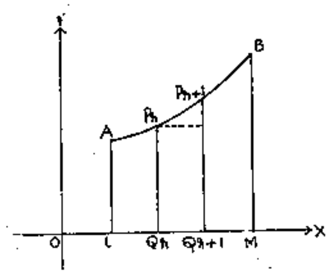

Fig.4

$\therefore$ The volume is $\rho \pi h y_r^2$ with its $C.G.$ at the mid point of $Q_r Q_{r+1}$, whose abscissa is $\frac{1}{2}(x_r + x_{r+1})$. Now we replace this later volume by a particle of equal mass at the mid point of $Q_r Q_{r+1}$. The abscissa of the $C.G.$ of this system of n particles is

$$\frac{\sum_{r=0}^{n-1} \rho \pi h \, y_r^2 \frac{1}{2}(x_r + x_{r+1})}{\sum_{r=0}^{n-1} \rho \pi h \, y_r^2}$$

This in the limiting case ($n \to \infty$ and $h \to 0$) will be $\dfrac{\int_a^b x\,y^2\,dx}{\int_a^b y^2\,dx}$

14.8 C.G. of a surface of revolution

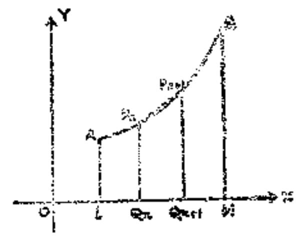

Fig. 5

Here we find the $C.G.$ of the surface of revolution obtained on revolving about x–axis the arc AB. We replace the part of the surface generated by the arc $P_r P_{r+1}$ by that generated by the chord $P_r P_{r+1}$ whose area is $\pi (y_r + y_{r+1}) P_r P_{r+1}$. (Ref. surface of revolution.)

Its $C.G.$ is at the mid point of $Q_r Q_{r+1}$. We replace this surface by a particle of equal mass at the mid point of $Q_r Q_{r+1}$. The $C.G.$ of this system of particles is a point on x–axis with abscissa

$$\frac{\Sigma\, \rho\, \pi\, (y_r + y_{r+1})\, P_r P_{r+1}\; \frac{1}{2}(x_r + x_{r+1})}{\Sigma\, \rho\, \pi\, (y_r + y_{r+1})\, P_r P_{r+1}}$$

which may be taken as

$$\frac{\sum_{r=0}^{n-1} h\,(y_r + y_{r+1}) \left(\frac{ds}{dx}\right)_{x = x_r} \frac{1}{2}(x_r + x_{r+1})}{\sum_{r=0}^{n-1} h\,(y_r + y_{r+1}) \left(\frac{ds}{dx}\right)_{x = x_r}}$$

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which in the limit, gives $\displaystyle\int_a^b xy\,\frac{ds}{dx}\,dx \Big/ \int_a^b y\,\frac{ds}{dx}\,dx$

Examples

Ex. 1 : Find the centre of gravity of a uniform lamina bounded by the coordinate axes and the arc of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in the first quadrant.

Sol : We know the $C.\,G.$ of a plane area bounded by a curve $y = f(x)$, x - axis and the two abscissae $x = a$, $x = b$ is

$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}, \quad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$$

Here the required uniform lamina is bounded by the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

i.e., $y^2 = b^2\left(1-\dfrac{x^2}{a^2}\right)$

$$= \frac{b^2}{a^2}(a^2-x^2),$$

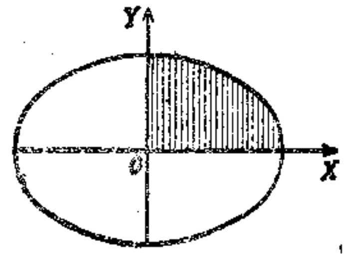

Fig - 6

x – axis and the two ordinates $x = 0$ and $x = a$.

$$\therefore\ \bar{x} = \frac{\int_0^a x\ \frac{a}{b}\sqrt{a^2-x^2}\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx}$$

$$= \frac{\int_0^a x\sqrt{a^2-x^2}\,dx}{\int_0^a \sqrt{a^2-x^2}\,dx} = \frac{4}{3}\frac{a}{\pi}.$$

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$$\bar{y} = \frac{\frac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx} = \frac{\frac{1}{2}\int_0^a \frac{b^2}{a^2}\cdot(a^2-x^2)\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx} = \frac{4}{3}\frac{b}{\pi}$$

$$\therefore\ C.G. = \left(\frac{4a}{3\pi},\frac{4b}{3\pi}\right).$$

Ex. 2 : Find the $C.G.$ of a uniform solid hemisphere of radius *'a'*.

Sol. : The hemisphere, we take as generated by revolving through x - axis the arc of the circle $x^2+y^2=a^2$ lying in the first quadrant. We know the $C.G.$ of the volume of revolution obtained by revolving the curve $y=f(x)$ about x - axis intercepted between two points with abscissae a, b is

$$\bar{x} = \frac{\int_a^b xy^2dx}{\int_a^b y^2dx}$$

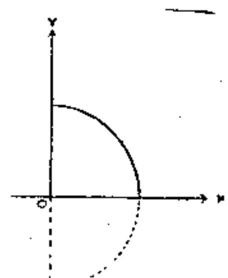


Fig - 7

and $\overline{y} = 0$ (since the volume is symmetrical w.r.t x - axis, $C.G$ lies on x - axis).

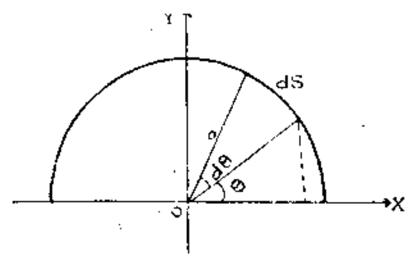
Here, in this problem, the volume of revolution is obtaind by revolving the portion of the curve $y = \sqrt{a^2 - x^2}$ intercepted between two points with abscissae 0 and a about x - axis

$$\therefore \quad \overline{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx} = \frac{\int_0^a x(a^2 - x^2)\, dx}{\int_0^a (a^2 - x^2)\, dx} = \frac{3a}{8}.$$

Ex. 3 : Find the $C.G$ of a thin wire bent into the form of a semi circle of radius 'a'.

Sol : Consider the small portion dS of the curve $dS = a d\theta$. Since the curve is symmetrical w.r.t. y - axis, $C.G.$ lies on y - axis. $\therefore \overline{x} = 0$. We know the $C.G.$ of a uniform plane curve is given by

$$\overline{x} = \frac{\int x\, dS}{\int dS}$$



$$\overline{y} = \frac{\int y\, dS}{\int dS}$$

Fig - 8

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$$\overline{y} = \frac{\int y\, dS}{\int dS} = \frac{\int_0^\pi a \sin\theta \, a\, d\theta}{\int_0^\pi a\, d\theta} = \frac{2}{\pi} a \quad (\because y = a \sin\theta)$$

14.9 Moment of Inertia

Moment of Inertia ($M.I.$) is of particular importance in the study of the motion of rigid bodies. $M.I.$ is analogous to mass. It is a measure of the rotational inertia of a body relative to some fixed axis of rotation, just as mass is a measure of translational inertia of a body.

$M.I.$ of a particle of mass 'm' about any given line is mR^2 where R is the perpendicular distance of the particle from the line. If $m_1, m_2, \ldots\ldots, m_n$ are masses of particles with $R_1, R_2, \ldots\ldots, R_n$ as perpendicular distances of the particles from the given line, then the $M.I.$ of the system is $\sum_{i=1}^{n} m_i R_i^2$. For a continuous distribution of matter $M.I.$ is obtained by limiting process, i.e., we can replace the summation by an integration over the body. $\int R^2\,dm$, where dm is the element of mass, given by a density factor multiplied by an appropriate differential (volume, area or length).

Now we will consider two important theorems concerning to *M.I.*

14.9.1 Perpendicular axis theorem

Statement : The moment of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axis passing through the given axis and lying in the plane of the lamina.

Proof: Consider a rigid body which is in the form of a thin plane lamina of any shape. Let us take up the lamina in the xy –plane.

The M.I. about z –axis, which we will denote by I_z is given by

$$I_z = \Sum m_i R_i^2$$

$$= \Sum m_i (x_i^2 + y_i^2)$$

$$= \Sum m_i x_i^2 + \Sum m_i y_i^2$$

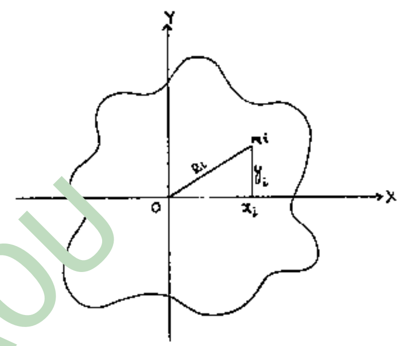

Fig. 9

But $\Sum m_i x_i^2$ is the *M.I.* about y –axis, i.e., I_y and $\Sum m_i y_i^2$ is the *M.I.* about x–axis i.e., I_x (since z_i is zero for all particles).

$\therefore \quad I_z = I_x + I_y$, which proves the theorem.

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14.9.2 Parallel axis theorem

The moment of inertia of a rigid body about any axis is equal to the moment of inertia about parallel axis passing through the centre of gravity plus the product of the mass of the body and the square of the distance between the two axes.

Proof : Consider the moment of inertia about z–axis,

$$\text{i.e., } I_z = \Sum m_i R_i^2 = \Sum m_i (x_i^2 + y_i^2)$$

From fig.10, we can express x_i, y_i interms of coordinates of C.G. i.e., $(\bar{x}, \bar{y}, \bar{z})$ and the coordinates relative to the C.G. i.e., (x'_i, y'_i, z'_i)

i.e., $R_i = \overline{R}_{CG} + R'_i$ gives

$$x'_i = \bar{x} + x'_i$$

$$y'_i = \bar{y} + y'_i$$

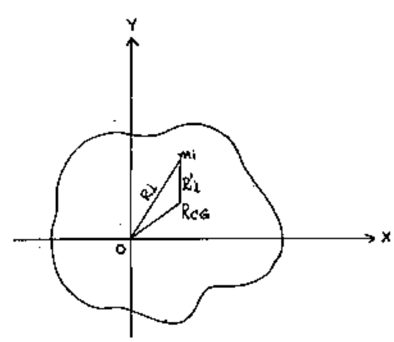


Fig-10.

$$\therefore I_z = \Sum m_i \left[\left(\bar{x} + x'_i\right)^2 + \left(\bar{y} + y'_i\right)^2 \right]$$

$$= \Sum m_i \left(x_i'^2 + y_i'^2\right) + \Sum m_i \left(\bar{x}^2 + \bar{y}^2\right) + 2\bar{x} \Sum m_i x'_i + 2\bar{y} \Sum m_i y'_i$$

$\Sigma m_i \left(x_i'^2 + y_i'^2\right)$ is the M.I. about an axis parallel to the z -axis passing through the C.G, and denote it by I_{CG}.

$\Sigma m_i \left(\bar{x}^2 + \bar{y}^2\right)$ is equal to the mass of the body multiplied by the square of the distance between the C.G. and z - axis and denote it by l.

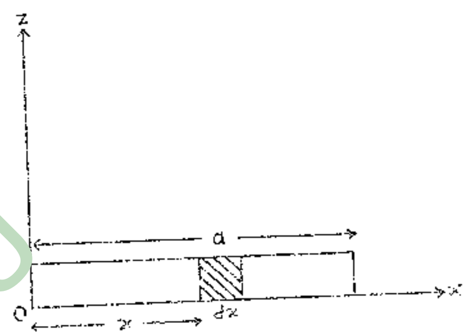
From the definition of the C.G, $\Sigma m_i x_i' = \Sigma m_i y_i' = 0$

$\therefore I_z = I_{CG} + ml^2$, which proves the theorem.

Exmaples

Ex.1: Find the moment of inertia of a uniform rod of length 'a' and mass 'm' (i) about an axis perpendicular to the rod at one end (ii) about an axis perpendicular to the rod at the centre of the rod.

Sol. i) Consider the uniform rod of length 'a' as shown in Fig. 11(a) Consider the small portion of length dx which is at a distance 'x' from the axis.



Let ρ be the density.

M.I. of small portion $= x^2 \rho\, dx$.

M.I. of the rod about z - axis $= \int_0^a x^2 \rho\, dx$.

$$= \frac{\rho a^3}{3} = \frac{1}{3} ma^3,$$

Where mass of the rod $= m = \rho a$.

(ii) *M.I.* of small portion $= \rho\, dx.\, x^2$.

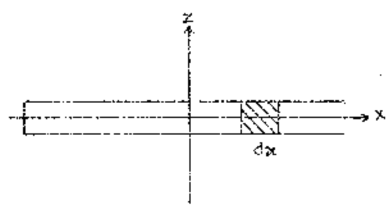


Fig-11(b)

$\therefore$ *M.I.* of the rod about the axis at the centre of the rod.

$$= \int_{-a/2}^{a/2} x^2 \rho\, dx = \frac{1}{12} ma^2.$$

Ex.3: Calculate the *M.I.* of a uniform circular disc of radius 'a' about an axis through the centre of the disc normal to the plane face.

Sol. Consider a circular ring of small thickness 'dr' which is at a distance 'r' from the centre of the disc. Then, mass of the circular ring $= dm = \rho\, 2\pi r\, dr$ where ρ is the density.

M.I. of the circular ring $= r^2 \rho\, 2\pi r\; dr$.

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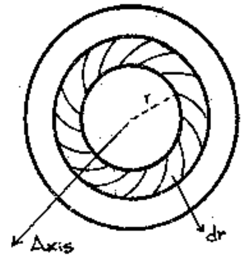
$$M.I. \text{ of the disc } = \int_0^a \rho r^3 \, 2\pi dr$$



$$= \frac{2\pi \rho a^4}{4} = \frac{1}{2} ma^2,$$

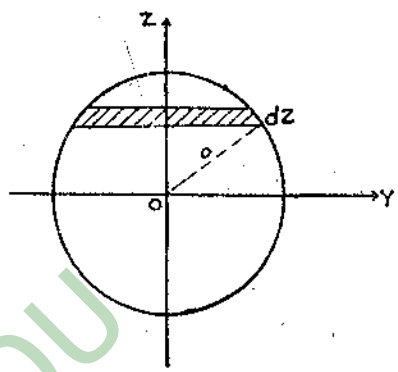
where $m = \rho \pi a^2$ is the mass of the disc.

Fig - 12

Ex.3 : Find the moment of intertia of a uniform solid sphere of radius '*a*' and mass '*m*' about an axis passing through the centre.

Sol. Divide the sphere into thin circular discs of thickness dz. Consider a disc which is at a distance z.

Mass of the disc $= \rho \pi (a^2 - z^2) \, dz$.



We know *M.I.* of circular disc of radius '*a*' about an axis through the centre of the disc normal to the plane face is $\frac{1}{2} ma^2$ (Example 2).

$$\therefore M.I. \text{ of the disc} = \frac{1}{2} \rho \pi (a^2 - z^2) \, dz. \, (a^2 - z^2),$$

$$M.I. \text{ of the sphere} = \int_{-a}^{a} \frac{1}{2} \pi \rho (a^2 - z^2)^2 dz = \frac{8}{15} \pi \rho a^5.$$

Fig-13.

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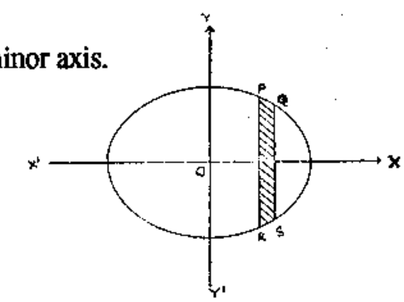
and mass of the sphere $= \frac{4}{3} \pi a^3 \rho = m.$

$$\therefore M.I. \text{ of the sphere} = \frac{2}{5} ma^2.$$

Ex4 : Find the *M.I.* of an Elliptic disc about major axis and minor axis.

Sol. Let $PQRS$ be an elementary strip of breadth 'dx' at a distance 'x' from 'O', the centre of the disc.



$M.I.$ of the strip about $OX = 2y \, dx \, \rho \frac{y^2}{3}$

(Ref. Example -1).

Fig-14.

$$M.I. \text{ of the elliptic lamina about } X'OX = \int_{-a}^{a} 2y \, \rho \, \frac{y^2}{3} dx. = \frac{2\rho}{3} \int_{a}^{a} y^3 \, dx.$$

But for the elliptic disc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$$\Rightarrow y = b \left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$\therefore \; M.I. \text{ about } X'OX = \frac{2\rho b^3}{3} \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right)^{3/2} dx = \frac{\pi\, ab^3 \rho}{4}$$ (using the substitution $x = a \sin \phi$ and reduction formula).

Mass of ellipse $= \pi\, ab\, \rho = m$.

$M.I.$ about $X'OX = \dfrac{mb^2}{4}$.

Similarly $M.I.$ about $Y'OY = \dfrac{ma^2}{4}$.

Ex.5 : Find the $M.I.$ of right circular cylinder (i) about the central axis. (ii) about a line through its C.G. perpendicular to its axis.

Sol. Let 'a' be the radius of the base, 'h' be the height of the cylinder. Consider any elementary disc of breadth 'dx' at a distance 'x' from the C.G 'G'.

$M.I.$ of this disc about OL, where 'O' is the centre of the disc is (Ref. Example.2).

$$= \pi\, a^2 \rho\, dx\, \frac{a^2}{2},$$

where ρ is the density.

$M.I.$ of the cylinder about OL

$$= \int_{-h/2}^{h/2} \frac{\rho\, \pi\, a^4}{2}\, dx$$

$$= \frac{\rho\, \pi\, a^4}{2} h = \frac{ma^2}{2}.$$

where $m = \pi a^2 \rho\, h$.

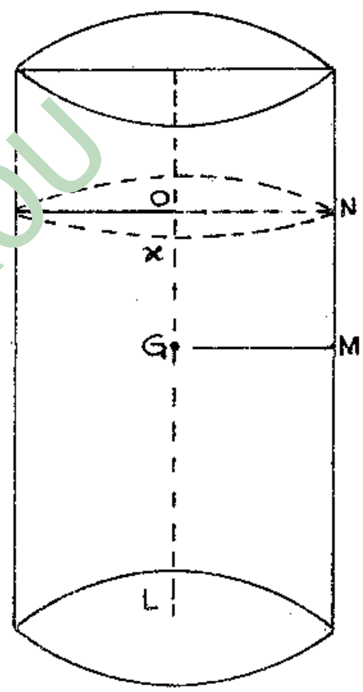


Fig-15

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(b) $M.I.$ of the elementary disc about GM (i.e., line through $C.G$ and perpendicular to the axis) $= M.I.$ of the disc about ON + Mass of the disc x square of the perpendicular distance from 'O' to the $C.G.$

$$= \rho\, \pi\, a^2\, dx\, \frac{a^2}{4} + \left(\rho\, \pi\, a^2\, dx\right) x^2.$$

$$= \rho\, \pi\, a^2 \left[\frac{a^2}{4} + x^2\right] dx.$$

$M.I.$ of the cylinder about $GM = \pi a^2 \rho \int_{-h/2}^{h/2} \left[\frac{a^2}{4} + x^2\right] dx.$

$$= \frac{m}{4}\left[a^2 + \frac{h^2}{3}\right]$$

where $m = \pi a^2 \rho h$ is the mass of the cylinder.

14.10 Summary

(i) The coordinates of the centres of gravity of a uniform plane curve $y = f(x)$ are

$$\left(\frac{\int_a^b x \frac{ds}{dx} dx}{l}, \frac{\int_a^b y \frac{ds}{dy} dx}{l}\right)$$

(ii) A uniform plane area bounded by a curve $y = f(x)$, x-axis and the two abscissae $x = a$, $x = b$ are

$$\left(\frac{\int_a^b xy\, dx}{\int_a^b y\, dx}, \frac{\frac{1}{2}\int_a^b y^2 dx}{\int_a^b y\, dx}\right)$$

iii) The abscissa of C.G of a volume of revolution of the curve $y = f(x)$ intercepted between a and b, revolve about x-axis is $\dfrac{\int_a^b x y^2 dx}{\int_a^b y^2 dx}$

(iv) The abscissa of the C.G of a surface of revoluation about the x- axis is

$$\int_b^a xy \frac{ds}{dx} dx \Big/ \int_b^a y \frac{ds}{dx} dx$$

(v) The $M.I.$ of a body is analogus to mass of the body. The $M.I.$ of a body of mass M, about a line which is at a distance R to the body is MR^2. We use parallel axis theorem and perpendicular axis theorem appropriately for obtaining the $M.I.$ of bodies with respect to various axes.

14.11 Sample Examination Questions

Answer the following questions in detail.

i) a) Obtain the expression for Centre of Gravity of a uniform plane curve.

b) Find the $C.G.$ of a uniform circular wire of radius a in the form of a quadrant of a circle.

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ii) a) Explain moment of inertia and state and prove the perpendicular axis therorem.

b) Find the *M.I.* of a uniform circular ring about an axis through '*O*' perpendicular to the plane of the ring.

iii) a) State and prove the parallel axis theorem of *M.I.*

b) Find the *M.I.* of a hollow sphere about its diameter.

II Briefly answer the following.

i) Show that the centroid of the arc of the cycloid $x = a\ (\theta + \sin\theta), y = a\,(1 - \cos\theta)$ in the positive quadrant is $\left[\left(\pi - \frac{4}{3}\right)a\ ,\ \frac{2}{3}\right]$

ii) A quardrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, revolves about the major axis, find the C.G. of the solid thus obtained.

iii) What are the coordinates of the C.G of the area under one arch of the sine curve $y = \sin x$.

iv) Find the *C.G.* of the arc of astroid $x = a\ \sin^3\theta, y = a\ \cos^3\theta$, lying in the first quadrant.

v) Find the *C.G.* of a solid uniform right circular cone of height '*h*'.

vi) Find the *M.I.* of a uniform solid right circular cone of mass '*m*' about the central axis.

vii) Find the *M.I.* of a circular disc of radius '*a*' about its diameter.

III Answer the following questions in about five lines.

i) Obtain the coordinates of centre of gravity of a discrete distibution of n points.

ii) Define Moment of Inertia.

Answers

I. (ib) $\overline{x} = \overline{y} = \frac{2a}{\pi}$ (iib) Ma^2 (iiib) $\frac{2}{3}Ma^2$

II. (ii) $\overline{x} = \frac{3}{8}a$ (iii) $(\pi/2, \pi/8)$ (iv) $\overline{x} = \overline{y} = \frac{2a}{5}$

(v) $\frac{h}{4}$ from base (vi) $\frac{3}{10}Ma^2$ (vii) $\frac{Ma^2}{4}$.

(Author of Units 8-14: Dr. D. Rama Murthy)

REFERENCES

1. *Integral Calculus* — Shantinarayana
2. *A text book of Integral Calculus vol - II* — H.C. Sinha (Ram Prasad Sons, Agra)
3. *Integral Calculus* — W.L. Ferror, Oxford University Press
4. *Calculus vol – II* — Narayan and Pillay, S. Viswanadhan Pvt. Ltd.

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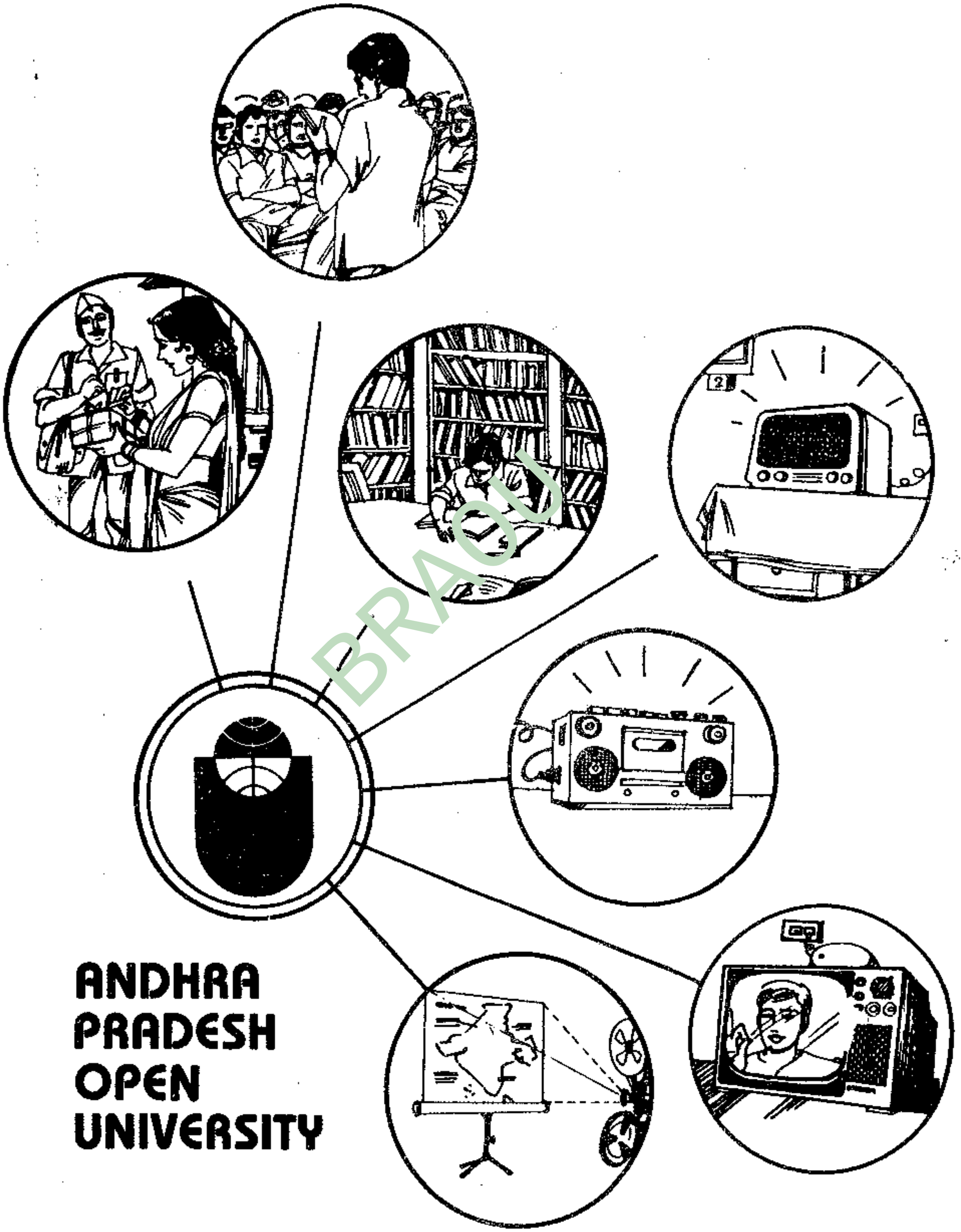

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DIFFERENTIAL EQUATIONS & MATRIX THEORY

BLOCKS : 5-8

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MATHEMATICS

COURSE – I

Calculus, Differential Equations and Matrix Theory

Blocks : V – VIII

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Dr. B.R. AMBEDKAR OPEN UNIVERSITY
HYDERABAD
1992

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Hyderabad.

First published in 1984.

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Further details may be obtained from :

The Director (Academic), Andhra Pradesh Open University, Somajiguda, Hyderabad – 500 482.

Printed and Sruthi Graphics (P)Ltd,Sri Krishna Nagar,Jubilee Hills xRoad Phone:24849

PREFACE

This book deals with the topics in Calculus, Differential Equations and Matrix theory included in the syllabus for the second year of the B.Sc. course offered by the Andhra Pradesh Open University. These topics cover the 'core area' of the subject to be studied in the Second Year of the Three Year Degree Course in Science (B.Sc.). The syllabus, for the sake of convenience, is divided into *blocks*, each of which comprises a number of units. Each block generally covers a specific area of the subject. The units are prepared by specialists in accordance with a format so designed as to enable the student read and understand them without much difficulty.

Man strives ceaselessly to discover the laws that govern nature. Scientists are trying to study the movements of the planets; the different forces that bring about changes in different objects of nature; the birth, multiplication and destruction of organisms. This book describes the changes that take place in interrelated unknown quantities, the rate at which changes take place and the manner in which they can be discussed through Calculus and Differential Equations.

The real functions – their limits and continuity, differentiation and applications of differentiation are explained in the first two blocks. Topics like Integration, evaluation of areas and volumes through integration are given in Blocks 3 and 4. How the ordinary and partial differential equations are formed, through what methods they could be solved, in what form such problems arise in geometry, Physics and Biology, these questions are discussed in Blocks 5, 6 and 7. Block 8 explains Matrix theory.

The University hopes that this course material will help the student to get acquainted with the concepts and principles of Mathematics.

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Contents

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BLOCK – 5 : DIFFERENTIAL EQUATIONS – 1

Introduction

A physical system or a process can be studied in several ways. One among them is to create a similar situation in a laboratory and then study the required properties of the system. Based on these results a theoretical explanation can be given to describe the system. This is only a means of fitting a theory for observed facts. Another method is to propose a theory and verify it experimentally. These are well known procedures usually followed. In either case the common feature is 'theorectical formulation'. In many of the situations such formulation leads to **'Differential Equations'**. Thus, the role of differential equations in studying physical phenomena, becomes significant. This branch of mathematics called **'differential equations'** is like a bridge, connecting mathematics and science with its applications. For example, we could try to compute the position of a moving practicle from a knowledge of its velocity or acceleration. A radioactive substance may be disintegrating at a known rate and we may be required to determine the amount of material present after a given time. In examples like these, we are trying to determine an unknown function from the given information expressed in the form of an equation involving at least one of the derivatives of the unknown function. These equations are called **differential equations** and their study forms one of the most challenging branches of mathematics.

Unit - 15 : Formation of Differential Equations

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15.0 Contents

15.1 Aims and Objectives

After going through this unit you must be able to:

(i) acquint yourself with the order and degree of a differential equation

(ii) distingwish the linear and non-linear differential equations

(iii) fomulate the differential equation by eliminating the arbitrary constants involved in the given function.

15.2 Introduction

Let a process be represented by a function $y = f(x)$. Frequently, it may not be possible to establish directly the types of dependence of y and x, but it is possible to give the relation between the quantities x and y and the derivatives of y with respect to x in the form of a differential equation. The name 'differential equation' itself suggests that these are equations where in the unknowns are connected through the derivatives. The differential equations are mainly classified as (i) Ordinary and (ii) Partial. An equation involving ordinary derivatives of an unknown function (function of a single independent variable) is called an '**ordinary differential equation**'.

For examole,

$$e^x dx + e^y dy = 0 \qquad \text{(i)}$$

$$\frac{d^2x}{dt^2} + n^2 x = 0, \text{ (} n \text{ is a constant)} \qquad \text{(ii)}$$

$$y = x\frac{dy}{dx} + x \Big/ \frac{dy}{dx} \qquad \text{(iii)}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = C \frac{d^2y}{dx^2}, (C \text{ is constant}) \qquad \text{(iv)}$$

are ordinary differential equations.

A **Partial differential equation** is one in which at least two (or more) independent variables occur and the partial differential coefficients occuring in these have reference to either of these variables.

For example.

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z \qquad \text{(v)}$$

$$\frac{\partial^3 y}{\partial t^3} = k \left(\frac{\partial^2 y}{\partial x^2}\right), (k \text{ is } a \text{ constant}) \qquad \text{(vi)}$$

are partial differential equations.

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15.3 Order and Degree of a Differential Equation

The order of a differential equation is the order of the highest order derivative involved in the differential equation.

The degree of a differential equation is the power of the highest order derivative involved in the equation, when the equation has been made rational and integral as far as the derivatives are concerned. According to these definitions of order and degree, equation

(i) is of first order to first degree
(ii) is of second order and first degree
(iii) is the first order and second degree.
After rotionalising the equation (iv) can be written as

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = C^2 \left(\frac{d^2y}{dx^2}\right)^2 \text{ is of second order and second degree.}$$

(v) is of first order, first degree
(vi) is of third order, first degree.

Linear and Non-linear differential equations

A differential equation is said to be linear if : (i) the dependent variable and every derivative involved occurs in the first degree only, and (ii) no products of dependent variables and/or derivatives occur. A differential equation which is not linear is called a non-linear differential equation.

Equations (i), (ii), (v), (vi) are linear and (iii), (iv) are non-linear.

We deal with the ordinary differential equasions in blocks 5 and 6. The partial differential equations are explained in Block 7.

15.4 Solution of a differential equation

Def: Any relation between dependent and independent variables satisfying the given diffential equation is called a solution of the differential equation.

A solution of a differential euqation does not contain the derivatives of the dependent variable with respect to the independent variables.

Def: The solution of an ordinary differential equation of n^{th} order which contains n arbitrary independent constants is called the complete primitive or the general solution or the complete Integrel.

For exmaple,

Consider the differential equation $\frac{d^2y}{dx^2} = 2$. Integrating the differential equation twice we get $y = x^2 + c_1x + c_2$. Here c_1 and c_2 are integration (arbitary) constants and the relation $y = x^2 + c_1x + c_2$ is called the solution of the differential equation $\frac{d^2y}{dx^2} = 2$. The number of arbitary constants, is two, which is equal to the order of the differential equation. Hence the solution is a general solution of the differential equation.

Def: A singular solution is a solution which cannot be obtained from the general solution by specifying values of the arbitary constants.

Ex. The general solution of $y = xy' - y'^2$ is $y = cx - c^2$. However, as seen by substitution another solution is $y = \frac{x^2}{4}$ which can not be obtained from the general solution for any constant c. This second solution is a singular solution.

Def: Any solution which is obtained from the complete primitive by giving particular values to the arbitrary constants is called a particular solution.

The solution $y = x^2 - 3x + 2$ is a particular solution of $\frac{d^2y}{dx^2} = 2$, and is obtained from the general solution $y = x^2 + c_1x + c_2$ by putting $c_1 = -3$ and $c_2 = 2$.

15.5 Formation of Differential equations

The following examples show, how a differential equation is formulated from a physical situation..

Ex.1 : A body of mass m is dropped from some height. It is required to establish the law according to which the velocity v will vary as the body is acted upon by the decelerating force of the air, which is proportional to the velocity (with constant of proportionality k); in other words it is required to find v as a function of t

Sol: Newton's second law states $m \frac{dv}{dt} = F$

Here F is the resultant of the two forces : the gravity mg and the air resistance $-kv$. The minus sign to the air resistance is due to its opposite direction to the velocity.

So we have $$m \frac{dv}{dt} = mg - kv$$

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This relation connecting the unknown function v and its derivative $\frac{dv}{dt}$ is an ordinary differential equation of first order and first degree. This is the equation of motion of certain types of parachutes. We can easily verify that any function of the form

$v = c\, e^{-(k/m)t} + \frac{mg}{k}$, satisfy the above differential equation. Here c is an arbitrary constant.

Ex. 2: Growth problem : This problem occurs in various fields like economic growth, growth of bacteria, decay of radioactive elements and so on. Today the burning problem is the growth of population and the increase in pollution.

Sol: An assumption is made that the rate of growth of $x(t)$ at any time t is proportional to $x(t)$. i.e.,

$$\frac{d}{dt}x(t) \propto x(t)$$

$$\text{or } \frac{d}{dt}x(t) = k\,x(t),$$

where k is the constant of proportionality, assumes +ve values if the problem deals with 'growth' and is negative in the case of 'decay'. This is a first order differential equation connecting the unknown variable x and its derivative.

Ex. 3: Pendulum Problem : This example is devoted to set up the equation of motion for a simple pendulum. The friction due to the air is neglected. The basic assumption is the conservation of energy.

Sol. Let a pendulum–bob be suspended from a point O and be at rest. Let OA be the vertical position of the pendulum. P denotes the position of the bob at time t and let OP make an angle x with OA. The maximum displacement of the pendulum is denoted by the angle that OB makes with OA i.e., a. The work done to change x to the value a is the work required to raise the pendulum bob through a vertical distance $l\cos x - l\cos a$, where l is the length of the pendulum. Since a denotes the angle for maximum displacement, the velocity v (with which the pendulum is swinging) is zero at $x = a$. It can be seen from classical Mechanics, that the velocity v at any time is given by $v = l\,\frac{dx}{dt}$. The conservation of energy implies that $\frac{1}{2}m\,v^2 = \frac{1}{2}m\,l^2\left(\frac{dx}{dt}\right)^2 = mg\,l\,(\cos x - \cos a)$, where m is the mass of the simple pendulum. Since l and m are never zero,

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$$\frac{1}{2}l\left(\frac{dx}{dt}\right)^2 = g(\cos x - \cos a),$$

Differentiating the equation both sides,

$$l\frac{dx}{dt}\frac{d^2x}{dt^2} = -g\sin x\,\frac{dx}{dt}$$

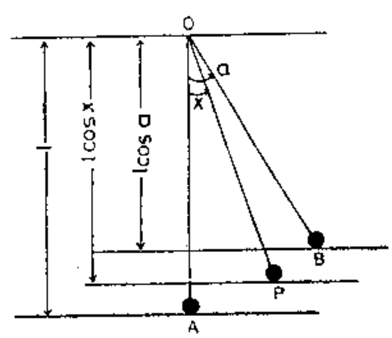

Fig.1

For a swinging pendulum $\frac{dx}{dt}$ is not identically zero and hence the equation of motion for the pendulum is governed by

$$\frac{d^2x}{dt^2} + \frac{g}{l}\sin x = 0$$

which is a second order differential equation.

Thus depending upon the physical situation, the resulting differential equation can be of any order and degree. The following are some of the geometrical applications of differential equations.

Ex. 4 : Family of Curves

(1) Consider a system of straight lines, $y = mx + 2$, where m is the slope. Differentiating the equation both sides with respect to x,

we get $\frac{dy}{dx} = m$, i.e., slope $m = \frac{dy}{dx}$.

Thus, $y = \frac{dy}{dx}\, x + 2$ or $\frac{dy}{dx} = \frac{y-2}{x}$, respresents the above family of lines.

(2) Consider the system of curves given by

$$y = A \sin x + B \cos x \qquad (1)$$

Differentiating twice,

$$\frac{dy}{dx} = A \cos x - B \sin x \qquad (2)$$

$$\frac{d^2y}{dx^2} = A \sin x - B \cos x \qquad (3)$$

By eliminating the arbitrary constants A and B we get the differential equation $\frac{d^2y}{dx^2} + y = 0$. Thus the above differential equation possesses as solution the family of curves with which the analysis was started. In fact (1) represents the general solution of $y'' + y = 0$.

In formulating the differential equations (of n^{th} order) from a given function containing n arbitrary constants, we often adopt the following procedure :

Differentiate the given function n times so as to get n additional equations containing the n arbitrary constants and derivatives. Now, we eliminate n arbitrary constants from these $(n + 1)$ equations and obtain an equation involving a derivative of n^{th} order. Thus we form a differential equation of n^{th} order.

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Ex. 5 : Form the differential equation associated with the circles of given radius r,

$$(x-a)^2 + (y-b)^2 = r^2 \qquad (1)$$

Sol : Here a and b are the constants. They are required to be eliminated. To achieve this differentiate the equation (1) with respect to x twice, getting

$$(x-a) + (y-b)\frac{dy}{dx} = 0 \qquad (2)$$

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad (3)$$

Eliminating a and b, we get

$$r^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

Ex. 6 : Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to the x - axis.

Sol : If the vertex of any parabola of the system be (α, β), the equation of the parabola is

$$(y-\beta)^2 = 4a(x-\alpha)$$

We have to eliminate α and β from this equation and the derived equations.

Differentiating twice with respect to x,

$$(y-\beta)\frac{dy}{dx} = 2a, \tag{2}$$

$$(y-\beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \tag{3}$$

Eliminating β between (2) and (3)

$$2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0, \text{ is the required equation.}$$

So, we conclude that in majority of cases it is possible to prove that the given system of curves represent the general solution of the desired differential equation.

15.6 Summary

The order of a differential equation is the order of the highest order derivative involved in the differential equation.

The Degree of the differential equation is the power of the highest order derivative involved in the equation, when the equation has been made rational and integral as far as the derivatives are concerned.

when you are given a function containing n arbitrary contants, we differentiate it n times to get n additional equations containing n arbitrary constants and derivatives. We eliminate n constants from the $(n+1)$ equations, and obtain an equation involving a derivative of n^{th} order. Thus we form a differential equation of n^{th} order.

15.7 Sample Examination Questions

I Answer the following in detail

(i) (a) Given a function explain the general method of formulating a differential equation for it.

(b) Form the differential equation for $y = ax + bx^2$

II Briefly answer the following

(i) Find the differential equation of the family of curves $y = a\,e^{2x} + b\,e^{-2x}$

(ii) Obtain the differential equation for which $xy = ae^x + b\,e^{-x} + x^2$ is a solution.

(iii) Find the differential equation of all circles passing through origin and having their centres on the x–axis.

(iv) Form the differential equation for $y = a\sin^{-1} x$

(v) Form the differential equations for $y = ax\cos\left(\frac{n}{x} + b\right)$

III Answer the following in about 5 lines.

(i) Define order and degree of a differential equation.

(ii) Explain the terms *'solution'* and *'general solution'* of a differential equation.

(iii) When do you say that a solution is a particular solution?

Answers

I (ib) $x^2y'' - 2xy' - 2y = 0$

II (i) $y'' = 4y$; (ii) $xy'' + 2y^2 - xy + x^2 - 2 = 0$; (iii) $2xy' = y^2 - x^2$;

(iv) $y = \sqrt{1-x^2}\,y'$ (v) $x^4y'' = -2n^2y$.

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Unit – 16 : Differential Equations of First order and First Degree

16.0 Contents

16.1 Aims and Objectives

After going through this unit you must be able to recognize various types of first order differential equations for which exact solution may be obtained and apply the corresponding methods of solutions.

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16.2 Introduction

The general form of a first order differential equation is

$$F(x, y, y') = 0$$

This can be solved for y', to give

$$y' = f(x, y)$$

The problem in elementary differential equations is essentially that of recovering the solution satisfying the equation. It is not always possible to solve the equations in general. The conditions under which a differential equation is solvable are given by the Existence and Uniqueness theorems.

Theorem : If in the equation $y' = f(x, y)$, the function $f(x, y)$ and its partial derivatives with respect to y, $\frac{\partial f}{\partial y}$, are continuous in some domain D, in xy – plane, containing some point (x_0, y_0), then there is a unique solution to this equation $y = \phi(x)$ which satisfies the condition $y = y_0$ at $x = x_0$.

The geometrical meaning of this theorem is that there exists one and only one such $y = \phi(x)$, the graph of which passes through a given point (x_0, y_0). It follows that eqn. (1) has an infinite number of solutions. For example, a solution the graph of which passes through (x_0, y_0); another solution whose graph passes through (x_1, y_1); through (x_2, y_2); another solution whose graph passes through (x_1, y_1); through (x_2, y_2) etc., provided these points lie in the domain.

Here the condition that the function y is equal to y_0 at $x = x_0$ is called the initial condition. It is frequently written in the form $y_{x=x_0} = y_0$.

Def : The problem satisfying an initial condition is called an **Initial value problem.**

Ex. 1 : Suppose a particle moves along a straight line in such a way that its velocity at time t is $2 \sin t$. Determine its position at time t.

Sol : If $y(t)$ denotes the distance at tme t measured from starting point, then the derivative $\frac{dy}{dt}$ represents the velocity at time t.

$$\text{We are given } \frac{dy}{dt} = 2 \sin t.$$

Integrating, $y(t) = 2 \int \sin t \, dt + c = -2 \cos t + c$. Some other piece of information is needed to fix the position function. We can determine c if we know the value of y at some particular instant. For example; $y(0) = 0 \Rightarrow c = 2$ and the position function $y(t) = 2 - 2 \cos t$.

The example just solved is typical of what happens in general. In the process of solving a first order differential equation, an integration is required to remove the derivative y' and in this step an arbitrary constant (integration constant) c appears. The way in which the constant c enters into the solution will depend on the nature of the given differential equation. It may appear as additive constant, as in the above example, but it is more likely to appear in some other way. For example, when we solve the equation $y' = y$, we shall find that every solution has the form $y = c e^x$.

In searching for the general solution we often arrive at a relation like $\phi(x, y, c) = 0$ which is not solved for y. Solving this relationship for y, we get the general solution. However, it is not always possible to solve for y explicitly. In such cases the general solution is left in implicit form. An equation of the form $\phi(x, y, c) = 0$ which gives an implicit general solution is called the complete integral of the differential equation.

To solve (or to integrate) a differential equation means :

(1) to find its general solution or complete integral (if any initial conditions are not specified)

(2) to find a particular solution of the equation that will satisfy the given initial condition (if such exists).

In this Unit we consider equations for which exact solutions may be obtained by definite procedures. The aim of this chapter is to gain ability to recognise these various types and to apply corresponding methods of solutions. The important types that we consider here are (i) exact equations (ii) separable equations (iii) linear equations. The remaining types are of various very special forms, and the corresponding methods of solution involve various devices. This unit may be treated as a collection of special 'methods', 'devices' and tricks of solving the differential equation.

The general form of a first order differential equation of first degree can be written as

$$\frac{dy}{dx} = f(x, y) \text{ or}$$

$$M(x, y)\, dx + N(x, y)\, dy = 0.$$

We can readily convert one of these forms into other form.

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16.3 Solution by inspection

Some times, by rearranging the terms of the given equation and/or by dividing by a suitable function of x and y, the equation will contain several parts integrable easily. In this connection the following list of exact differentials should be noted carefully.

(i) $d(xy) = x\,dy + y\,dx$

(ii) $d(y/x) = \dfrac{x\,dy - y\,dx}{x^2}$

(iii) $d[\log(xy)] = \dfrac{x\,dy + y\,dx}{xy}$

(iv) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{(x^2 + y^2)}$

(v) $d[\log(y/x)] = \dfrac{x\,dy - y\,dx}{xy}$

(vi) $d\left[\dfrac{e^y}{x}\right] = \dfrac{x\,e^y\,dy - e^y\,dx}{x^2}$

Examples

Ex. 1 : Solve $y\,dx - x\,dy + (1 + x^2)\,dx + x^2 \sin y\,dy = 0.$

Sol : Dividing each term of the given equation by x^2, we get

$$\frac{y\,dx - x\,dy}{x^2} + \frac{1 + x^2}{x^2}\,dx + \sin y\,dy = 0.$$

or $$-\frac{x\,dy - y\,dx}{x^2} + \left[\frac{1}{x^2} + 1\right] dx + \sin y\,dy = 0.$$

i.e. $$-d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right) dx + \sin y\,dy = 0.$$

Integrating $-\dfrac{y}{x} + x - \dfrac{1}{x} - \cos y = C,$

where C is an arbitrary constant.

i.e. $x^2 - y - x\cos y - 1 = Cx,$

a complete integral of the given equation.

Ex. 2 : Solve $y(2xy + e^x)\,dx = e^x\,dy$

Sol : Rewriting the given equation $2xy^2\,dx + ye^x\,dx - e^x\,dy = 0$

i.e., $$2x\,dx + \frac{ye^x\,dx - e^x\,dy}{y^2} = 0$$

i.e., $$2x\,dx + d\left[\frac{e^x}{y}\right] = 0$$

Integrating, $x^2 + \dfrac{e^x}{y} = C$ or $yx^2 + e^x = Cy$

a general solution of the given equation.

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SAQ 1 : Solve the following equations

(i) $x\,dy - y\,dx = (x^2 + y^2)\,dx$

(ii) $y\,(axy + e^x)\;dx - e^x\;dy = 0$

(iii) $x\,dy - y\,dx = xy^2\,dx$

(iv) $x\,dy - y\,dx + 2x^3\,dx = 0$

(v) $y\,(2x^2 y + e^x)\,dx - (e^x + y^3)\,dy = 0$

16.4 Variables Separable

Consider a differential equation of the form

$$\frac{dy}{dx} = f_1(x)\,f_2(y) \qquad (1)$$

where the right side is a product of two functions: one is dependent only on x and the other dependent only on y.

This equation can be written as

$$\frac{1}{f_2(y)}\,dy = f_1(x)\,dx \qquad [\text{provided } f_2(y) \neq 0] \qquad (2)$$

Integrating the left side with respect to y and the right with respect to x, we obtain

$$\int \frac{1}{f_2(y)}\,dy = \int f_1(x)\,dx + C \qquad (3)$$

which is a relationship connecting the solution y, the independent variable x and an arbitrary constant C; we have thus obtained a general solution (complete integral) of the equation (1). An equation of the form (2) is called an equation with separated variables.

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Examples

Ex. 1 : Solve $x\,dx + y\,dy = 0$.

Sol : The given equation is an equation with separated variables and on integration,

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1.$$

Since the left side of this equation is non - negative, the right side is also non - negative. Denoting $2c_1$ by c^2, we have $x^2 + y^2 = c^2$ which is an equation of a family of concentric circles with centre at the origin and radius c.

An equation of the form $M_1(x)\,N_1(y)dx + M_2(x)\,N_2(y)dx = 0$ is also called an equation with separable variables. It can be reduced to an equation with separated variables by dividing both sides by the expression $N_1(y)\,M_2(x)$:

$$\frac{M_1(x)\,N_1(y)}{N_1(y)\,M_2(x)}\,dx + \frac{M_2(x)\,N_2(y)}{N_1(y)\,M_2(x)}\,dy = 0.$$

(provided $N_1(y)$ and $M_2(x)$ are non - zero)

$$\text{or} \quad \frac{M_1(x)}{M_2(x)}\,dx + \frac{N_2(y)}{N_1(y)}\,dy = 0,$$

an equation with separated variables.

Ex. 2 : Solve the equation $\frac{dy}{dx} = -y/x$.

Sol : Separating the variables, $\frac{dy}{y} = -\frac{dx}{x}$

Integrating, we get $\log y = -\log x + \log c$.

[Here, we denoted the arbitrary constant by $\log c$, keeping in view the subsequent transformations. This is permissible since $\log c$ (when $c \neq 0$) can take on any value, and we make $c > 0$]

$$\text{or} \quad \log y = \log c/x$$

$$\text{or} \quad y = c/x,$$

the required general solution of the given equation.

Ex. 3 : Solve $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

Sol : By separating the variables, we have

$$\frac{dy}{\sqrt{(1-y^2)}} + \frac{dx}{\sqrt{(1-x^2)}} = 0.$$

Integrating $\sin^{-1}y + \sin^{-1}x = c$

SAQ 2 : Solve the following equations

(i) $\frac{dy}{dx} = \frac{y+2}{x-1}$

(ii) $x\sqrt{(1+y^2)} + y\sqrt{(1+x^2)}\,\frac{dy}{dx} = 0$

(iii) $\tan y \sec^2 x\, dx + \tan x \sec^2 y\, dy = 0$

(iv) $\sqrt{(1+x^2)}\, dx + \sqrt{(1+y^2)}\, dy = 0$

16.5 Homogeneous Equations

Def. 1 : The function $f(x, y)$ is called a homogeneous function of degree n in the variables x and y, if for any λ the following identity is true :

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

For example, $f(x, y) = xy - y^2$ is a homogeneous function of degree two, since

$$f(\lambda x, \lambda y) = (\lambda x)(\lambda y) - (\lambda x)^2 = \lambda^2 xy - \lambda^2 y^2$$
$$= \lambda^2 [xy - y^2] = \lambda^2 f(x, y).$$

The function $f(x, y) = (x^2 - y^2)/xy$ is a homogeneous function of degree zero, since

$$\frac{(\lambda x)^2 - (\lambda y)^2}{(\lambda x)(\lambda y)} = \frac{x^2 - y^2}{xy}$$

i.e., $f(\lambda x, \lambda y) = f(x, y)$ or $f(x\lambda, y\lambda) = \lambda^0 f(x, y)$

Def. 2 : A differential equation of first order $\frac{dy}{dx} = f(x, y)$ is called homogeneous in x and y if the function $f(x, y)$ is a homogeneous function of degree zero in x and y.

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Solution of Homogeneous Equation

It is given that $f(\lambda x, \lambda y) = f(x, y)$. (1)

Putting $\lambda = \frac{1}{x}$ in this identity,

we have $f(x, y) = f(1, y/x)$

Thus a homogeneous function of degree zero is dependent only on the ratio of the arguments. Hence the equation takes the form

$$\frac{dy}{dx} = f(1, y/x). \quad (2)$$

Making the substitution $u = y/x$ or $y = ux$

we get $\frac{dy}{dx} = u + x\frac{du}{dx}$

Substituting the expression of the derivative into equation (2) we obtain

$$u + x\frac{du}{dx} = f(1, u) \quad (3)$$

This is an equation with variables separable, and separating the variables

$$\frac{du}{f(1, u) - u} = \frac{dx}{x}$$

and on integration $\int \frac{du}{f(1, u) - u} = \int \frac{dx}{x} + c$

Substituting $u = y/x$ after integration, we get the integral of the equation (2).

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Examples

Ex. 1 : Solve $\frac{dy}{dx} = xy / (x^2 - y^2)$

Sol : On the right is a zero degree hemogeneous function which means that we have a homogeneous equation.

Substituting $y/x = u$.

we have $y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx}$

then the given equation reduces to $u + x\frac{du}{dx} = \frac{u}{1 - u^2}$

or $x\frac{du}{dx} = u^3/(1 - u^2)$.

Separating variables, we obtain

$$\frac{1 - u^2}{u^3} du = \frac{dx}{x}$$

or $\left(\frac{1}{u^3} - \frac{1}{u}\right)du = \frac{dx}{x}$

Integrating $-\frac{1}{2u^2} - \log u = \log x + \log c$

or $-\frac{1}{2u^2} = \log(u\ x\ c)$

Substituting $u = y/x$, we get

$$-\frac{x^2}{2y^2} = \log(cy) \qquad (1)$$

a complete integral of the given equation.

Ex. 2 : Solve $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

Sol : The given equation can be written as

$$\frac{dy}{dx} = -\frac{(x^3 + 3xy^2)}{(y^3 + 3x^2y)},$$ a homogeneous equation.

Substituting $y/x = u$ and

$$\frac{dy}{dx} = u + x\frac{du}{dx},$$

the given equation reduces to

$$u + x\frac{du}{dx} = -(1 + 3x^2)/(u^2 + 3u)$$

or $$x\frac{du}{dx} = -\frac{1 + 3u^2}{u^3 + 3u} - u = -\frac{(u^4 + 6u^2 + 1)}{u^3 + 3u}$$

Separating the variables $$\frac{dx}{x} = -\frac{u^3 + 3u}{u^4 + 6u^2 + 1}\,du$$

or $$4\frac{dx}{x} = -\frac{4u^3 + 12u}{u^4 + 6u^2 + 1}\,du$$ (multiplying both sides by 4)

Integrating, $4\log x = -\log(u^4 + 6u^2 + 1) + \log C$

or $\log x^4 = \log\left[C/(u^4 + 6u^2 + 1)\right]$

or $x^4\,[u^4 + 6u^2 + 1] = C$

Substituting $u = y/x$, $y^4 + 6x^2y^2 + x^4 = C$

a complete integral of the given equation.

SAQ 3 : Solve the following equations.

(1) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

(2) $xdy - ydx = \sqrt{(x^2 + y^2)}\,dx$

(3) $(x^2 - y^2)\frac{dy}{dx} = 2xy$, given that $y = 1$, when $x = 1$

(4) $x\,dx + y\,dy = 2(x\,dy - y\,dx)$

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16.6 Non-homogeneous equations Reducible to Homogeneous equations

Equations of the type $\frac{dy}{dx} = \frac{Ax + By + C}{ax + by + c}$ (1)

where a, b, c, A, B, C are constants and $ax + by + c \neq 0$, are reducible to homogeneous equations.

If $c = 0 = C$, then equation (1) is obviously homogeneous. Now let c and C (or one of them) be different from zero.

Make a change of variables :

$$x = X + h, y = Y + k, \qquad h, k \text{ are constants} \tag{2}$$

Then $$\frac{dy}{dx} = \frac{dY}{dX}.$$

and the given equation (1) becomes

$$\frac{dY}{dX} = \frac{AX + BY + Ah + Bk + C}{aX + bY + ah + bk + c} \tag{3}$$

If h and k are chosen to satisfy $\left.\begin{array}{l} ah + bk + c = 0 \\ ah + Bk + C = 0 \end{array}\right\}$ (4)

the equation (3) becomes homogeneous :

$$\frac{dY}{dX} = \frac{AX + BY}{aX + bY} \tag{5}$$

Solving this equation and passing once again to x and y by formulas (2), we obtain the solution of the equation (1).

Examples

Ex. 1 : Solve $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

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Sol : To convert the given equation into a homogeneous equation, make the substitution

$$x = X + h, \; y = Y + k$$

The given equation becomes $\frac{dY}{dX} = \frac{X + 2Y + h + 2k - 3}{2X + Y + 2h + k - 3}$ (1)

We choose h and k, such that they satisfy the equations

$$\left.\begin{array}{l} h + 2k - 3 = 0 \\ 2h + k - 3 = 0 \end{array}\right\} \tag{2}$$

Solving these equations we get $h = 1, \; k = 1$ and the equation (1) reduces to

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \quad \text{a homogeneous equation.}$$

To solve this equation put $Y = uX$, so that

$$\frac{(2 + u)}{1 - u^2}\,du = \frac{dX}{X}$$

i.e., $\left[\frac{1}{2}\frac{1}{1+u} + \frac{3}{2}\frac{1}{1 - u}\right]du = \frac{dX}{X}.$

Integrating, $\frac{1}{2}\log(1+u) - \frac{3}{2}\log(1-u) = \log X + \log c$

$$\text{or} \qquad (1+u) = c^2 X^2 (1-u)^3$$

$$(X+Y) = c^2 (X-Y)^3$$

Now, substituting $X + 1 = x,\ Y + 1 = y$

$$(x+y-2) = c^2\ (x-y)^3 \text{ a complete integral of the equation.}$$

Note 1 : The above method of solution succeeds if h and k can be solved from the system (4).

i.e., if $\frac{a}{A} \neq \frac{b}{B}$. If, however $\frac{a}{A} = \frac{b}{B}$. and $\frac{c}{C}$ be different from each of these fractions, h and k cannot be obtained from (4) hence the following method is adopted :

Put $\frac{a}{A} = \frac{b}{B} = \frac{1}{m}$.

The given equation is transformed to

$$\frac{dy}{dx} = \frac{m\ (ax + by) + C}{ax + by + c}$$

Then by substitution $ax + by = v$

we have $\frac{dv}{dx} = a + b\ \frac{dy}{dx} = a + b\frac{mv + C}{v + c}$

which is an equation with variables separable and can be solved by the known method.

Note 2 : suppose further that $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \frac{1}{m}$

the given equation reduces to $\frac{dy}{dx} = m$

and the solution is given by $y = mx + k$.

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Ex. 2 : Solve $\frac{dy}{dx} = \frac{2x + y - 1}{4x + 2y + 5}$

Sol : This equation can not be solved by substitution $x = X + h$, $y = Y + k$ since in this case the set of equations that serves to determine h and k is insolvable

because $\frac{a}{A} = \frac{b}{B}$.

By substituting $v = 2x + y$ and

$$\frac{dv}{dx} - 2 = \frac{dy}{dx}$$

the given equation transformed to

$$\frac{dv}{dx} - 2 = \frac{v - 1}{2v + 5} \quad \text{or} \quad \frac{dv}{dx} = \frac{5v + 9}{2v + 5}$$

Solving this equation, $\frac{2}{5}v + \frac{7}{25}\log\ (5v + 9) = x + c$

Since $v = 2x + y$, we obtain the solution of the given equation in the form

$$10y - 5x + 7\ \log(10x + 5y + 9) = c_1$$

SAQ 4 : Solve the following

(i) $(2x + y + 6)\,dx = (y - x - 3)\,dy$

(ii) $\dfrac{dy}{dx} + \dfrac{10x + 8y - 12}{7x + 5y - 9} = 0$

(iii) $\dfrac{dy}{dx} = \dfrac{x + 7y + 2}{3x + 5y + 6}$

(iv) $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

16.7 Exact Equations

If M and N are continuous functions of x and y, the equation

$M(x, y)\,dx + N(x, y)\,dy = 0$ is called exact if there exists a function f of x and y such that

$d\,[f(x, y)] = M(x, y)\;dx + N(x, y)\;dy$.

For example, the differential equation $y^2\,dx + 2xy\,dy = 0$ is an exact differential equation, for there exists a function xy^2 such that

$$d\,(xy^2) = \frac{\partial}{\partial x}(xy^2)\;dx + \frac{\partial}{\partial y}(xy^2)\;dy$$

$$= y^2\;dx + 2xy\;dy = 0$$

Therefore the given equation can be written as $d\,(xy^2) = 0$. This on integration yields the general solution, $xy^2 = c$, where c is an arbitrary constant.

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In practice, however we shall not be able to determine $f(x, y)$ so easliy. But the method explained here will be often useful. Note that, in the above illustration, if $xy^2 = c$ is merely differentiated to get the given equation. Thus an exact differential equation can always be recovered from its general solution directly by differentiation.

Necessary and Sufficient Condition for a differential equation to be exact :

A necessary and sufficient condition for a differential equation of first order and first degree,

$$M\,dx + N\,dy = 0 \text{ to be exact is}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The Condition is Necessary :

Let the equation $M\,dx + N\,dy = 0$ (1)

be exact; We have to show that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. (2)

By definition, there must exist a function $f(x, y)$

such that $d\,[f(x, y)] = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy = M\,dx + N\,dy$ (3)

so that $M = \dfrac{\partial f}{\partial x}$ (4)

$N = \dfrac{\partial f}{\partial y}$ (5)

Differentiating (4) and (5) with respect to y and x respectively, we get

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}, \text{ and}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$\text{since } \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$\text{we have } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus if (1) is exact M and N satisfy the condition (2).

The Condition is sufficient :

We now assume the condition (2) holds and show that the equation (1) is exact. To do this we must find a function $f(x, y)$ such that $d\,[f\,(x,\,y)] = M\,dx + N\,dy$.

Let $\int M\,dx = u$, where y is supposed constant while performing integration.

$$\text{Then } \frac{\partial}{\partial x}\left(\int M\,dx\right) = \frac{\partial u}{\partial x};$$

$$\text{and } \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

$$\text{Since } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ and } \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

$$\text{we have } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Integrating both sides with respect to x (taking y as constant)

$$N = \frac{\partial u}{\partial y} + \psi(y)$$

where $\psi(y)$ is a function of y alone.

$$\therefore \quad M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \left[\frac{\partial u}{\partial y} + \psi\,(y)\right]dy$$

$$= \left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right] + \psi(y)\,dy$$

$$= du + \psi(y)\,dy$$

$$= d\left[u + \int \psi\,(y)\,dy\right] \text{ an exact differential.}$$

$\therefore\ M\,dx + N\,dy = 0$ is an exact differential equation.

Working rule for solving an exact differential equation :

1. Compare the given equation with $M\,dx + N\,dy = 0$ and find out M and N. Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then we conclude that the given equation is exact.

2. Then integrate M with respect to x treating y as a constant and

3. integrate N with respect to y treating x as a constant and omit those terms which have been already obtained by integrating M (that is, terms which occur in step 2).

4. On adding the terms obtained in steps 2 and 3 and equating to an arbitrary constant, we get the required solution.

Ex. : Solve $y \sin 2x\,dx - (y^2 + \cos^2 x)\,dy = 0$

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$,

we get $M = y \sin 2x$ and $N = -(y^2 + \cos^2 x)$.

$$\frac{\partial M}{\partial y} = \sin 2x. \quad \frac{\partial N}{\partial x} = 2 \cos x \sin x = \sin 2x.$$

$$\text{i.e., } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{The given equation is exact.}$$

$$\text{Now} \int M\,dx = \int y \sin 2x\,dx = -y\,\frac{\cos 2x}{2}$$

$$\text{and} \int N\,dy = -\int (y^2 + \cos^2 x)\,dy = -\frac{y^3}{3} - y \cos^2 x$$

$$= -\frac{y^3}{3} - y\left(\frac{1 + \cos 2x}{2}\right)$$

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Omitting $-\frac{1}{2} y \cos 2x$ in this, which has been already obtained in $\int M\,dx$, we get the required solution,

$$-\frac{1}{2} y \cos 2x - \frac{y^3}{3} - \frac{y}{2} = C_1$$

i.e., $y \cos 2x + \frac{2}{3} y^3 + y = C$ where $C\,(= -2C_1)$ is an arbitrary constant.

SAQ 5: Solve the following equations

(i) $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]\,dy = 0$

(ii) $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$

(iii) $x\,dx + y\,dy = \frac{a^2 (x\,dy - y\,dx)}{x^2 + y^2}$

(iv) $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$

(v) $\left[1 + e^{x/y}\right] dx + e^{x/y}\,[1 - x/y]\,dy = 0$

16.8 Integrating Factor : (Equations not exact but can be made exact)

Let the equation $M\,dx + N\,dy = 0$ (1)

be not an exact differential equation. It is some times possible to choose a function $\mu(x, y) \neq 0$ such that after multiplying both sides of the equation (1) by it, the equation is converted into an exact differential equation. The general solution of the equation thus obtained coincides with the general solution of the original equation; the function $\mu(x, y)$ is called an **integrating factor** of the given equation.

In order to find the integrating factor μ; multiply both sides of the given equation by μ, to get $\mu M\,dx + \mu N\,dy = 0$.

For this equation to be an exact differential equation, it is necessary and sufficient that

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\text{i.e., } \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

$$\text{or } M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

dividing both sides of the equation by μ, and rearranging the terms we get

$$M\frac{\partial}{\partial y}(\log\mu) - N\frac{\partial}{\partial x}(\log\mu) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad (2)$$

It is obvious that any function $\mu(x, y)$ that satisfies this equation is an integrating factor of equation (1). Equation (2) is a partial differential equation in the unknown function μ dependent on the two variables x and y. It can be proved that under certain conditions it has an infinite number of solutions and we say that equation (1) has an integrating factor. But in the general case, the problem of finding $\mu(x, y)$ from equation (2) is harder than the original problem of integrating equation (1). Only in certain particular cases does one manage to find the functions $\mu(x, y)$.

For example, let the equation (1) have an integratring factor dependent only on y. Then

$$\frac{\partial}{\partial x}(\log\mu) = 0$$

and the equation (2) reduces to an ordinary differential eqution

$$\frac{d}{dy}(\log\mu) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Big/ M,$$ from which we determine $\log\mu$

and hence μ. It is clear that this can be done only if the expression $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Big/ M$ is a constant k (say) or a function of y alone say $f(y)$, and the respective integrating factors will be

$$e^{\int k\,dy} \text{ or } e^{\int f(y)\,dy}$$

Similarly, if the expression $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Big/ N$ is a constant, say k or a function of only x, say $f(x)$, then the integrating factor will be $e^{\int k\,dx}$ or $e^{\int f(x)\,dx}$ respectively.

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Examples

Ex. 1: Solve the equation $(y + xy^2)\, dx - x\, dy = 0$.

Sol. Here, $M = y + xy^2, N = -x, \quad \frac{\partial M}{\partial y} = 1 + 2xy, \quad \frac{\partial N}{\partial x} = -1$, so $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and hence the given differential equation is not exact. Let us verify whether this equation allows for an integrating factor dependent only on y or not. Noting that

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Big/ M = \frac{-1 - 1 - 2xy}{y + xy^2} = -\frac{2}{y} \quad \text{a function of } y \text{ alone.}$$

The integrating factor is given by

$$e^{\int -\frac{2}{y} dy} = e^{-2\log y} = \frac{1}{y^2}$$

Multiplying the given equation by $\frac{1}{y^2}$, we get

$$\left[\frac{1}{y} + x\right] dx - \frac{x}{y^2} dy = 0 \text{ as an exact differential equation.}$$

Solving this equation, we get, $\frac{x}{y} + \frac{x^2}{2} + c = 0$

$$\therefore y = \frac{-2x}{x^2 + 2c}$$

Ex. 2: Solve $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$

Sol. Here $M = x^2 + y^2 + 2x$ and $N = 2y$.

$$\therefore \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 0 \Rightarrow \text{the given equation is not exact}$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = 1, \text{ a constant.}$$

$$I.F. = e^{\int 1 dx} = e^x.$$

multiplying the given equation by e^x we get

$$(x^2 e^x + e^x y^2 + 2x e^x)\, dx + 2y e^x\, dy = 0$$

which must be exact. From this new equation we get

$$\int x^2 e^x\, dx + y^2 \int e^x\, dx + 2\int x e^x\, dy = C$$

$$\text{or } e^x (x^2 + y^2) = C$$

SAQ 6. Solve the following equations

(i) $(3xy - 2ay^2)\, dx + (x^2 - 2axy)\, dy = 0$

(ii) $(x^3 - 2y^2)\, dx + 2xy\, dy = 0$

(iii) $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

(iv) $(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)\, dy = 0$

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16.9 Linear Equations

The standard form of a linear equation of the first order, commanly known as **Leibnitz's Linear equation** is

$$\frac{dy}{dx} + Py = Q \qquad (1)$$

where P and Q are given continuous functions of x.

Solution of the the linear equation :

To make the left side of the equation (1) to be an exact differential, we multiply the equation by $e^{\int P\,dx}$, the integrating factor of the equation, so that

$$\frac{dy}{dx} e^{\int P\,dx} + y\, e^{\int P\,dx} P = Q\, e^{\int P\,dx}$$

$$\Rightarrow \frac{d}{dx}\left[y\, e^{\int P\,dx}\right] = Q\, e^{\int P\,dx}$$

Integrating both sides, we get

$$y e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + C$$

is the solution of the given linear equation.

Examples

Ex. 1: Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

Sol. Here $P = \cos x$, $Q = \frac{1}{2}\sin 2x$,

$$\int P\,dx = \sin x,\ e^{\int P\,dx} = e^{\sin x}$$

The solution of the given equation can be directly written as

$$y\, e^{\sin x} = \int \frac{1}{2}\sin 2x\, e^{\sin x}\, dx + C$$

$$= \int e^{\sin x} \sin x.\cos x\, dx + C$$

$$= \int e^{z} z\, dz + C, \quad \text{(on putting } z = \sin x\text{)}$$

$$= e^{z}(z-1) + C$$

$$= e^{\sin x}(\sin x - 1) + C$$

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Remark : Sometimes a differential equation cannot be put in the form of a linear equation. Then we regard y as the independent variable and x as the dependent variable and may get a differential equation of the form $\frac{dx}{dy}+P_1 x = Q_1$, where P_1 and Q_1 are functions of y alone. Then the Integrating Factor will be $e^{\int P_1 dy}$ and the solution is given by, $x\, e^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C$.

Ex. 2: Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$.

Sol. This equation contains y^2 and $\tan^{-1}y$ and is, therefore, not linear in y, it can be written as

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\text{or } \frac{dx}{dy} + \frac{1}{1+y^2}.x = \frac{\tan^{-1}y}{1+y^2}$$

which is a Leibnitz's linear equation in x.

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

The solution is given by $$x\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy + C$$

$$\text{or } x = \tan^{-1}y - 1 + C\,e^{-\tan^{-1}y}$$

SAQ 7: Solve the following equations

(i) $x\frac{dy}{dx} + y\log x = e^x x^{1-\frac{1}{2}\log x}$

(ii) $(1-x^2)\frac{dy}{dx} + 2xy + x\sqrt{(1-x^2)}$, given that $y = 0$, at $x = 0$,

(iii) $x(x^2+1)\frac{dy}{dx} = y(1-x^2) + x^3\log x$

(iv) $\frac{dy}{dx} + y\cot x = \sin x$

16.9.1 Equations reducible to linear equations

An equation of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$ (A)

where P and Q are functions of x only, is reducible to linear equation by putting $f(y) = v$ so that

$$f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

and (A) becomes $\frac{dv}{dx} + Pv = Q$ a linear equation in v and x and its solution can be obtained. Replacing v by $f(y)$ we get the solution interms of x and y.

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Ex. : Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Sol. Dividing the given equation throughout by $\cos^2 y$,

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \text{ which is of the form } (A)$$

or $\sec^2 y \frac{dy}{dx} + \frac{2x \sin y \cos y}{\cos^2 y} = x^3.$

Put $\tan y = z$, then $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$.

Then the equation (1) becomes $\frac{dz}{dx} + 2xz = x^3$

a linear equation in x and z.

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

and the solution is given by

$$z e^{x^2} = \int e^{x^2} x^3 \, dx + C = \frac{1}{2} e^{x^2} (x^2 - 1) + C$$

Replacing z by $\tan y$, we get

$$\tan y = \frac{1}{2}(x^2 - 1) + C e^{-x^2}$$

which is the required solution.

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SAQ 8. Solve the following equations

(i) $\frac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$

(ii) $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$

(iii) $\frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x^2} (\log z)^2$

(iv) $\frac{dy}{dx} + \frac{1}{x} \tan y = \frac{1}{x^2} \tan y \sin y$

16.10 Bernoulli's Equation

The equation is of the form

$$\frac{dy}{dx} + P y = Q y^n \quad \text{...(1)}$$

where P and Q are functions of x and $n \neq 0$ and $n \neq 1$ (otherwise we would have a linear or an exact equation), is called **Bernoulli's Equation.**

Equation (1) reduces to a linear equation by the following transformation :

Dividing both sides of the equation (1) by y^n, we get

$$y^{-n} \frac{dy}{dx} + P y^{-n+1} = Q \quad \text{...(2)}$$

Making the substitution $z = y^{-n+1}$

we have $\frac{dz}{dx} = (-n+1)\, y^{-n} \frac{dy}{dx}$

substituting this value in (2), we get

$$\frac{dz}{dx} + (-n+1)\, Pz = (-n+1)\, Q$$

Finding its complete integral and substituting the expression y^{-n+1} for z, we get the complete integral of the Bernoulli's equation.

Ex: Solve $x\frac{dy}{dx} + y = x^3 y^6$ (1)

Sol. Dividing throughout by $x\, y^6$

$$y^{-6} \frac{dy}{dx} + \frac{1}{x} y^{-5} = x^2$$

Put $y^{-5} = z$ then $-5y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$,

then the equation (1) reduces to

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$$

$$\text{or} \quad \frac{dz}{dx} - \frac{5}{x} z = -5\, x^2 \qquad (2)$$

which is linear in z.

$$\text{I.F.} = e^{-\int \frac{5}{x} dx} = e^{-5 \log x} = x^{-5}$$

and the solution of (2) is given by

$$zx^{-5} = \int (-5x^2) x^{-5} dx + C$$

$$\text{or } y^{-5} x^{-5} = -5\frac{x^{-2}}{-2} + C$$

$$\text{or } x^3 y^{-5} (2.5 + c\, x^2) = 1$$

which is the required solution.

SAQ. 9: Solve the following

(i) $\frac{dy}{dx} - y \tan x = \frac{\sin x \cos^2 x}{y^2}$

(ii) $(x+1)\frac{dy}{dx} + 1 = 2\, e^{-y}$

(iii) $xy' + y = y^2 \log x$

16.11 Summary

(i) In the equation $y' = f(x, y)$, if $f(x, y) = f_1(x).f_2(y)$; then we can separate the x and y variables and solve the equation.

(ii) If $f(x, y)$ is a homogeneous function of x and y, then we substitute $y = ux$ in the given equation and we solve it by separating the variables.

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(iii) An equation of the form $M\,dx + N\,dy = 0$ is called an exact equation if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. To solve the exact equation: Integrate M with respect to x treating y as a constant and integrate N with respect to y treating x as constant and omit those terms which have been already obtained by integrating M. Adding these two and equating it to an arbitrary constant, we get the solution.

(iv) $\frac{dy}{dx} + P(x)\,y = Q(x)$ is called a linear equation and its solution is given by

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + C$$

16.12 Sample Examination Questions

I. Answer the following in detail.

(i) Explain the method of separation of variables and solve the equation

$$(1-x^2)\frac{dy}{dx} + xy = 5x$$

(ii) a) Define a homogeneous function of degree n and explain the method of finding the solution of a homogeneous equation.

b) Solve $x\frac{dy}{dx} + \frac{y}{x} = y$

(iii) a) How do you solve a non-homogeneous differential equation of first order reducible to homogeneous equation ?

b) Solve $\frac{dy}{dx} = \frac{2x+y+6}{y-x-3}$

(iv) When do you say that a differential equation is exact ? State and prove the necessary and sufficient conditions of an equation to be exact.

(v) What is an integrating factor ? Explain how to find an integrating factor of a differential equation.

II. Briefly answer the following

(i) How do you solve the Leibnitz's Linear equation ?

(ii) Solve $(x+2y^3)\frac{dy}{dx} = y$

(iii) Solve $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$

(iv) Solve $\frac{dx}{dy} - \frac{2}{3}xy = x^4y^3$

III. Answer the following questions in about 5 lines

(i) What is an initial value problem ?

(ii) When do you say that a differential equation is homogeneous ?

(iii) How do you solve an exact differential equation ?

(iv) State the Bernoulli's equation.

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Answers

I. (i) $5 - y = c\sqrt{(1-x^2)}$

(iib) $\log \frac{y}{x} - \frac{1}{x} = c$

(iiib) $y^2 - y(x+3) = 2(x+3)^2 + c$

II (ii) $\frac{x}{y} = y^2 + c$

(iii) $x^2 \tan y = \frac{1}{6}x^3 + c$

(iv) $\frac{1}{x^4} = \frac{3}{2}(1 - y^2) + ce^{-y^2}$

16.13 Answers to SAQ's

SAQ 1 : All the equations can be solved by first inspectin of the equations and the solutions are :

(i) $\tan^{-1}(y/x) = x + c$

(ii) $ax^2y + 2e^x = 2cy$

(iii) $yx^2 + 2x = 2cy$

(iv) $y + x^3 = cx$

(v) $\frac{2}{3}x^3 - \frac{1}{2}y^2 + \frac{e^x}{y} = c$

SAQ 2 : The euations are solvable by the method of variables seperable. The solutions of the equations are :

(i) $y + z = c\,(x -$ [illegible]

(ii) $\sqrt{(1+x^2)} + \sqrt{(1+y^2)} = c$

(iii) $\tan x \,.\, \tan y = c$

(iv) $x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)} + \log\left(x + \sqrt{1+x^2}\right)$

$+ \log\left(y + \sqrt{1+y^2}\right) = c.$

SAQ 3 : All the equations come under Homegeneous equations and the solutions are :

(i) $y = c\,e^{y/x}$

(ii) $y + \sqrt{(x^2+y^2)} = cx^2.$

(iii) $x^2 + y^2 = 2y$

(iv) $\log r = 2\theta + c,\ r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$

SAQ 4 : These are non-homegeneous equations but can be reducible to homogeneous equations. The solutions are :

(i) $y^2 - 2xy - 2x^2 - 6y - 12x - 18 = c$

(ii) $(x + y - 1)^2(y + 2x - 3)^3 = c$

(iii) $x + 5y + 2 = c\,(x - y + 2)^4$

(iv) $(y - x + 1)^2(y + x - 1)^5 = c$

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SAQ 5: These are exact equations. The solutions are :

(i) $yx + y \log x + x \cos y = c$

(ii) $ax^2 + 2hxy + by^2 + 2gx + 2fy = c$

(iii) $x^2 + 2a^2 + \tan^{-1}\left(\frac{x}{y}\right) + y^2 = c$

(iv) $x^3 - 3axy + y^3 = c$

(v) $x + y e^{x/y} = c$

SAQ 6 : These are not exact equations but can be converted into exact equations by proper choice of an integrating factor. The solution are :

(i) $x^3y - ax^2y^2 = c$

(ii) $x + \frac{y^2}{x^2} = c$

(iii) $\left(y + \frac{2}{y^2}\right)x + y^2 = c$

(iv) $3x^2 y^4 + 6xy^2 + 2y = c$

SAQ 7 : All are linear equations and the solutions are :

(i) $y x^{1/2 \log x} = e^x + c$

(ii) $\frac{y}{1-x^2} = \frac{1}{\sqrt{(1-x^2)}} - 1.$

(iii) $\frac{y(x^2+1)}{x} = \frac{x^2}{2} \log x - \frac{x^2}{4} + c$

(iv) $y \sin x = \frac{x}{2} - \frac{1}{4} \sin 2x + c$

SAQ 8: The equations are not linear. But can be converted into linear equations. The solutions are:

(i) $\tan^{-1} y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

(ii) $e^y = e^x - 1 + c e^{-e^x}$

(iii) $\frac{1}{x(\log z)} = \frac{1}{2x^2} + c$

(iv) $2x = \sin y \, (1 + 2cx^2)$

SAQ 9 : These equations are of the form of Bernoullis equation. The solution are :

(i) $y^3 \cos^3 x = -\frac{1}{2}\cos^6 x + c$

(ii) $e^y(x+1) = 2x + c$

(iii) $y(cx + 1 + \log x) = 1$

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Unit - 17: Equations of First Order and of Degree Higher Than the First

17.0 Contents

17.1 Aims and Objectives

After going through this unit, you will be able to recognise various forms of differential equations of first order but the degree higher than one and apply various methods for solving them.

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17.2 Introduction

The most general form of a differential equation of the first order but of n^{th} degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + ... + P_n = 0 \qquad (1)$$

where p denotes $\frac{dy}{dx}$ and $P_1, P_2, ... P_n$ are functions of x and y. This equation in its most general form cannot be solved. Three main methods of solving such equations in certain cases will be presented here. The first consists in solving the differential equation for the derivative i.e., p, so as to replace the original equation by two or more equations each of the first degree. The two remaining methods apply to cases in which it is not convenient to solve for p.

17.3 Factorization of Equations

Suppose the L.H.S. of equation (1) can be resolved into factors of the first degree and the equation can be written as

$$(p - R_1)(p - R_2) \; ... \; (p - R_n) = 0 \qquad (2)$$

Each factor is equated to zero to give rise to equations of first order and first degree. Solving these first degree equations, suppose the solutions are given by

$$\phi_i(x, y, c_i) = 0 \, , \, i = 1,2 \, ... \, n$$

Where c_i's are arbitrary constants. Without loss of generality we can replace $c_1, c_2, ... c_n$ by c. Here $\phi_i \; (x, y, c_i) = 0$ is a solution of $p - R_i = 0$ and hence is a solution of equation (2). Collecting all these solutions, the solution of (1) can be put in the form

$$\phi_1(x, y, c)\, \phi_2(x, y, c) \, ... \, \phi_n(x, y, c) = 0 \qquad (3)$$

Examples

Ex. 1: Solve $x^2 p^2 + 3xy\ p + 2y^2 = 0$

Sol.: Solving for p (assuming the given equation to be a quadratic in p)

$$p = \frac{-3xy \pm \sqrt{(9x^2y^2 - 8x^2y^2)}}{2x^2} = \frac{-3xy \pm xy}{2x^2}$$

i.e. $p = -\frac{y}{x}$ or $p = -2\frac{y}{x}$.

$\frac{dy}{dx} = -\frac{y}{x}$ gives $xy = c$

and $\frac{dy}{dx} = -2\frac{y}{x}$ gives $yx^2 = c$

Hence the general solution is $(xy - c)(yx^2 - c) = 0$

Ex 2. If the curve whose differential equation is $p^2 + 2py\ \cot x - y^2 = 0$, passes through $(\pi/2, 1)$, show that the equation of the curve is given by

$$\left(2y - \sec^2\frac{x}{2}\right)\left(2y - \operatorname{cosec}^2\frac{x}{2}\right) = 0$$

Sol. Solving $p^2 + 2py\ \cot x - y^2 = 0$ for p,

$$p = \left[-2y\ \cot x \pm \sqrt{(4y^2\cot^2 x + 4y^2)}\right]\Big/2$$

or $\frac{dy}{dx} = -y\ (\cot x \pm \operatorname{cosec} x)$,

Taking the $+ve$ and $-ve$ signs, the resulting equations are

(i) $\frac{dy}{dx} = \frac{1 - \cos x}{\sin x}\, y$ (ii) $\frac{dy}{dx} = -\frac{1 + \cos x}{\sin x}\, y$

separating variables in (i), we have

$$\frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\, y$$

$$\text{or} \quad \frac{dy}{y} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\, dx$$

Integrating, $\log y = -2\log\cos\frac{x}{2} + \log c$ or $y = c\ \sec^2\frac{x}{2}$

From (ii) $y = c\ \operatorname{cosec}^2\frac{x}{2}$

Hence the general solution is

$$\left(y - c\sec^2\frac{x}{2}\right)\left(y - c\operatorname{cosec}^2\frac{x}{2}\right) = 0 \qquad (2)$$

(2) has to pass through $(\pi/2, 1)$ implies

$$(1 - 2c)(1 - 2c) = 0 \Rightarrow c = \frac{1}{2}$$

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Putting this value of c in (2) and simplifying, we get

$$\left(2y - \sec^2 \frac{x}{2}\right)\left(2y - \text{cosec}^2 \frac{x}{2}\right) = 0$$ which is the desired form of the curve.

Solve the following Equations *Answers*

(i) $p^2 + p - 2 = 0$ $[(y + 2x + c)(y - x + c) = 0]$

(ii) $p^2(1 - x^2) + y^2 - 1 = 0$ $[(\sin^{-1}y + \sin^{-1}x + c)(\sin^{-1}y - \sin^{-1}x + c) = 0]$

(iii) $y = x\left[p + \sqrt{(1 + p^2)}\right]$ $\left[x^2 + y^2 = cx\right]$

(iv) $p^2 - 2p\cosh x + 1 = 0$ $\left[\left(y - e^x - c\right)\left(y + e^{-x} - c\right) = 0\right]$

17.4 Equations solvable for x

Let the differential equation (1) of 17.2 be put in the form $f(x, y, p) = 0$. When it can not be resolved into rational linear factors as in 17.3 it may be either solved for y or x.

When the given equation is of first degree in x,

then it can be put in the form $x = F(y, p)$ (1)

Differentiating this equation with respect to y, gives us an equation of the form

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \left(\because \frac{dx}{dy} = \frac{1}{p}\right)$$

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This is an equation in two variables y and p and will give a solution of the form

$$\psi(p, y, c) = 0 \qquad (2)$$

eliminating p between (1) and (2) gives the required solution.

Examples

Ex. 1 : Solve $x = 2p^3 + \dfrac{y}{p}$

Sol : $x = 2p^3 + y/p$ (1)

Differentiating (1) with respect to y, we get

$$\frac{1}{p} = 6p^2 \frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{dp}{dy}\left[6p^2 - \frac{y}{p^2}\right] = 0.$$

$$\Rightarrow \frac{dp}{dy} = 0 \text{ or } 6p^2 - \frac{y}{p^2} = 0$$

$$\therefore \frac{dp}{dy} = 0 \Rightarrow p = c.$$

substituting this value in (1), we get

$x = 2c^3 + \dfrac{y}{c}$, (2) the required general solution of the given equation.

Here we have omitted the other factor, $6p^2 - \dfrac{y}{p^2}$ which does not involve a derivative of p with respect to x or y. Such factors always lead to singular solution.

Note 1 : In case the elimination of p is not possible between (1) and (2), they are solved to give results in the form $x = F_1(p, c),\ y = F_2(p, c)$

in which p is regarded as a parameter. Sometimes even this is not possible. In that case (1) and (2) may be regarded as giving x and y in terms of p, that is (1) and (2) are said to constitute the solution in parametric form,

Note 2 : This method is specially useful for equations in which y is absent.

Ex : Solve $x = y^2 + \log p$ (1)

Sol : Differentiating the given equation with respect to y,

$$\frac{1}{p} = 2y + \frac{1}{p}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{dp}{dy} + 2py = 1$$

This is linear in p and hence

$$p\, e^{y^2} = \int e^{y^2} dy + c \qquad (2)$$

(It must be noted that the integral on the R. H. S. cannot be integrated in finite terms).

The eliminant of p between (1) and (2) give solution.

In this problem, the solution has not been got explicitly by eliminating p. So, the solution will be given by the two parametric equations (1) and (2).

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Solve the following equations *Answers*

(i) $y^2 \log y = xyp + p^2$ $[\log y = xc + c^2]$

(ii) $p^2 y + 2px = y$ $[y^2 = 2cx + c^2]$

(iii) $y = 2px + y^2 p^3$ $[y^2 = 2cx + c^3]$

17.5 Equations solvable for y

When the given equation is of first degree in y, it can be put in the form $y = F(x, p)$ (1)

Differentiating with respect to x

$$p = \phi\left(x, p, \frac{dp}{dx}\right)$$

This being an equation in the two variables p and x, and is solved to give

$$\psi(x, p, c) = 0 \qquad (2)$$

The elimination of p between (1) and (2) gives the solution.

If elimination is not possible, we do as explained in Note 1 of above case. This method is specially useful for equations in which x is absent.

Ex : Solve $y = -px + x^4 p^2$

Sol : Differentiating the given equation with respect to x,

$$p = -p - x\frac{dp}{dx} + 4x^3 p^2 + 2x^4 p\frac{dp}{dx}$$

$$\Rightarrow \qquad x\frac{dp}{dx}(1 - 2x^3 p) + 2p\,(1 - 2x^3 p) = 0$$

$$\Rightarrow \qquad (1-2x^3p)\left(x\frac{dp}{dx}+2p\right) = 0$$

We omit the first factor since it will lead to singular solution. Thus for general solution, we have

$$x\frac{dp}{dx}+2p = 0 \text{ or } 2\frac{dx}{x}+\frac{dp}{p} = 0$$

$$\text{Integrating, } 2\log x+\log p = \log c.$$

$$\text{or} \quad p = \frac{c}{x^2},$$

Putting this value of p in the given equation, we get

$$y = -\frac{c}{x^2}x+x^4\frac{c^2}{x^4}$$

$$= -\frac{c}{x}+c^2$$

Solve the following equations — *Answers*

(1) $x^3p^2+x^2yp+4=0$ — $[cxy+4x+c^2=0]$

(2) $4y^3p^2-4xp+y=0$ — $[y^4=4c(x-c)]$

(3) $xp^3-2py^2+4x^2=0$ — $[x^2=4c(y-8c^2)]$

(4) $p^2-xp-y=0$ — $[y=cx-c^2]$

17.6 Lagrange's Equation

The differential equation of the form

$$y = xF(p)+f(p) \tag{1}$$

is known as **Lagrange's equation.**

Clearly (1) is an equation of the form which has been already discussed under 17.4. We shall solve (1) by that method. In fact all problems based on the present article can be solved by using the method discussed in 17.4. However, for the sake of convenience we have considered all problems of special form (1) here separately.

Differentiating the equation (1) with respect to x, we get

$$p = F(p)+xF'(p)\frac{dp}{dx}+f'(p)\frac{dp}{dx}$$

$$p-F(p) = [xF'(p)+f'(p)]\frac{dp}{dx}$$

$$\text{or} \quad \frac{dx}{dp} = \frac{xF'(p)+f'(p)}{p-F(p)}$$

$$\text{or} \quad \frac{dx}{dp}-\frac{F'(p)}{p-F(p)}x = \frac{f'(p)}{p-F(p)}$$

which is linear in x and p and can be solved to give a relation of the form

$$x = \phi(p,c) \tag{2}$$

Then we eliminate p between (1) and (2) to get the required solution.

If p cannot be easily eliminated, we put the value of x in (1) and get

$$y = \phi(p,c)F(p)+f(p) \tag{3}$$

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Now (2) and (3) together give us the required solution in parametric form.

Ex : Solve $y = -xp \log p + (2 + 3\log p)\, p^3$ (1)

Sol : The given equation is in the form of Lagrange's equation.

Differentiating with respect to x, and simplifying we get

$$(1 + \log p)\left[p + x\frac{dp}{dx} - 9p^2\frac{dp}{dx}\right] = 0.$$

Omitting the first factor, we have

$$p + x\frac{dp}{dx} - 9p^2\frac{dp}{dx} = 0$$

$$\text{or} \quad \frac{dp}{dx}(x - 9p^2) + p = 0$$

$$\text{or} \quad \frac{dx}{dp} + \frac{1}{p}x = 9p,$$

a linear equation in x and p and the solution is given by

$$xe^{\int (1/p)\,dp} = \int 9p\, e^{\int (1/p)\,dp}\, dp + c$$

$$\text{or} \quad xp = 3p^3 + c \quad (2)$$

Eliminating p between (1) and (2) we get the solution. Here the elimination is not so easy. Therefore, we express the solution in parametric equations,

$$x = \frac{c}{p} + 3p^2$$

$$y = -c \log p + 2p^3$$

where p is treated as a parameter.

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Solve the following equations. *Answers*

(i) $y = (1 + p)x + p^2$ $\quad [x = 2(1-p) + ce^{-p},\ y = 2 - p^2 + c(1 + p)e^{-p}]$

(ii) $y = p^2x + p^4$ $\quad \left[\begin{array}{l} x = c(p-1)^{-2} - \frac{1}{3}p^3(3p-4)(p-1)^{-2} \\ y = cp^2(p-1) - 2 - \frac{1}{3}p^5(3p-4)(p-1)^{-2} + p4 \end{array}\right]$

(iii) $x - yp = ap^2$ $\quad \left[\begin{array}{l} x = \dfrac{p}{\sqrt{(p^2-1)}}(c - a\cosh^{-1} p) \\ y = \dfrac{1}{\sqrt{(p^2-1)}}(c - a\cosh^{-1} p) - ap \end{array}\right]$

(iv) $p^2 - py + x = 0$ $\quad$ [Hing : Take $a = -1$ in 3]

17.7 Clairaut's equation

If we replace $F(p)$ by p in the Lagrange's equation, we get the differential equation of the form

$$y = xp + f(p) \quad (1)$$

This equation is called **Clairaut's equation.**

Method of Solution

Differentiating the equation (1) with respect to x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\text{or} \quad [x+f'(p)]\frac{dp}{dx} = 0$$

Omitting the first factor, we get $\frac{dp}{dx} = 0$

$$\text{or} \quad p = c. \tag{2}$$

Elimination of p between (1) and (2), we get the solution,

$$y = cx + f(c),$$

Observe that the general solution of Clairaut's equation is obtained by replacing p by c, where c is an arbitrary constant.

Ex : Solve $y = px + 1 + p + p^2$

Sol : The given equation is in the form of Clairaut's equation. The solution will be obtained by replacing p by c and therefore the solution is given by $y = cx + 1 + c + c^2$.

Solve the following equations. *Answers*

(i) $y - x\frac{dy}{dx} = exp\left(\frac{dy}{dx}\right)$ $[y - cx = e^c]$

(ii) $y = px + \frac{a}{p}$ $[y = cx + \frac{a}{c}]$

(iii) $y = xp + p - p^3$ $[y = cx + c - c^3]$

(iv) $y = xp + \sqrt{(1+p^2)}$ $\left[y = cx + \sqrt{(1+c^2)}\right]$

17.7.1 Equations reducible to Clairaut's form

By using suitable substitutions, some equations can be put in the Clairaut's form. There is no general method of deciding about the proper substitution in a certain problem. These can be learned by practice. However, the students are advised to remember the following two substitutions.

I. If the equation is of the form

$y^2 = pxy + f\left(\frac{py}{x}\right)$, then put $x^2 = u$ and $y^2 = v$

Ex : Solve $x^2(y - px) = yp^2$

Sol : Multiplying the given equation by y throughout

$$x^2y^2 - px^3y = y^2p^2$$

$$\text{or} \quad y^2 = pxy + \left(\frac{py}{x}\right)^2 \tag{1}$$

which is of the form $y^2 = pxy + f\left(\frac{py}{x}\right)$

Put $x^2 = u, y^2 = v$

$$\Rightarrow \frac{2y}{2x}\frac{dy}{dx} = \frac{dv}{du}$$

$$\Rightarrow \quad \frac{y}{x} p = P, \qquad p = \frac{dy}{dx}, \; P = \frac{dv}{du}.$$

$$\therefore \quad pxy = \frac{x}{y} P.xy = x^2P = uP,$$

Putting these values of pxy and $p\frac{y}{x}$ in (1), we get $v = uP + P^2$ which is of Clairaut's form and hence its general solution is

$$v = uc + c^2 \text{ or } y^2 = cx^2 + c^2.$$

II. If the equation is of the form

$$e^{by}(a - bp) = f\left(p\, e^{by-ax}\right),$$

then put $e^{ax} = u$, $e^{by} = v$,

Ex : Solve $e^{3x}(p-1) + p^3 e^{2y} = 0.$

Sol : Rewriting, $(p-1)\,e^{3x} = -p^3 e^{2y}$

$$\text{or} \qquad (1-p) = p^3 e^{2y-3x}$$

$$\text{or} \qquad e^y(1-p) = p^3 e^{3(y-x)}$$

$$\text{or} \qquad e^y(1-p) = \left(p\, e^{(y-x)}\right)^3$$

which is of the form given in II.

Put $e^x = u$, $e^y = v$ [here $a = 1$, $b = 1$]

$$\Rightarrow \quad \frac{e^y}{e^x}\frac{dy}{dx} = \frac{dv}{du} \text{ or } \frac{v}{u}p = P \text{ or } p = \frac{u}{v}P$$

Putting these in the given equation, we get

$$u^3\left(\frac{uP}{v} - 1\right) + \frac{u^3P^3}{v^3}v^2 = 0$$

$$\Rightarrow \quad uP - v + P^3 = 0$$

$$\text{or} \qquad v = uP + P^3,$$

which is of the Clairaut's form and the general solution is given by

$$y = uc + c^3 \text{ or } e^y = c\,e^x + c^3$$

Solve the following equations. *Answers*

(i) $e^{4x}(p-1) + e^{2y}p^2 = 0$ $\quad [e^{2y} = ce^{2x} + c^2]$

(ii) $x^2y = x^3p + yp^2$ $\quad [y^2 = ce^x + c^3]$

(iii) $y^2 = 2xyp + y^3p^3$ $\quad \left[y^2 = cx + \frac{1}{8}c^3\right]$

(iv) $y + xp = x^4p^2$ $\quad \left[xy + c = c^2x\right]$

17.8 Equations that do not contain x explicitly

Suppose an equation is of the form

$$f(y, p) = 0 \tag{1}$$

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If it is possible to solve for y in terms of p, we get the relation of the form $y = \psi(p)$ and can be solved according to 17.5.

If (1) can be solved for p, in terms of y, $p = \phi(y)$, it can be integrable immediately and the solution is given by

$$\int \frac{dy}{\phi(y)} = x + c$$

Ex : Solve $y = p + 2p^2$ (1)

Sol : Here x is absent in the given equation.

Differentiating the given equation with respect to x, we get

$$\frac{dp}{dx} = \frac{p}{1+4p}$$

$$\text{or} \quad \frac{1+4p}{p}\, dp = dx$$

The solution of this equation is given by

$$x = \log p + 4p - c \qquad (2)$$

Equations (1) and (2) constitute the general solution of the given equation in the parametric form.

17.9 Equation that do not contain y explicitly

Let the equation be $f(x, p) = 0$ (1)

If it is possible to solve for x in terms of p say, $x = \psi(p)$ (2)

then the equation can be solved according to 17.4. If the given equation is solvable for p,

say $p = \phi(x)$ (3)

then it can be directly integrable and the solution is given by

$$y + c = \int \phi(x)\, dx \qquad (4)$$

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Ex : Solve $2x = p^2 - 3p$

Sol : The given equation $\quad 2x = p^2 - 3p$ (1)

do not contain y explicitly.

Differentiating (1) with respect to y,

$$\frac{2}{p} = 2p\frac{dp}{dy} - 3\frac{dp}{dy} \qquad (2)$$

$$\text{or} \quad \frac{dp}{dy} = \frac{2}{p(2p-3)}$$

Integrating, $\quad 2y + c = \frac{2}{3}p^3 - \frac{3}{2}p^2$ (3)

(1) and (3) constitute the general solution of the given equation in parametric form.

Solve the following equations. *Answers*

(i) $x\sqrt{(p^2-1)} + p = a\sqrt{(p^2-1)}$ $\quad [c^2 - 2cy + y^2 - (a-x)^2 + 1 = 0]$

(ii) $p^2y^2 + 1 = 2py + y^4$ $\quad [c^2 - 2c(\cosh 2x - y^2 \sinh 2x) + 1 - y^4 = 0]$

(iii) $y^2 = p^2 + 1$; $\left[x+c-\log\left\{y+\sqrt{(y^2-1)}\right\}\right]\left[x+c+\log\left\{y+\sqrt{(y^2-1)}\right\}\right] = 0$

(iv) $x^2 = p + 1$ $\qquad 3y = x^3 - 3x + c$

17.10 Equations Homogeneous in x and y

Let the equation be $f(y/x, p)) = 0$ (1)

If this is solvable for p, say, $p = F\left(\frac{y}{x}\right)$ then it can be immediatly integrable.

If (1) is solvable for $\left(\frac{y}{x}\right)$, $y = x\,F(p)$, (2)

then we proceed as in Lagrange's equation.

Differentiating (2) with respect to x, we have

$$p = F(p) + x\,F'(p)\,\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{F'(p)\,dp}{p - F(p)}$$

This can be integrated directly to obtain the relation between x, p and c, and the eliminant of p between this relation and the equation (1) will be the required solution.

Ex : Solve $xy\,p^2 + p\,(3x^2 - 2y^2) - 6xy = 0$.

Sol : This is homogeneous in x and y and solvable for p.

$$p = \frac{2y}{x} \text{ or } -\frac{3x}{y}$$

$$\Rightarrow \quad \frac{dy}{y} = 2\,\frac{dx}{x} \text{ or } y\,dy + 3x\,dx = 0$$

$$\therefore \quad y = cx^2 \text{ or } y^2 + 3x^2 = c$$

The solution is given by

$$(y - cx^2)\,(y^2 + 3x^2 - c) = 0$$

17.11 Second order Differential Equations reducible to first order equations

Some times it is possible to reduce the second order differential equation which does not contain either x or y explicitly, into a first order differential equation by means of some transformation. We will discuss here two such cases.

Consider the equation of the form

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \qquad \text{..... (1)}$$

in which y is absent. This is a second order equation.

$$\text{Now let } \frac{dy}{dx} = p \qquad (2)$$

$$\text{then } \frac{d^2y}{dx^2} = \frac{dp}{dx} \qquad (3)$$

From (1), (2) and (3)

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$$f\left(x, p, \frac{dp}{dx}\right) = 0$$

a first order equation in x and p.

Similarly, we can also solve the equation of the type

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0, \text{ in which } x \text{ is absent.}$$

Taking $\frac{dy}{dx} = p$, we get $f\left(y, p, p\frac{dp}{dy}\right) = 0$, a first order equation.

Examples

Ex. 1 : Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x = \sec x + \cos x$ (1)

Sol : The given equation is a second order differential equation in which y is absent. Taking $\frac{dy}{dx} = p$, the equation reduces to

$$\frac{dp}{dx} + p\tan x = \sec x + \cos x \quad (2)$$

a first order equation in p and x. The integrating factor of this equation is

$exp\ (\int \tan x\, dx) = exp\ (-\log \cos x) = \sec x.$

Multiplying equation (2) by this factor and integrating, we get

$$p = x\cos x + \sin x + C\cos x \quad (3)$$

Integrating, $y = x\sin x + C\sin x + D$ (4)

where C and D are arbitrary constants. (4) is the required general solution of equation (1).

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Ex. 2 : Solve $(1 + x^2)\, y'' + 2xy' = 0.$

Sol : $(1 + x^2)\, y'' + 2xy' = 0.$ (1)

Here y is absent.

Taking $\frac{dy}{dx} = p$, (1) reduces to

$$(1 + x^2)\frac{dp}{dx} + 2xp = 0 \quad (2)$$

Separating the variables and Integrating,

$$\int \frac{dp}{p} + \int \frac{2x}{1 + x^2}\, dx = c \quad (3)$$

$$\Rightarrow p = \frac{c_1}{1 + x^2} \Rightarrow \frac{dy}{dx} = \frac{c_1}{1 + x^2}$$

again integrating, $y = c_1 \tan^{-1} x + c_2$

The required general Solution

If, however the particle be moving along a curve, then

(i) its velocity $v = \frac{ds}{dt}$

and (ii) its acceleration $= \frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$

b) Newton's second Law states Mass x Acceleration = Net force

$$\text{i.e., } m\frac{dv}{dt} = F.$$

c) Hook's law states that tension of an elastic string (or a spring) is proportional to the extension of the string (or the spring) beyond its natural length.

Thus $T = \lambda \frac{e}{l}$

where e is the extension beyond the natural length l and λ is the modulus of elasticity. Sometimes for a spring, we write $T = ke$ where e is the extension beyond the natural length and k is the stiffness of the spring.

Ex. 6: *Motion of a boat across a stream* ; A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current at any point is directly proportional to the product of the distances of the point from the two banks, find the path of the boat and the distance down-stream to the point where it lands.

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Sol.: Taking the origin at the point from where the boat starts, let the axes be chosen as in figure-4.

At any time t after its start from O, let the boat be at $P(x, y)$, so that $\frac{dx}{dt}$ = velocity of the current $= ky(a-y)$ and $\frac{dy}{dt}$ = velocity with which the boat is being rowed $= u$.

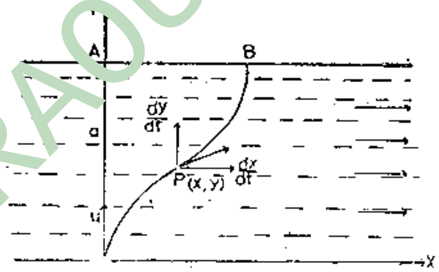

Fig-4.

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{u}{ky(a-y)} \qquad (1)$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Separating the variables in (1),

$$y(a-y)\,dy = \frac{u}{k}\,dx.$$

Integrating, we get

$$\frac{ay^2}{2} - \frac{y^3}{3} = \frac{u}{k}x + c$$

Since $y = 0$ when $x = 0$, $\therefore c = 0$.

Hence the equation to the path of the boat is

$$x = \frac{k}{6u}y^2(3a - 2y) \qquad (2)$$

The distance AB is obtained by putting, $y = a$, and is equal to $ka^3/6u$.

Ex 7. Resisted Motion : A moving particle of unit mass is opposed by a force cx and resistance bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.

Sol. By Newton's second law, the equation of motion of the particle is,

$$v\frac{dv}{dx} = -cx - bv^2$$

$$\text{or } v\frac{dv}{dx} + bv^2 = -cx \qquad (1)$$

This is Bernoulli's equation

$\therefore$ Put $v^2 = z$ and $2v\dfrac{dv}{dx} = \dfrac{dz}{dx}$

so that (1) becomes $\dfrac{dz}{dx} + 2bz = -2cx$ (2)

This is Leibnitz's linear equation and $I.F. = e^{2bx}$.

$\therefore$ The solution of (2) is

$$ze^{2bx} = -\int 2cxe^{2bx}dx + c'$$

$$= -2c\left[x\,\frac{e^{2bx}}{2b} - \int 1.\,\frac{e^{2bx}}{2b}dx\right] + c'$$

$$= -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

$$\text{or } v^2 = \frac{c}{2b^2} + c'\,e^{-2bx} - \frac{cx}{b} \qquad (3)$$

Initially the velocity is zero, i,e. $v = 0$ when $x = 0$.

$$\therefore 0 = \frac{c}{2b^2} + c' \Rightarrow c' = \frac{-c}{2b^2}$$

$$\therefore v = \frac{c}{2b^2}\left[1 - e^{-2bx}\right] - \frac{cx}{b}$$

Ex. 8: Resisted vertical Motion : A Particle of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that

$$\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

If the particle be moving with velocity v after having fallen through a distance x, then its equation of motion is

$$mv\frac{dv}{dx} = mg - kv^2$$

$$\text{or } mv\frac{dv}{dx} = k(a^2 - v^2) \qquad (1)$$

Ex. 3 : Solve $yy'' = (y')^2$

Sol : $y y'' = (y')^2$ (1)

$$\text{Put } \frac{dy}{dx} = p$$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}.$$

Then Equation (1) reduces to

$$yp\frac{dp}{dy} = p^2 \qquad (2)$$

$$\Rightarrow \quad p = 0 \qquad (3)$$

$$\text{or} \quad y\frac{dp}{dy} = p, \qquad (4)$$

$$p = 0 \Rightarrow y = c \qquad (5)$$

Only one arbitrary constant is involved, in this solution and hence it cannot be a solution for (1).

Then (4) can be wrttten as

$$\frac{dy}{y} = \frac{dp}{p} \qquad (6)$$

$$\text{Integrating,} \quad p = c_1 y \qquad (7)$$

This can be written as $c_1 dx = \frac{dy}{y}$.

On integration,

$$c_1 x + c_2 = \log y$$

$$\text{or} \quad y = c_2 e^{c_1 x} \qquad (8)$$

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the required general solution.

If $c_1 = 0$ in (8), we get $y = c_2$ which is in the form of (5)

Solve the following differential equations *Answers*

(i) $(1+x^2)\, y'' + 2xy' = 0$ $\left[y = c_1 \tan^{-1}x + c_2\right]$

(ii) $\left(\frac{d^2x}{dt^2}\right)^2 - t = 3$ $\left[15x = 4\,(t+3)^{5/2} + c_1 t + c_2\right]$

(iii) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ $\left[y = c_1 \log x + c_2\right]$

(iv) $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 1$ $\left[c_1 e^{-y} = \cos(x + c_2)\right]$

(v) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$ $\left[y = \log \sin(x - c_1) + c_2\right]$

17.12 Summary

(1) The general form of the differential equations of first order and nth degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0$$

If this equation can be factorised as

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$(p - R_1)(p - R_2) \ldots (P - R_n) = 0$, each factor equated to zero gives rise to differential equations of first order and first degree and if the solutions of each of the equations are given by $\phi_i(x, y, c) = 0, i = 1, 2 \ldots n$ then the solution of the given equation is $\phi_1(x, y, c)\, \phi_2(x, y, c) \ldots \phi_n(x, y, c) = 0$.

(ii) The differential equation of the form $y = x F(p) + f(p)$ is called Lagranges equation and its solution is obtained by eliminating p between Lagranges equation and a relation of the type $x = \phi(p, c)$.

(iii) The equation of the form $y = xp + f(p)$ is called Clairaut's equation and its solution is obtained by replacing p by c in the equation.

17.13 Sample Examination Questions

I. *Answer the following in detail.*

(i) How do you solve the equation of the type

$x = F(y, p)$ and solve $2px = 2 \tan y + p^3 \cos^2 y$

(ii) Explain the method of solution of the equation of the form

$y = x F(p) + f(p)$. Solve $y = 2px + p^2$

(iii) What is the form of solution of a Clairaut's Equation ?

Solve $y - 2xp + ayp^2 = 0$.

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II. *Solve the following equations.*

(i) $p^3 - p^2(1 + x + x^2) + p(x + x^2 + x^3) - x^3 = 0$.

(ii) $y = x^6 p^3 - xp$

(iii) $y = p(x - b) + \frac{a}{p}$

(iv) $x = p^2 + 1$

(v) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$.

Answers

I (i) $2cx = 2\sin y + c^3$

(ii) $x = -\frac{2p}{3} + \frac{c}{p^2}, y = -\frac{7p^2}{3} + \frac{2c}{p}$

(iii) $y^2 = cx - \frac{1}{4}ac^2$.

II. (i) $(y - x + c)^2 \left(y - \frac{x^3}{2} + c^2\right)\left(y - \frac{x^3}{3} + c\right) = 0$

(ii) $xy = c\left(c^2 x - 1\right)$

(iii) $y = c(x - b) + \frac{a}{c}$;

(iv) $9(y + c)^2 = 4(x - 1)^3$

(v) $y = \log \sin\left(x - c_1\right) + c_2$

Unit – 18 : Applications of First Order Differential Equations

18.0 Contents

18.1 Aims and Objectives

After going through this unit, you will be acquianted with some of the applications of differential equations of first order.

18.2 Introduction

In the previous units we have learnt how to formulate and solve a differential equation. In this unit we shall consider some of the practical problems which give rise to differential equations of first order. The statement of certain physical laws can be expressed in terms of rates of changes of one variable with respect to another. We call $\frac{dy}{dx}$ first rate of change of y with respect to x, and $\frac{d^2y}{dx^2}$ as the second rate of change of y with respect to x etc.

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18.3 Geometric Applications

Determination of curves with given properties

Recall some of the important results which are useful in the Applications.

(a) Cartesian Coordinates

Let $p(x, y)$ be any point on the curve $y = f(x)$, then from figure 1,

(i) Slope of the tangent at $P = \tan\psi = \frac{dy}{dx}$

(ii) Equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

so that its x – intercept $OT = x - y\frac{dx}{dy}$

and y – intercept $OT' = y - x\frac{dy}{dx}$

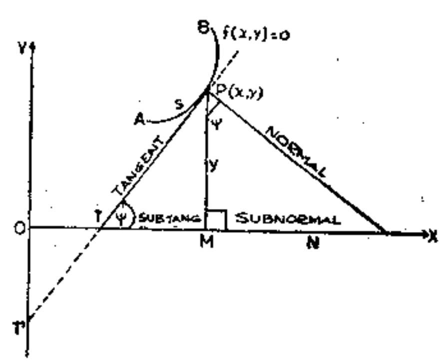

Fig. 1

(iii) Equation of the normal at P is

$$Y - y = \frac{dx}{dy}(X - x)$$

(iv) Length of the tangent $= PT = y\sqrt{\left(1 + \left(\frac{dx}{dy}\right)^2\right)}$

(v) Length of the normal = $PN = y\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$

(vi) Length of the sub-tangent = $TM = y\,\frac{dx}{dy}$

(vii) Length of the sub-normal = $MN = y\,\frac{dy}{dx}$

(viii) The radius of curvature at P $= \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \left(\frac{d^2y}{dx^2}\right)$

(b) Polar Coordinates

The equation of the curve C (interms of polar coordinates, r, θ) is $r = f(\theta)$,then from figure 2

(i) $\psi = \theta + \phi$.

(ii) $\tan\phi = r\,\frac{d\theta}{dr}$, $p = r\sin\phi$

(iii) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(iv) Polar subtangent $(= OT) = r^2\,\frac{d\theta}{dr}$

(v) Polar subnormal $(= ON) = \frac{dr}{d\theta}$

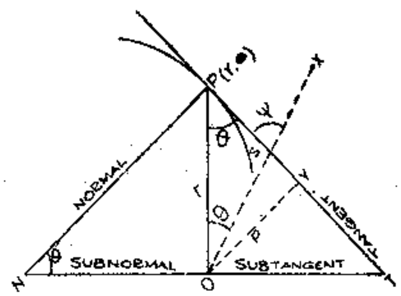

Fig. 2

Ex. 1: Find the curves for which the cartesian subnormal is a constant.

Sol : If x, y are the coordinates in the cartesian system, the subnormal

$$= y\frac{dy}{dx} = C \text{ (a constant)}$$

This is a first order, first degree differential equation in which the variables can be separated for solving it.

The solution of the equation is

$$y^2 = 2Cx + K \text{ (where } K \text{ is the arbitrary constant),}$$

a parabola. Therefore, the equations of the curves for which the subnormal is a constant are the parabolas, with x – axis as their axis.

Ex. 2: Find the curves whose polar sub-tangent is equal to the polar sub-normal.

Sol : Polar sub-tangent = polar sub-normal

$$\Rightarrow r^2\frac{d\theta}{dr} = \frac{dr}{d\theta}$$

$$\text{or } \frac{dr}{d\theta} = \pm r$$

The solution of this equation is

$$\log r = \pm\theta + \log c, \text{ where } c \text{ is an arbitrary constant.}$$

This solution can also be written as

$$r = ce^{\pm\theta}, \text{ the required equation of the curve.}$$

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Ex. 3: Find the curve for which the normal makes equal angles with the radius vector and the initial line.

Sol: Let PT and PN be the tangent and normal at $P(r, \theta)$ of the curve so that

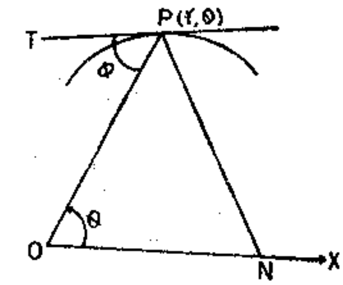

Fig. 3

$$\tan\phi = r\frac{d\theta}{dr}$$

By the condition of the problem,

$$\angle OPN = 90° - \phi = \angle ONP$$

$$\therefore \theta = \angle PON = 180° - (180 - 2\phi) = 2\phi$$

$$\text{or } \frac{\theta}{2} = \phi$$

$$\tan\frac{\theta}{2} = \tan\phi = r\frac{d\theta}{dr}$$

Here the variables are separable;

$$\therefore \frac{dr}{r} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta.$$

Integrating both sides,

$$\log r = 2\log\sin\frac{\theta}{2} + \log c$$

$$\text{or } r = c\sin^2\frac{\theta}{2} = \frac{1}{2}c\,(1 - \cos\theta)$$

Thus the curve is the cardioid $[r = a(1 - \cos\theta)]$.

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Orthogonal Trajectories

The families of curves such that every member of either family cuts each member of the other family at right angles are called Orthogonal Trajectories of each other. The concept of orthogonal trajectories is of wide-spread use in applied mathematics, especially in field problems. For instance in problems concerning the flow of electricity, the paths along which the current flows are orthogonal trajectories of the equipotential curves and vice versa. The lines of heat flow for a body are every where perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of a first order differential equation.

The straight lines passing through the centre, i.e., the diameters are orthogonal trajectories to the concentric circles.

To find the orthogonal trajectories of the family of curves $f(x, y, c) = 0$;

(i) Form its differential equation in the form $f\left(x,\ y,\ \frac{dy}{dx}\right) = 0$, by eliminating c.

(ii) Replace in the differenial equation $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ (so that the product of their slopes at each point of intersection is -1)

(iii) Solve the differential equation of the orthogonal trajectories, i.e., $f\left(x,\ y, -\frac{dx}{dy}\right) = 0$

Ex. 4 : Find the orthogonal trajectories of hyperbolas $xy = c$.

Sol :
$$xy = c \qquad (1)$$

Differentiating (1) with respect to x, we get

$$x\frac{dy}{dx} + y = 0 \qquad (2)$$

as the differential equation of the given family (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we obtain

$$x\left(-\frac{dx}{dy}\right) + y = 0$$

$$\text{or} \quad y\,dy - x\,dx = 0 \qquad (3)$$

as the differential equation of the orthogonal trajectories. Integrating (3), we get $y^2 - x^2 = c$, as the required orthogonal trajectories of (1).

Ex. 5: Find the orthogonal trajectories of a system of confocal and coaxial parabolas.

Sol : The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \qquad (1)$$

Differentiating (1), with respect to x,

$$y\frac{dy}{dx} = 2a \qquad (2)$$

Substituting the value of 'a' from (2) in (1), we obtain

$$y^2 = 2y\frac{dy}{dx}\left[x + \frac{1}{2}y\frac{dy}{dx}\right]$$

$$\text{or} \quad y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \qquad (3)$$

as the differential equation of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3) we obtain

$$y\left(\frac{dx}{dy}\right)^2 - 2x\left(\frac{dx}{dy}\right) - y = 0$$

$$\text{or} \quad y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0 \qquad (4)$$

which is the same as (3).

Thus we see that a system of confocal and coaxial parabolas is self orthogonal, i.e., each member of the family (1) cuts every other member of the same family orthogonally.

18.4 Physical Application

a) Let a particle of mass m start moving from O along the straight line OX under the action of a force F. After any time t, let it be moving at P, where $OP = x$, then

(i) its velocity $v = \frac{dx}{dt}$

(ii) its acceleration $= \frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$

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separating the variables and intergrating, we get

$$\int v \frac{dv}{a^2 - v^2} = \int \frac{k}{m} dx + c$$

$$\Rightarrow -\frac{1}{2} \log (a^2 - v^2) = \frac{kx}{m} + c \qquad (2)$$

Initially, when $x = 0, v = 0$

$$\therefore -\frac{1}{2} \log a^2 = c. \qquad (3)$$

Subtracting (3) from (2), we get

$$\frac{1}{2}[\log a^2 - \log (a^2 - v^2)] = \frac{kx}{m}$$

$$\Rightarrow 2\frac{kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Note : When the resistance becomes equal to the weight, the acceleration becomes zero and the particle continues to fall with a constant velocity, called the limiting or terminal velocity. From (1) it follows that the acceleration will become zero when $v = a$. Thus, the limiting velocity, i.e. the maximum velocity which the particle can attain is a.

Ex 9. Escape Velocity : Find the initial velocity of a particle which is fired in radial direction from the earth and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.

According to Newton's law of gravitation, the acceleration a of the particle is proportional to $1/r^2$ where r is the variable distance of the particle from the earth's centre. Thus

$a = v \dfrac{dv}{dr} = -\mu/r^2$, where v is velocity at a distance r from the earth's centre. The acceleration is negative because v is decreasing.

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When $r = R$, the earth's radius, then $a = -g$, the acceleration of gravity at the earth's surface.

$$\therefore -g = -\mu/R^2 \Rightarrow \mu = gR^2$$

$$\therefore v\frac{dv}{dr} = -gR^2/r^2$$

Separating the variables and integrating, we obtain

$$\int v\, dv = -gR^2 \int \frac{dr}{r^2} + c$$

$$\text{i.e., } v^2 = \frac{2gR^2}{r} + 2c$$

on the earth's surface $r = R$, and $v = v_0^2$ (say), the initial velocity.

Then $v_0^2 = 2gR + 2c$, i.e., $2c = v_0^2 - 2gR$.

Substituting this value of c in (1),

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$$

When v vanishes, the particle stops and the velocity will change from positive to negative and the particle will return to the earth. Thus the velocity will remain positive, if and only if $v_0^2 \geq 2gR$ and then the particle projected from the earth with this velocity will escape from the earth. Hence the minimum of such velocity of projection $v_0 = \sqrt{(2gR)}$ is called the velocity of escape from the earth. (This is equal to 11.2651 km/sec.)

Heat Flow : The fundamental principles involved in the problems of heat conduction are :

(i) Heat flows from a higher temparature regions to the lower temperature regions.

(ii) The quantity of heat in a body is proportional to its mass and temperature.

(iii) The rate of heat flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.

If q (cal/sec) be the quantity of heat that flows across an area α (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (iii) above.

$$q = -k\,\alpha \frac{dT}{dx}$$

Where k is a constant depending upon the material of the body and is called the thermal conductivity.

Ex 10. A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm, thick for which $K = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.

Sol. Let q cal/sec be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm and length 1 cm. Then the area of the lateral surface $= 2\pi x$.

The equation $q = -k\,\alpha \dfrac{dT}{dx}$ gives

$$q = -k \,.\, 2\pi x \frac{dT}{dx}$$

$$\text{or } dT = -\frac{q}{2\pi k}\frac{dx}{x}$$

Integrating, we have $T = -\dfrac{q}{2\pi k}\log x + c.$

Since $T = 150$, when $x = 10$,

$$150 = -\frac{q}{2\pi k}\log 10 + c. \qquad (1)$$

Again since $T = 40$, when $x = 15$,

$$40 = -\frac{q}{2\pi k}\log 15 + c. \qquad (2)$$

Subtracting (2) from (1), $110 = \dfrac{q}{2\pi k}\log(1.5)$ (3)

Let $T = t$, when $x = 12.5$

$$\therefore t = \frac{q}{2\pi k}\log(12.5) + c \qquad (4)$$

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Subtracting (1) from (4),

$$t - 150 = -\frac{q}{2\pi k} \log (1.25) \qquad (5)$$

Dividing (5) by (3),

$$\frac{t - 150}{110} = -\frac{\log 1.25}{\log 1.5}$$

$$\Rightarrow t = 89.5°C.$$

Radioactive Disintegration

The nuclii of radioactive substances are highly unstable. α, β, γ rays will be emitted continuously and spontaneously from such materials. This process continues till a new non-radio active matter is formed. The result of the decay is the decrease of mass M of the substance. According to the Rutherford's theory, the rate of decrease of mass at any time t is directly proportional to the mass of the substance at that time.

$$\text{i.e., } \frac{dM}{dt} = -\lambda M \qquad (1)$$

λ is the proportionality constant, depends on the substance.

The solution of this equation.

$$M(t) = Ce^{-\lambda t} \qquad (2)$$

C is the arbitary constant.

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To decide the constants C and λ, two conditions are needed. Let M_1 and M_2 be the masses at times t_1 and t_2 respectively.

Then $M_1 = Ce^{-\lambda t_1}, M_2 = Ce^{-\lambda t_2}$

so that $\lambda = (\log M_1 - \log M_2)/(t_2 - t_1)$

$$C = e^{\left\{ (t_2 \log M_1 - t_1 \log M_2)/(t_2 - t_1) \right\}} \qquad (3)$$

Substituting the values of λ and C in (2), we get

$$M(t) = \left[M_1^{(t_2 - t)} M_2^{(t - t_1)} \right]^{1/(t_2 - t_1)} \qquad (4)$$

and the time, at which the mass of the substance is M is,

$$t = \left[t_2 \log \left(\frac{M_1}{M} \right) + t_1 \log \left(\frac{M}{M_2} \right) \right] \Big/ (t_2 - t_1) \qquad (5)$$

From (4), it can be seen that the initial mass of the substance is

$$M_0 = M(0) = \left[M_1^{t_2} / M_2^{t_1} \right]^{1/(t_2 - t_1)}$$

Half Life Period

The time taken for disintigration of the substance which reduces the mass to half of the orginial mass of the substance is called Half life period of that substance.

If M_0 is the initial mass of the substance, from (2) $C = M_0$

$$\therefore \quad M(t) = M_0 e^{-\lambda t}$$

Let T be the half life period,

$$M(T) = \frac{M_0}{2} \Rightarrow \frac{M_0}{2} = M_0 e^{-\lambda T}$$

$$\Rightarrow T = \frac{1}{\lambda}\log 2 = \frac{(t_2 - t_1)\log 2}{\log\left(\frac{M_1}{M_2}\right)}$$

Ex 11: If the half life period of Radium is 1600 years, what is the time taken for 100 mg of radium to get reduced to 90 mg of radium.

Sol. We have seen that the differential equations governing the mass of radioactive substance with time is

$$\frac{dM(t)}{dt} = -\lambda M \qquad (1)$$

The conditions given here are

$$M(0) = 100 \text{ mg}, M(1600) = 50 \text{ mg} \qquad (2)$$

The solution of equation (1) is given by

$$\log M = -\lambda t + C$$

To obtain the constants λ and C, we use (2), $\log 100 = C$.

$$\log 50 = -1600\,\lambda t + C$$

$$\therefore \lambda = \frac{\log 2}{1600} = 0.0004332;\ C = \log 100$$

$$\Rightarrow \log M = -0.0004332t + \log 100$$

$$\Rightarrow \quad M = 100.\, e^{-0.0004332\, t}$$

$$\therefore \quad t = \frac{\log 100 - \log M}{0.0004332}$$

Time taken for $M = 90$ is, $t = \dfrac{\log 100 - \log 90}{0.0004332} = 243.3$ years

18.5 Application to population growth (Decay)

The change in the population is given by the differential equation

$$\frac{dp}{dt} = f(p) \qquad (1)$$

Here $f(p)$ is a function indicating population growth or decay.

In ordinary circumstances the growth or decay of population are relatively constant and therefore $f(p)$ can be taken as a linear function of P.

So, in ordinary circumstances,

$$\frac{dp}{dt} = \alpha p, \qquad (2)$$

where α is a constant, and is positive if it is a growth in population, negative if it is a decay in population and $\alpha = 0$ if the growth and decay is the same.

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The solution of the equation (2) is given by

$$p(t) = p_0 e^{\alpha t}, \text{ where } p_0 \text{ is the population presented initially.}$$

The decay of population is too high during wars, epidemic etc. In that case $f(p)$ can not be taken as αp. Using the statistical back ground we take the function $f(p) = \alpha(p - \epsilon p^2)$

where $\alpha > 0, \epsilon > 0$.

In that case the differential equation is

$$\frac{dp}{dt} = \alpha(p - \epsilon p^2) \qquad (3)$$

and the solution is $p(t) = p_0 e^{\alpha t} \Big/ \left\{ 1 + \epsilon p_0 \left(e^{\alpha t} - 1 \right) \right\}$

Ex 13: If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants?

Sol. Let y be the population at time t and let y_0 be the intitial population.

Then $\quad \dfrac{dy}{dt} = \alpha y \Rightarrow \dfrac{dy}{y} = \alpha\, dt,$

where α is the proportionality constant.

Thesolution is given by

$$y = c\, e^{\alpha t}$$

$$\text{or } y = y_0 e^{\alpha t} \qquad \left(\because y(0) = y_0 \right)$$

At $t = 50, y = 2y_0,$

$$\therefore 2y_0 = y_0 e^{50\alpha} \text{ or } e^{50\alpha} = 2$$

When $y = 3y_0,$

$$3y_0 = y_0 e^{\alpha t} : \text{raising the power to 50 on both sides,}$$

$$3^{50} = e^{50\alpha t} = \left(e^{50\alpha}\right)^t = 2^t$$

$$\therefore t = 79 \text{ years}$$

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18.6 Summary

(i) Geometrical Applications : Orthogonal Trajectories

The families of curves such that every member of either family cuts each member of the other family at right angles are called orthogonal tajectories of each other. To find the orthogonal trajectories of the family of curves $f(x, y, c) = 0$; form its differential equation by eliminating c and replace $\dfrac{dy}{dx}$ *by* $-\dfrac{dx}{dy}$ in the differential equation and solve it.

(ii) Physical Application

a) Newton's II law states that $m\dfrac{dv}{dt} = \mathbf{F}$, where **V** is the velocity of the particle and **F** is the force acting and m is the mass of the particle.

b) The differential equation for the resisted motion of a particle is given by

$$v\frac{dv}{dx} = -cx - bv^2.$$

c) The heat flow equation is $q = -K \alpha \frac{dT}{dx}$, where q is quantity of heat that flows across an area α and thickness δx in one second where the difference of temparature at the face is δT and K is the thermal conductivity.

d) The rate of change of population is given by the diffrential equation $\frac{dp}{dt} = f(p)$; where $f(p)$ is a function indicating population growth or decay.

18.7 Sample Examination Questions

I. Answer the following questions in detail

(i) a) How do you find the orthogonal trajectories of a given family of curves?

b) Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda y + c = 0$, λ being the parameter.

(ii) a) Find the initial velocity of a particle which is fired in radial direction from the earth and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.

b) A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\frac{m}{k} \log\left(1 + \frac{kv}{mg}\right)$.

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II. Briefly answer the following

(i) Determine the curve in which the length of the subnormal is proportional to the square of the ordinate.

(ii) Find the curve whose polar subtangent is constant.

(iii) Find the orthogonal trajectories of the cardioids $r = a\,(1 + \cos\theta)$

(iv) When a thermometer is placed in a hot liquid both at temperature T, the temperature θ indicated by the thermometer rises at a rate proportional to $T - \theta$. For a bath at temparature 95°, the thermometer reads 15° at a certain instant ($t = 0$) and 35° at $t = 10$ secs. What will be its reading at $t = 20$ sec?

(v) Radium decomposes at a rate propotional to the amount present. If P % of the original amount disappears in l years how much will remain at the end of $2l$ years.

Answers

I (i) b) $x^2 + y^2 + 2\mu x - c = 0$

II (i) $y = a e^{cx}$

(ii) $r(\theta - \alpha) = c$

(iii) $r = c\,(1 - \cos\theta)$

(iv) 50°C

(vi) $\left(10 - \frac{P}{10}\right)^2$

BLOCK - 6 : DIFFERENTIAL EQUATIONS – II

Unit – 19 : Linear equations of Second and Higher order with constant coefficients

19.0 Contents

19.1 Aims and Objectives

After going through this unit, you will be able to : (i) solve the linear differential equations with constant coefficients of second and higher order equations (ii) obtain the particular integral of the given differential equation using the method of variation of parameters.

19.2 Introduction

The ordinary differential equations may be divided into two large classes, the linear equations and non-linear equations. While non-linear equations (of second and higher orders) are difficult, linear equations are much simpler in many respects, because, various properties of their solutions can be characterised in a general way and standard methods are available for solving many of these equations. In the present unit we shall consider linear differential equations with constant coefficients. These equations play an important role in mechanical vibrations, electric circuits and networks etc.

The general form of the linear differential equation of nth order is given by

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1} \frac{dy}{dx} + a_ny = h(x) \qquad (1)$$

in which the coefficients $a_1, a_2 \dots\dots, a_n$ are constants.

As we have already mentioned in unit 15 the distingwishing character of such an equation is the absence of products or non-linear functions of the dependent variable y and its derivatives.

Using the symbolic notation D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$ etc.,

We write the equation (1) as

$$D^ny + a_1D^{n-1}y + \dots + a_ny = h(x)$$

or $$\left(D^n + a_1 D^{n-1} + \dots + a_n\right)y = h(x) \qquad (2)$$

It can also be written as $f(D)y = h(x)$ (3)

where $f(D)$ is a linear operator, $D^n + a_1 D^{n-1} + \dots + a_n$. Before going for the solution of this equation it is required to introduce the concepts of linear independence and dependence of solutions. By a linear combination of n solutions, $y_1, y_2, \dots y_n$, is meant an expression of the form

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

where c's are constants. The solutions, $y_1, y_2, \dots, y_n$ are said to be linearly independent iff

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0 \Rightarrow$$

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all the c's are equal to zero. Otherwise the solutions are said to be linearly dependent.

$$W\,(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \text{ is called}$$

the Wronskian of solutions $y_1, y_2, \dots y_n$.

The necessary and sufficient condition for the linear independence of the solutions $y_1, y_2, \dots y_n$. is that their wronskian should not vanish. For example, e^x and e^{-x} are linearly independent since

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0.$$

19.3 Solutions

If we replace $h(x)$ by zero in equation (3), we get

$$f(D)\,y = 0 \qquad (4)$$

This equation is called homogeneous equation since each term in the equation involves the first power of y or of one of its derivatives.

If $y = y_1(x)$ is a solution of the equation (4), then $cy_1(x)$ (c is an arbitrary constant) is also a solution, since by subsitituting $cy_1(x)$ in the equation (4)

$$f(D)\,[c\,y_1] = c\,f(D)\,y_1 = 0$$

[$\because$ y_1 is a solution and $f(D)$ is a linear operator].

Similarly, if $y_1, y_2, \dots, y_n$ are solutions of the equation (4),

we have $f(D)\,y_1 = 0,\; f(D)\,y_2 = 0 \dots f(D)\,y_n = 0$ and

$$f(D)\,[c_1\,y_1 + c_2\,y_2 + \dots + c_n\,y_n] = c_1 f(D)\,y_1 + c_2 f(D)\,y_2 + \dots + c_n f(D)\,y_n$$
$$= c_1.0 + c_2.0 + \dots + c_n.0 = 0.$$

$\Rightarrow$ $c_1\,y_1 + c_2\,y_2 + \dots + c_n y_n$ is also a solution of (4).

If $y_1, y_2 \dots y_n$ are a set of independent solutions of the equation (4) (an nth order homogeneous equation), then $y = c_1\,y_1 + c_2\,y_2 + \dots + c_n\,y_n$ is called the general solution or the primitive of the equation (4). It can be shown that any solution can be represented in the form $c_1\,y_1 + c_2\,y_2 + \dots + c_n\,y_n$ for some $c_1, c_2, \dots c_n$. If $y = p(x)$ is a particular integral of the equation

$$f(D)\,y = h(x) \qquad (5)$$

then $y = c_1\,y_1 + c_2\,y_2 + \dots + c_n\,y_n + p(x)$ is called the complete solution (or the primitive) of the equation (5).

The general solution of the homogeneous equation (4) namely $c_1\,y_1 + c_2\,y_2 + \dots + c_n\,y_n$ is called the complementary function of the equation (5). So, the complete solution of the equation (5) is the sum of the complementary function and the particular Integral.

19.4 Auxiliary Equation

Consider the homogeneous equation, $\left(D^n + a_1 D^{n-1} + \dots + a_n\right) y = 0.$

Assume that $y = e^{mx}$ is a solution of this equation.

Since $Dy = m.e^{mx}$, $D^2y = m^2e^{mx}, \dots D^n e^{mx} = m^n e^{mx}$, and substituting these values in the above equation, we get.

$$\left(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n\right) e^{mx} = 0,$$

which will hold only if

$$m^n + a_1 m^{n-1} + \dots + a_n = 0. \qquad (6)$$

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This equation (6) is called the auxiliary equation. It should be noticed that (6) is obtained from the associated homogeneous equation by formally replacing D^k by m^k. On solving the auxiliary equation we shall get n roots. Three cases arise according as the roots of the auxiliary equation are (i) distinct (ii) repeated (iii) complex.

Case (i) Suppose the auxiliary equation has n-distinct roots, say $m_1, m_2 \dots m_n$.

Then $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, \dots y_n = e^{m_n x}$ are the n independent solutions and the general solution is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

Ex. 1 : Solve the equation $\left(D^2 + 7D + 12\right) y = 0.$

Sol : Let the solution be $y = e^{mx}$

then $\frac{dy}{dx} = me^{mx}, \ \frac{d^2y}{dx^2} = m^2 e^{mx}$

substituting these values in the given equation, we get

$$m^2 e^{mx} + 7m.e^{mx} + 12e^{mx} = 0$$

$$\Rightarrow \quad (m^2 + 7m + 12)\, e^{mx} = 0$$

$$\Rightarrow m^2 + 7m + 12 = 0$$

$$\Rightarrow (m+3)(m+4) = 0.$$

$$\Rightarrow m = -3 \text{ or } -4$$

The general solution is given by $y = c_1 e^{-4x} + c_2 e^{-3x}$, where c_1 and c_2 are arbitrary constants.

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Ex. 2 : Solve $\frac{d^2y}{dx^2} - 16y = 0.$

Sol : The auxiliary equation of the given differential equation is

$m^2 - 16 = 0 \Rightarrow m = \pm 4.$

$\therefore$ The general solution is $y = c_1 e^{4x} + c_2 e^{-4x}$.

Case (ii) If one or more of roots of the auxiliary equation is repeated, less than n-independent solutions to the given nth order homogeneous differential equation are obtained. But, for the general solution we must have n independent solutions. To find the missing solutions we proceed as follows.

Suppose $m = m_1$ is a double root of the auxiliary equation. Then

$$f(D)\, e^{mx} = (m - m_1)^2 (m - m_3) \dots (m - m_n)\, e^{mx}$$

It follows that not only the right hand side number itself but also its derivative with respect to m must vanish when $m = m_1$ and hence

$$f(D)\, e^{mx}\Big|_{m=m_1} = 0, \ \frac{d}{dm}\left[f(D)\, e^{mx}\right]_{m=m_1} = 0$$

i.e., $$f(D)\left[\frac{d}{dm} e^{mx}\right]_{m=m_1} = f(D)\, x e^{m_1 x} = 0$$

$$\left[\text{Here we made use of fact that } \frac{d}{dm}\left(\frac{d^k e^{mx}}{dx^k}\right) = \frac{d^k}{dx^k}\left(\frac{d}{dm} e^{mx}\right)\right]$$

Therefore $x e^{m_1 x}$ is also a solution.

Thus the part of the solution corresponding to a double root m_1 can be written in the form $c_1 e^{m_1 x} + c_2 x e^{m_1 x} = (c_1 + c_2 x)\, e^{m_1 x}$ and the general solution is $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \ldots + c_n e^{m_n x}$. By a simple extension of this argument, it can be shown that the part of the solution corresponding to a root m_1 repeated k times is of the form $\left(c_1 + c_2 x + c_3 x^2 + \ldots + c_k x^{k-1}\right) e^{m_1 x}$.

Hence to each of the n-roots of the auxiliary equation, repeated roots being counted separately there is a corresponding known solution and the general solution of the homogeneous differential equation is determined as a linear combination of these n independent solutions.

Ex. 3 : Solve the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Sol : In the symbolic form, the given equation is $(D^3 - 3D + 2)\, y = 0$.

so that the auxiliary equation is $m^3 - 3m + 2 = 0$

$$\text{Factorising } m^2(m-1) + m(m-1) - 2(m-1) = 0.$$

$$\text{or} \quad (m-1)\,(m^2 + m - 2) = 0$$

$$(m-1)(m-1)(m+2) = 0$$

The roots are $m = 1,\ 1, -2$

Here $m = 1$ is a repeated (double) root and the general solution is given by

$y = (c_1 + c_2 x)\, e^{x} + c_3 e^{-2x}$ where c_1, c_2 and c_3 are arbitrary constants.

Ex. 4 : Solve $(D^3 - 3D^2 + 3D - 1)\ y = 0$

Sol : The auxiliary equation is $m^3 - 3m^2 + 3m - 1 = 0$ or $(m-1)^3 = 0$

i.e., $m - 1$ repeated thrice.

The general solution is $y = (c_1 + c_2 x + c_3 x^2)\, e^{x}$.

Case (iii) If the auxiliary equation has imaginary roots (complex roots), the roots must occur in conjugate pairs.

Thus, if $m_1 = a + ib$ is one root, a second root must be $m_2 = a - ib$.

The part of the solution corresponding to these two roots can be written in the form

$$Ae^{(a+ib)x} + Be^{(a-ib)x} = e^{ax}\left[Ae^{ibx} + Be^{-ibx}\right].$$

By making use of the Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the solution becomes,

$$e^{ax}\,[A\,(\cos bx + i\sin bx) + B\,(\cos bx - i\sin bx)]$$

It can be written in a more convenient form,

$$e^{ax}\,[c_1\cos bx + c_2\sin bx], \text{ where } c_1 = A + B,\ c_2 = i\,(A - B)$$

If $a \pm ib$ are k fold roots, the corresponding part of the general solution will be

$$e^{ax}\left[\left(c_1 + c_2 x + c_3 x^2 + \ldots + c_k x^{k-1}\right)\cos bx + \left(c_{k+1} + c_{k+2}\,x + \ldots + c_{2k} x^{k-1}\right)\sin bx\right]$$

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Ex. 5 : Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Sol : The auxiliary equation is $m^2 + 2m + 5 = 0.$

The roots are $m = \frac{-2 \pm \sqrt{(2^2 - 4 \times 5)}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$

The general solution is $y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

Ex. 6 : Solve $(D^4 + m^4)\, y = 0$

Sol : The auxiliary equation is $n^4 + m^4 = 0.$ (1)

Factorising, $n^4 + m^4 = (n^2 + m^2)^2 - 2n^2m^2$

$= \left(n^2 + m^2 + \sqrt{2}\,nm\right)\left(n^2 + m^2 - \sqrt{2}\,nm\right)$

$\Rightarrow \quad n^2 + m^2 + \sqrt{2}\,nm = 0$ .... (2)

or $\quad n^2 + m^2 - \sqrt{2}\,nm = 0$ .... (3)

(2) gives $n = -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$

and (3) gives $n = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$

The general solution is given by

$$y = e^{-mx/\sqrt{2}}\left(c_1 \cos\frac{mx}{\sqrt{2}} + c_2 \sin\frac{mx}{\sqrt{2}}\right) + e^{mx/\sqrt{2}}\left(c_3 \cos\frac{mx}{\sqrt{2}} + c_4 \sin\frac{mx}{\sqrt{2}}\right)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Exercise – I

I. Answer the following in detail.

(1) How do you obtain the general solution of a homogeneous equation (of n^{th} order) $f(D)\,y = 0$? Discuss various cases.

(2) (a) How do you obtain the general solution of a given homogeneous equation if a root is repeated k times?

(b) Solve the equation $(D^4 - 8D^2 + 16)\, y = 0.$

(3) (a) Obtain the general solution of a given homogeneous equation if a root is complex.

(b) Solve $(D^4 - 4D^3 + 14D^2 - 20D + 25)\, y = 0$

II. Briefly answer the following.

Solve (1) $(D^2 + 9D + 20)\, y = 0$

(2) $(D^2 - 2D + 24)\, y = 0$

(3) $(D^3 - 6D^2 + 11D - 6)\, y = 0$

(4) $(D^2 - 9)\, y = 0$

(5) $(D^4 - 5D^3 + 9D^2 - 7D + 2)\, y = 0$

(6) $(D^2 - 2D + 10)\, y = 0$

(7) $(D^2 + 4)\, y = 0$

(8) $(D^3 + 16D)\, y = 0$

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III. Answer the following in about 5 lines.

(1) If $y_1, y_2 \ldots. y_n$ are solutions of $f(D)y = 0$ show that their linear combination is also a solution.

(2) Obtain the auxiliary equation of $f(D)y = 0$.

Answers

I. (2) b) $y = (A+Bx)e^{2x} + (C+Dx)e^{-2x}$

(3) b) $y = e^{x}\left[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x\right]$

II. (1) $y = c_1e^{4x} + c_2e^{5x}$

(2) $y = c_1e^{4x} + c_2e^{-6x}$

(3) $y = c_1e^{x} + c_2e^{2x} + c_3e^{3x}$

(4) $y = A\cosh 3x + B\sinh 3x$

(5) $y = (c_1 + c_2x + c_3x^2)e^{x} + c_4e^{2x}$

(6) $y = e^{x}[A\cos 3x + B\sin 3x]$

(7) $y = A\cos 2x + B\sin 2x$

(8) $y = c_1 + c_2\cos 4x + c_3\sin 4x$

19.5 The Operators D and D^{-1}

We are using the operator D in the place of $\frac{d}{dx}$ and $f(D)$ in place of

$D^n + a_1D^{n-1} + \ldots.. + a_n$

The operator D^m follows the commutative, associative and distributive laws.

i.e., $(D^n + D^m)\phi(x) = (D^m + D^n)\phi(x) = D^n\phi(x) + D^m\phi(x)$

and $D^m . D^n\{\phi(x)\} = D^n . D^m\{\phi(x)\} = D^{m+n}\{\phi(x)\}$ etc.

If $f(D)$ and $\phi(D)$ are two operators, then the product $f(D) . \phi(D)$ is also an operator. When this is operated on a function, $\phi(D)$ should be operated first and then $f(D)$.

It can be seen that $f(D) . \phi(D) = \phi(D) . f(D)$

The operator D^{-1} is called the inverse operator of D,

i.e., $D^{-1}(x) = \frac{1}{D}(x)$

and is the integral of x. Similarly, if $\frac{1}{f(D)}$ is the inverse operator of $f(D)$

i.e., $\frac{1}{f(D)}\phi = \psi \Rightarrow f(D)\psi = \phi$

19.6 Determination of the particular integral

Consider the non-homogeneous equation $f(D)y = h(x)$.

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Let $\frac{1}{f(D)}h(x)$ denote some function of x, which operated upon by $f(D)$ produces $h(x)$. Clearly the equation is satisfied if we take $y = \frac{1}{f(D)}h(x)$ and therefore it is the particular integral of the equation. We shall now discuss some of the methods for finding out the particular integral of certain standard equations.

(i) **$h(x) = e^{ax}$ (a is constant)**

The differential equation is in the form $f(D)y = e^{ax}$, and the particular integral is $\frac{1}{f(D)}e^{ax}$.

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
....
$$D^ne^{ax} = a^ne^{ax} \quad \text{or} \quad f(D)e^{ax} = f(a)e^{ax}$$
$$\therefore \quad e^{ax} = \frac{1}{f(D)}f(a).e^{ax}$$
$$\text{or} \quad e^{ax} = f(a).\frac{1}{f(D)}e^{ax}$$

Dividing both sides by $f(a)$, $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$

$\therefore$ The particular integral $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, f(a) \neq 0$ (7)

If $f(a) = 0$, the above method fails and we proceed as follows.

$f(a) = 0 \Rightarrow a$ is a root of the equation

$D^n + a_1D^{n-1} + ... + a_n = 0$ and $(D - a)$ is a factor of $f(D)$.

Suppose $f(D) = (D - a)\phi(D)$, where $\phi(a) \neq 0$.

Then
$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)\phi(D)}e^{ax} = \frac{1}{D-a}\left(\frac{1}{\phi(a)}e^{ax}\right),$$

when $\phi(a) \neq 0$. (by using the above case)
$$= \frac{1}{\phi(a)}.\frac{1}{(D-a)}e^{ax}.$$

Let $\frac{1}{D-a}e^{ax} = y.$ i.e., $e^{ax} = (D-a)y$

or $Dy - ay = e^{ax}$

a solution of which is given by
$$y.e^{-ax} = \int e^{ax}e^{-ax}dx = \int dx = x.$$
$$\text{or } y = x.e^{ax}.$$

$\therefore$ The particular integral when $f(a) = 0$, is given by
$$\frac{1}{f(D)}e^{ax} = \frac{1}{\phi(a)}x.e^{ax}, \text{ provided } \phi(a) \neq 0.$$

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We have taken $f(D) = (D-a)\,\phi(D),$

$\therefore \quad f'(D) = (D-a)\,\phi'(D) + 1.\,\phi(D)$

or $\quad f'(a) = \phi(a)$

$$\frac{1}{f(D)} e^{ax} = \frac{x \,.\, e^{ax}}{f'(a)} \qquad (8)$$

provided $\quad f'(a) \neq 0$

If $f'(a) = 0$, it can be shown that

$$\frac{1}{f(D)} e^{ax} = x^2 \,.\, \frac{e^{ax}}{f''(a)} \qquad (9)$$

provided $\quad f''(a) \neq 0$ and so on.

Ex. 7: Solve $\left(D^2 + 5D + 6\right)y = e^x.$

Sol. For finding out the complementary function, we solve

$$\left(D^2 + 5D + 6\right)y = 0$$

The auxiliary equation is $m^2 + 5m + 6 = 0$, factorising

$(m+2)(m+3) = 0$ i.e., $m = -2$ or $-3.$

Therefore, the complementary function of the given equation is

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

The P.I. $= \dfrac{1}{\left(D^2 + 5D + 6\right)} e^x = \dfrac{e^x}{\left(1^2 + 5.1 + 6\right)} = \dfrac{e^x}{12}$

$\therefore$ The complete solution of the given equation is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}.$$

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Ex. 8: Solve $\left(D^2 + 4D + 3\right)y = e^{-3x}.$

Sol. The auxiliary equation is $\left(m^2 + 4m + 3\right) = 0$

or $(m+1)(m+3) = 0$

i.e., $m = -1$ or $m = -3$

The complementary function is $c_1 e^{-x} + c_2 e^{-3x}$

$$\text{The P.I.} = \frac{1}{(D+3)(D+1)} e^{-3x}$$

$$= \frac{1}{(D+3)} \cdot \frac{1}{(-3+1)} e^{-3x} \qquad \text{[Using (7)]}$$

$$= -\frac{1}{2} \cdot \frac{1}{D+3} e^{-3x}$$

$$= -\frac{1}{2} x \,.\, e^{-3x} \qquad \text{[Using (8)]}$$

$\therefore$ The complete solution is given by

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2} x e^{-3x}$$

Ex. 9: Solve $\left(D^2 + 8D + 16\right)y = e^{-4x}$.

Sol. The complementary function is $\left(c_1 x + c_2\right) e^{-4x}$

$$\text{P.I.} = \frac{1}{(D+4)^2} e^{-4x} = +\frac{x^2}{2} e^{-4x} \qquad \text{[Using (9)]}$$

$\therefore$ The complete solution is

$$y = \left(c_1 x + c_2\right) e^{-4x} + \frac{x^2}{2} e^{-4x}$$

Ex. 10: Solve $\left(D^2 + 4\right)y = \sin 3x$

Sol. The auxiliary equation $m^2 + 4 = 0$

$$\text{or} \quad m = \pm 2i$$

The complementary function is $y = c_1 \cos x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2 + 4} (\sin 3x)$$

$$= \frac{1}{D^2 + 4} \text{ (imaginary part of } e^{3ix})$$

$$= \text{Imaginary part of } \frac{e^{3ix}}{(3i)^2 + 4}$$

$$= \text{Img. p. of } \frac{e^{3ix}}{-9 + 4}$$

$$= \text{Img. p. of } \frac{\cos 3x + i \sin 3x}{-5} = -\frac{1}{5} \sin 3x$$

The complete solution is given by $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{\sin 3x}{5}$

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SAQ 1: Solve $\left(D^3 - 5D^2 + 7D - 3\right)y = e^{2x} \cosh x$

(ii) When $h(x) = \sin(ax + b)$ or $\cos(ax + b)$ where a and b are constants.

$$\text{Since} \quad D \sin(ax+b) = a \cos(ax+b)$$

$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

$$D^3 \sin(ax+b) = -a^3 \cos(ax+b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

$$D^2 \sin(ax+b) = \left(-a^2\right) \sin(ax+b)$$

$$\left(D^2\right)^2 \sin(ax+b) = \left(-a^2\right)^2 \sin(ax+b)$$

$$\left(D^2\right)^r \sin(ax+b) = \left(-a^2\right)^r \sin(ax+b)$$

$$\therefore f(D^2) . \sin(ax+b) = f(-a^2) \sin(ax+b)$$

Operating both sides by $\dfrac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)} . f(D^2) \sin(ax+b) = \frac{f(-a^2)}{f(D^2)} \sin(ax+b).$$

or $$\sin(ax+b) = \frac{f(-a^2)}{f(D^2)}\sin(ax+b)$$

Dividing by $f(-a^2)$ on both sides,

$$\frac{1}{f(-a^2)}\sin(ax+b) = \frac{1}{f(D^2)}\sin(ax+b). \qquad (10)$$

provided $f(-a^2) \neq 0$.

If $f(-a^2) = 0$, we proceed as follows :

$$\text{Since } e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b)$$

$$\frac{1}{f(D^2)}\sin(ax+b) = \text{img. p. of } \frac{1}{f(D^2)}e^{i(ax+b)}$$

$$= \text{img. p. of } x\frac{1}{f'(-a^2)}e^{i(ax+b)} \qquad \text{[Using Eq. 8]}$$

$$= x\frac{1}{f'(-a^2)}\sin(ax+b)$$

$\therefore$ When $f(-a^2) = 0$,

$$\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b) \qquad (11)$$

provided $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$,

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b) \qquad (12)$$

provided $f''(-a^2) \neq 0$ and so on

Similarly, $$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b) \qquad (13)$$

provided $f(-a^2) \neq 0$

If $f(-a^2) = 0$,

$$\frac{1}{f(D^2)}\cos(ax+b) = x\frac{1}{f'(-a^2)}\cos(ax+b), \text{ and so on.}$$

Ex. 11: Find out the particular integral of the differential equation, $(D^3+1)y = \cos(2x-1)$

Sol. $$\text{P.I.} = \frac{1}{(D^3+1)}\cos(2x-1)$$

$$= \frac{1}{D^2.D+1}\cos(2x-1)$$

$$= \frac{1}{-2^2.D+1}\cos(2x-1)$$

$$= \frac{1}{(1-4D)}\cos(2x-1)$$

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$$= \frac{(1+4D)}{(1-4D)(1+4D)} \cos(2x-1)$$

$$= \frac{(1+4D)}{65} \cos(2x-1)$$

$$= \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$$

Ex. 12: Solve $\left(D^2+9\right)y = \sin 3x$

Sol. The complementary function is

$$y = c_1 \cos 3x + c_2 \sin 3x \qquad \text{(verify it)}$$

$$\text{P.I.} = \frac{1}{\left(D^2+9\right)} \sin 3x$$

$\left(D^2+9\right) = 0$, if we substitute -3^2 in place of D^2.

$$\text{Therefore } \frac{1}{\left(D^2+9\right)} \sin 3x = \text{Imaginary part of } \frac{1}{\left(D^2+9\right)} e^{i3x}$$

$$= \text{Img. part of } \frac{1}{(D+3i)(D-3i)} e^{i3x}$$

$$= \text{Img. part of } \frac{1}{(3i+3i)(D-3i)} e^{i3x}$$

$$= \text{Img. part of } \frac{1}{6i} \frac{1}{(D-3i)} e^{i3x}$$

$$= \text{Img. part of } \frac{1}{6i} x \,.\, e^{i3x}$$

$$= \text{Img. part of } \frac{ix}{6i^2} [\cos 3x + i \sin 3x]$$

(multiplying the numerator and the denominator by i)

$$= \text{Img. part of } \left\{-\frac{ix}{6}[\cos 3x + i\sin 3x]\right\} \quad [\because i^2 = -1]$$

$$= -\frac{x}{6}\cos 3x$$

$\therefore$ The complete solution is $y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{6}\cos 3x$

(iii) When $h(x) = x^m$,

Particular integral is $\frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$. (14)

Expand $[f(D)]^{-1}$ as a power series in D. Since the $(m+1)^{th}$ and higher order derivatives of x^m are zero, we need not consider the terms in the expansion beyond D^m.

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Ex. 13: Solve $\left(D^2 + D + 1\right)y = x^2$

Sol. The Complementary function is $e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$

$$\text{P.I.} = \frac{1}{1+D+D^2}x^2 = (1+\overline{D+D^2})^{-1}x^2$$

$$= \left[1-(D+D^2)+(D+D^2)^2 - \ldots\right]x^2$$

$$= (1-D-D^2+D^2)\,x^2 = (1-D)\,x^2$$

$$= x^2-2x$$

$\therefore$ The complete solution of the given equation is

$$y = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + x^2-2x$$

Ex. 14: Find the particular integral of $(D^2+D)y = x^2+2x+4$

Sol.

$$\text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4)$$

$$= \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^{-2}\ldots)(x^2+2x+4)$$

$$= \frac{1}{D}\left[x^2+2x+4-(2x+2)+2\right]$$

$$= \int(x^2+4)\,dx = \frac{x^3}{3}+4x$$

(iv) When $h(x) = e^{ex}v$, where v is a function of x.

If u is a function of x, then $D\left[e^{ax}u\right] = e^{ax}Du + ua\,e^{ax} = e^{ax}[D+a]\,u$

$$D^2\left[e^{ax}u\right] = e^{ax}D^2u + 2a\,e^{ax}u + a^2e^{ax}u = e^{ax}[D+a]^2u$$

In general, it can be shown that

$$D^n\left(e^{ax}u\right) = e^{ax}(D+a)^n u$$

$$\text{and } f(D)\left[e^{ax}u\right] = e^{ax}f(D+a)u$$

Operating both sides by $\frac{1}{f(D)}$,

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax}f(D+a)\,u\right]$$

Put $\quad f(D+a)\,u = V$ or $u = \frac{1}{f(D+a)}V$

so that $\quad e^{ax}.\frac{1}{f(D+a)}V = \frac{1}{f(D)}\left[e^{ax}V\right]$

$$\therefore \quad \frac{1}{f(D)}\left[e^{ax}V\right] = e^{ax}\frac{1}{f(D+a)}V \qquad (15)$$

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Ex. 15: Find the P.I. of $(D^2 - 2D + 4)y = e^x \cos x$

Sol.

$$\text{P.I.} = \frac{1}{(D^2 - 2D + 4)} e^x \cos x$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x \qquad \text{[using (15)]}$$

$$= e^x \cdot \frac{1}{D^2 + 3} \cos x$$

$$= e^x \cdot \frac{\cos x}{-1 + 3}$$

$$= \frac{1}{2} e^x \cos x$$

SAQ 2: Solve $y'' - 4y = 3x\ e^{2x}$

19.7 Variation of Parameters

This method tells how to determine the **Particular Integral** of $f(D)y = h(x)$, if we know n independent solutions $y_1, y_2, \ldots, y_n$ of the n^{th} order equation $f(D)y = 0$. The method provides a particular integral of the form $y = u_1 y_1 + u_2 y_2 + \ldots\ldots + u_n y_n$, where $u_1, u_2, \ldots, u_n$ are functions of x that can be calculated interms of $y_1, y_2, y_3, \ldots, y_n$ and $h(x)$.

We illustrate the method with a second order linear equation,

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = h(x). \qquad (16)$$

Let the complementary function of the given equation is

$$y = c_1 y_1 + c_2 y_2 \qquad (17)$$

Assume that the P.I. of (16) is $y = U y_1 + V y_2$ (18)

where U and V are functions of x.

Differentiating (18) with respect to x, we get

$$y' = U y_1' + V y_2' + U' y_1 + V' y_2$$

$$= U y_1' + V y_2' \qquad (19)$$

on assuming $U' y_1 + V' y_2 = 0$ (20)

Differentiating w.r.t. x, and substituting in (16), we obtain

$$U' y_1' + V' y_2' = h(x) \qquad (21)$$

$$\left(\because y_1 \text{ and } y_2 \text{ satisfy } \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0\right)$$

Solving (20) and (21), we get U' and V' and integrating U' and V' we get U and V hence the particular integral.

Ex. 16: Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Sol. The complementary function is $y = c_1 \cos 2x + c_2 \sin 2x$

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To obtain the particular integral, replace c_1 and c_2 in y by U and V and we choose these functions in such a way that

$$U' \cos 2x + V' \sin 2x = 0 \tag{1}$$

Let the P.I. be $U \cos 2x + V \sin 2x$ (2)

Then $$y' = -2U \sin 2x + 2V \cos 2x \tag{3}$$

$$(\because U' \cos 2x + V' \sin 2x = 0)$$

$$y'' = -4U \cos 2x - 4V \sin 2x - 2U' \sin 2x + 2V' \cos 2x \tag{4}$$

Substituting the values of y'' and y in (2), we obtain

$$-2U' \sin 2x + 2V' \cos 2x = \tan 2x \tag{5}$$

Solving for U' and V' from (1) and (5), we obtain

$$U' = \frac{-\sin^2 2x}{2 \cos 2x} \text{ and } V' = \frac{\sin 2x}{2}$$

On integration, we get

$$U = \frac{1}{4} [\sin 2x - \log (\sec 2x + \tan 2x)]$$

$$V = -\frac{1}{4} \cos 2x$$

$$\therefore \text{ P. I.} = -\frac{1}{4} \log (\sec 2x + \tan 2x) \cos 2x,$$

and the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \log (\sec 2x + \tan 2x) \cos 2x.$$

SAQ 3 : Obtain the complete integral of the equation $(D^3 - 6D^2 + 11D - 6)\, y = e^x$ using variation of parameters.

19.8 Summary

A linear differential equation of nth order, can be represented by $f(D)\, y = h(x)$.

If $h(x) = 0$; $f(D)\, y = 0$ is called a homogeneous equation and its solution is obtained by solving the auxiliary equation $f(m) = 0$; if the roots of $f(m) = 0$ are distinct say $m_1, m_2 \dots m_n$, then the general solution of the equation $f(D)\, y = 0$ is a linear combination, $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.

If one root say m_1 of $f(m) = 0$ is repeated k times, then the solution corresponding to this repeated root is of the form $\left(c_1 + c_2 x + c_3 x^2 \dots + c_k x^{k-1}\right) e^{m_1 x}$.

If the auxiliary equation has imaginary roots, the roots must occur in pairs and if $m_1 = a + ib$ is one root then $m_2 = a - ib$ is another root, and the solution corresponding to those imaginary roots is given by $e^{ax}\, [c_1 \cos bx + c_2 \sin bx]$. The part of the solution accounting is the near homogeneous part of $f(D)\, y = h(x)$ is called the particular integral.

Depending upon the form of the function $h(x)$, we use different methods of obtaining Particular integral.

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A general method of obtaining particular integral is the method of variation of parameters. This method provides the particular integral of the form $u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$ where $y_1, y_2 \ldots y_n$ are solutions of $f(D)\, y = 0$ and $u_1, u_2 \ldots u_n$ are functions of x, determined interms of $y_1, y_2, \ldots y_n$ and $h(x)$.

19.9 Sample Examination Questions

I. Answer the following in detail

(i) a) How do you obtain the particular integral of the equation $f(D)\, y = h(x)$, when $h(x)$ is of the form e^{ax}?

b) obtain the complete solution of $(D^2 - 6D + 13)\, y = 5e^{2x}$.

(ii) a) Determine the particular integral of the equation $f(D)\, y = \cos(ax + b)$

b) Solve completely, the equation $(D^2 + 4)\, y = \sin 3x$.

(iii) a) If $h(x)$ is of the form $e^{ax}. V$ (where V is a function of x) in the equation $f(D)\, y = h(x)$, determine the particular integral.

b) Obtain the complete solution of $(D^2 + 4)\, y = x.e^{2x}$

(iv) a) Explain the method of variation of parameters.

b) Solve the equation $\frac{d^2y}{dx^2} + y = x \sin x$

II. Solve the following equations.

(i) $(D^2 + 5D + 6)\, y = e^{2x}$ given that $y = 0, \frac{dy}{dx} = 0$ when $x = 0$.

(ii) $(D^2 + 16)\, y = e^{-3x} + \cos 4x$

(iii) $(D^2 + n^2)\, y = \cos nx + 3 \sin nx$ (n is a constant)

(iv) $(D^2 + 1)^2\, y = \cosh x$

(v) $(D^2 - 4D + 3)\, y = 2xe^{3x} + 3e^x \cos 2x$

(vi) $(D^2 + a^2)\, y = \sec ax$

(vii) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

III. Answer the following in about 5 lines.

(i) What is the meaning of the operator D^{-1} ?

(ii) How do you obtain the particular integral, if the $h(x)$ part in the given differential equation is of the form x^m?

(iii) What is the use of variation of parameters ?

Answers

I. (i) (b) $y = e^{3x} [c_1 \cos 2x + c_2 \sin 2x] + e^{2x}$

(ii) (b) $y = A_1 \cos(2x + B) - \frac{1}{5} \sin 3x$

(iii) (b) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{2x}}{8} \left(x - \frac{1}{2}\right)$

(iv) (b) $y = c_1 \cos x + c_2 \sin x + \frac{x}{2} \sin x - \frac{x^2}{4} \cos x.$

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II. (i) $y = \frac{e^{-3x}}{5} - \frac{1}{4} e^{-2x} + \frac{1}{20} e^{2x}$

(ii) $y = c_1 \cos 4x + c_2 \sin 4x + \frac{e^{-3x}}{25}$

(iii) $y = A \cos (nx + B) + \frac{x}{2n} (\sin nx + 3 \cos nx)$

(iv) $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + \frac{3}{4} \cosh x$

(v) $y = c_1 e^x + c_2 e^{3x} + \frac{1}{2} e^{3x} (x^2 - x) - \frac{3}{8} e^x (\cos 2x + \sin 2x)$

(vi) $y = c_1 \cos ax + c_2 \sin ax + \frac{\cos ax}{a^2} \log (\cos ax) + \frac{x}{a} \sin ax$

(vii) $y = (c_1 + c_2 x) e^{3x} - e^{3x} \log x.$

19.10 Solutions to SAQ's

SAQ 1: The auxiliary equation is $m^3 - 3m^2 + 7m - 3 = 0$ and its roots are 1,1,3 . So the complimentary function is $(c_1 + c_2 x) e^x + c_3 e^{3x}$.

$$\text{Particular Integral} = \frac{e^{2x} . \cosh x}{(D-1)^2 (D-3)}$$

$$= \frac{1}{2} \frac{(e^{3x} + e^x)}{(D-1)^2 (D-3)} \quad \left[\because \cosh x = \frac{1}{2} \left(e^x + e^{-x} \right) \right]$$

$$\text{Now the P.I. of } e^x = \frac{1}{2} \frac{1}{(D-1)^2 (D-3)} e^x$$

$$= \frac{1}{2(1-3)} \frac{e^x}{(D-1)^2}$$

$$= -\frac{1}{4} \frac{x^2}{2} e^x$$

$$\text{P.I. of } e^{3x} = \frac{1}{2} \frac{1}{(D-3)(D-1)^2} e^{3x} = \frac{1}{8} \frac{e^{3x}}{(D-3)} = \frac{x e^{3x}}{8}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^x + c_3 e^{2x} + \frac{x e^{3x}}{8} - \frac{x^2 e^x}{8}$$

SAQ 2: The complimentary function is $c_1 e^{2x} + c_2 e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^2 - 4} 3x . e^{2x} = 3e^{2x} \frac{1}{(D+2)^2 - 4} x. \quad \left[\because \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{v}{f(D+a)} \right]$$

$$= \frac{3}{4} e^{2x} \frac{1}{D \left(1 + \frac{D}{4}\right)} x.$$

$$= \frac{3}{4} e^{2x} \frac{1}{D} \left(1 + \frac{D}{4}\right)^{-1} x.$$

$$= \frac{3}{4} e^{2x} \frac{1}{D} \left(1 - \frac{D}{4} + \ldots \right) x.$$

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$$= \frac{3}{4} e^{2x} \left(\frac{x^2}{2} - \frac{x}{4}\right).$$

$\therefore$ The complete Integral $= c_1 e^{2x} + c_2 e^{-2x} - \frac{3}{8} e^{2x} \left(x^2 - \frac{x}{2}\right)$

SAQ 3: The Complimentary function C. F $= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

We assume the particular integral

$$y_p = v_1 e^x + v_2 e^{2x} + v_3 e^{3x}.$$

where v_1, v_2, v_3 are to be determinated.

We have $y'_p = v_1 e^x + 2v_2 e^{2x} + 3v_3 e^{3x} + v'_1 e^x + v'_2 e^{2x} + v'_3 e^{3x}$.

We assume $v'_1 e^x + v'_2 e^{2x} + v'_3 e^{3x} = 0$ (1)

leaving $y'_p = v_1 e^x + 2v_2 e^{2x} + 3v_3 e^{3x}$ (2)

Now, $y''_p = v_1 e^x + 4v_2 e^{2x} + 9v_3 e^{3x}$ (3)

where we assumed $v'_1 e^x + 2v'_2 e^{2x} + 3v'_3 e^{3x} = 0$ (4)

and $y'''_p = v_1 e^x + 8v_2 e^{2x} + 27v_3 e^{3x} + v'_1 e^x + 4v'_2 e^{2x} + 9v'_3 e^{3x}$ (5)

Now substituting y'_p, y''_p and y'''_p in the given differential equation we get,

$v'_1 e^x + 4v'_2 e^{2x} + 9v'_3 e^{3x} = e^x$ (6)

we determine v'_1, v'_2 and v'_3 from (1), (4) and (6) and obtain $v'_1 = \frac{1}{2}$, $v'_2 = -e^{-x}$, $v'_3 = \frac{1}{2} e^{-2x}$.

$\therefore v_1 = \frac{x}{2}, v_2 = e^{-x}, v_3 = -\frac{e^{-2x}}{4}$.

So $y_p = \frac{1}{2} x e^x + e^{-x} \cdot e^{2x} - \frac{1}{4} e^{-2x} \cdot e^{3x} = \frac{1}{2} x e^x + \frac{3}{4} e^x$

$\therefore$ The complete integral $y = c'_1 e^x + c_2 e^{2x} + c_3 e^{3x} + \frac{1}{2} x e^x$

where $c'_1 = c_1 + \frac{3}{4}$.

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Unit – 20 : Linear Equations with variable coefficients

20.0 Contents

20.1 Aims and Objectives

After going through this unit, you will be able to slove the equations of the type

(i) Legender Equation

(ii) Linear Equation of 2nd order with variable coefficents.

20.2 Introduction

In the last unit we have discussed about the linear equations with constant coefficients. Here, we discuss the methods of solving the linear differential equations in which the coefficients are function of x.

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = h(x) \qquad (1)$$

or $\left(x^n D^n + a_1 x^{n-1} D^{n-1} + \dots + a_n\right) y = h(x)$ (1′)

where $a_1, a_2, \dots, a_n$ are constants and $h(x)$ is a function of x only, is called a **Homogeneous Linear Differential Equation**. Note that the index of x and the order of derivative is same in each term of such equations. This equation is also known as **Cauchy - Euler equation.**

The more general form of the Cauchy - Euler equaion is

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = h(x) \qquad (2)$$

If we put $a = 1, b = 0$ in (2), we get the equation (1). This equation (2) is called Legender Linear Equation.

20.3 Method of Solution

First Method : In this method, we change the variable from x to z by putting $x = e^z$, i.e., $z = \log x$.

This will transform (1) or (2) to a linear equation with constant coefficients and this new equation may be solved according to the procedures out lined in Unit-19. After getting the solution in terms of y and z, z is replaced by $\log x$ in order to get the solution interms of the original variables x and y.

Differentiating the equation, $z = \log x$ with respect to x,

$$\frac{dz}{dx} = \frac{1}{x}$$

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$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad (3)$$

$$\text{or } x\frac{dy}{dx} = \frac{dy}{dz}$$

Denote the operator $\frac{d}{dz}$ by D_1, then the $x\frac{dy}{dx} = D_1 y$.

$$\text{Now, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dz}\right]$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\text{i.e. } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = \left(D_1^2 - D_1\right)y = D_1\left(D_1 - 1\right)y.$$

Differentiating further, we obtain,

$$x^3\frac{d^3y}{dx^3} = D_1\left(D_1 - 1\right)\left(D_1 - 2\right)y.$$

....

$$\text{and } x^n\frac{d^ny}{dx^n} = D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\ldots\left(D_1 - n + 1\right)y.$$

Substituting these relations in equation (1), we get,

$$\Big[D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\ldots\left(D_1 - n + 1\right) + a_1 D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\ldots\left(D_1 - n + 2\right)$$

$$+ \ldots + a_{n-1}D_1 + a_n\Big]\, y = h(e^z) \qquad (4)$$

This is a linear equation with constant coefficients and can be solved by the known methods.

If we substitute $(ax + b) = e^z$, the Legender linear equation can also be reduced to the linear equation with constant coefficients.

Examples

Ex. 1: Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

Sol: Let $x = e^z$ or $\log x = z$,

differentiating with respect to x, on both sides we get

$$\frac{1}{x} = \frac{dz}{dx}$$

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$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} = \frac{1}{x} D_1 y$$

$$\text{or } x \frac{dy}{dx} = D_1 y$$

Similarly, we obtain $x^2 \dfrac{d^2y}{dx^2} = D_1 \left(D_1 - 1\right) y.$

$$x^3 \frac{d^3y}{dx^3} = D_1 \left(D_1 - 1\right)\left(D_1 - 2\right) y.$$

Substituting these relations in the given differential equation, we get

$$\left[D_1 \left(D_1 - 1\right)\left(D_1 - 2\right) + 3D_1 \left(D_1 - 1\right) - 2D_1 + 2\right] y = 0$$

or $\left(D_1^3 - 3D_1 + 2\right) y = 0.$

[here D_1^3, D_1^2 etc., stand for differentiation thrice, twice etc.]

This is a differential equation with constant coefficients, and the solution is given by,

$$y = c_1 e^z + c_2 z e^z + c_3 e^{-2z}$$

Substituting $e^z = x$, $\quad y = c_1 x + c_2 x \log x + \dfrac{c_3}{x^2}.$

Ex. 2: Solve $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 2y = x \log x.$

Sol.: Using the relations $e^z = x$ and $\dfrac{d}{dz} = D_1$, the given equation transforms to

$$\left[D_1(D_1 - 1) - D_1 + 2\right] y = z.e^z$$

or $\left[D_1^2 - 2D_1 + 2\right] y = z.e^z$ which is a linear equation with constant coefficients.

The complementary function is given by $e^z (c_1 \cos z + c_2 \sin z)$

and the particular integral is $z.e^z$

Therefore the required complete solution of the equation is

$$y = e^z (c_1 \cos z + c_2 \sin z) + z e^z$$

In terms of x, $y = x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + x \log x.$

Ex. 3: Solve $(x+2)^2 \dfrac{d^2y}{dx^2} - (x+2) \dfrac{dy}{dx} + y = 3x + 4.$

Sol. : The given equation is in the Legender Linear form :

$$\text{Let } x + 2 = e^z$$

$$\text{or } \log(x+2) = z$$

$$\text{Since } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{(x+2)}$$

$$\text{or } (x+2) \frac{dy}{dx} = \frac{dy}{dz} = D_1 y \qquad (1)$$

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$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{1}{(x+2)}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{1}{(x+2)}\right)\cdot\frac{dz}{dx}$$

$$= \frac{d^2y}{dz^2}\,\frac{1}{(x+2)^2} + \frac{dy}{dz}\left(-\frac{1}{(x+2)^2}\right)\cdot\frac{dx}{dz}\cdot\frac{dz}{dx}$$

$$= \frac{1}{(x+2)^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dz}\right)$$

$$\text{or } (x+2)^2\frac{d^2y}{dx^2} = D_1^2y - D_1y = D_1(D_1-1)y \qquad (2)$$

Substituting (1) and (2) in the given equation, we obtain

$$\left[D_1(D_1-1) - D_1 + 1\right]y = 3\left[e^z - 2\right] + 4 = 3e^z - 2$$

$$\text{or } (D_1-1)^2y = 3e^z - 2, \text{ an eqn. with constant coefficients.}$$

The complementary function is = $(c_1 + c_2z)\,e^z$

and the particular integral is given by $\frac{3}{2}z^2e^z - 2$

Substituting $z = \log(x+2)$

the general solution of the given equation is

$$y = \left[c_1 + c_2\log(x+2)\right](x+2) + \frac{3}{2}\left[\log(x+2)\right]^2(x+2) - 2$$

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20.3.1 Second Method

Without transforming equation (1) of 20.2 into a linear eqn. with constant coefficients, an independent method can be given. In this method we first solve for the complementary function and then for P.I.

To find the complementary function, we solve

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_ny = 0$$

If x^m, for some value of m, be taken as a tentative solution, and substituting into the equation, we get

$$\left[m(m-1)(m-2)\dots(m-n+1) + a_1m(m-1)\dots(m-n+2) + \dots + a_n\right]x^m = 0$$

$$\Rightarrow \quad m(m-1)(m-2)\dots(m-n+1) + a_1m(m-1)(m-2)\dots(m-n+2) + \dots + a_n = 0 \qquad (5)$$

This is an n^{th} degree equation in m and let the roots of the equation be $m_1, m_2, \dots m_n$. Then the complementary function will be

$$c_1x^{m_1} + c_2x^{m_2} + \dots + c_nx^{m_n}$$

Here $c_1, c_2, \dots c_n$ are arbitrary constants. If one root is repeated twice we obtain the concerned part of the complementary function as follows.

Let $m_1 = m_2$ is the root repeated.

Putting $m_2 = m_1 + \epsilon$, where $\epsilon \to 0$, the corresponding part of the complementary function is

$$x^{m_1}\left[c_1 + c_2 x^{\epsilon}\right] = x^{m_1}\left[c_1 + c_2 \cdot e^{\epsilon \log x}\right]$$

(Expanding $e^{\epsilon \log x}$ and neglecting ϵ^2 and higher powers)

$$= x^{m_1}\left[c_1 + c_2 \cdot (1 + \epsilon \log x)\right]$$

$$\text{Let } c_2 \epsilon = B, \text{ and } c_1 + c_2 = A.$$

Then the part of the complementary function arising from the two equal roots is $x^{m_1}(A + B \log x)$. If m_1 is a root repeated k times, the corresponding part of the complementary function will be

$$x^{m_1}\left[c_1 + c_2(\log x) + c_3(\log x)^2 + \ldots + c_k (\log x)^{k-1}\right]$$

If $(\alpha \pm i\beta)$ is a pair of complex roots, the corresponding part of the complementary function is

$$y = x^{\alpha}\left[c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)\right]$$

To find the particular integral :

We write the equation (1) of 20.2 in the operator notation, i.e.,

$$f(D_1)y = h(x)$$

$$\text{where } f(D_1) = \left[D_1(D_1 - 1)(D_1 - 2)\ldots(D_1 - n + 1) + a_1 D_1 (D_1 - 1) \ldots\ldots(D_1 - n + 2) + \ldots + a_n\right]$$

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The particular integral $= \dfrac{1}{f(D_1)} h(x)$.

(i) To obtain the value of $\dfrac{1}{f(D_1)} h(x)$ some times the operator $\dfrac{1}{f(D_1)}$ may be broken up into partial fractions, then

$$\text{P.I.} = \frac{1}{f(D_1)} h(x) = \left(\frac{A_1}{D_1 - \alpha_1} + \frac{A_2}{D_1 - \alpha_2} + \ldots + \frac{A_n}{D_1 - \alpha_n}\right) h(x).$$

In obtaining the result, the following formulae will be useful.

$$\left.\begin{aligned} \frac{1}{D_1 - \alpha} h(x) &= x^{\alpha} \int x^{-\alpha - 1} h(x)\, dx \\ \frac{1}{D_1 + \alpha} h(x) &= x^{-\alpha} \int x^{\alpha - 1} h(x)\, dx \end{aligned}\right\} \quad \text{(A)}$$

(ii) Factorising $\dfrac{1}{f(D_1)} h(x)$,

$$\frac{1}{f(D_1)} h(x) = \frac{1}{(D_1 - \alpha_1)(D_1 - \alpha_2) \ldots (D_1 - \alpha_n)} h(x)$$

Here the operations indicated by factors are to be taken in succession, beginning with the first on the right.

Ex. 4: Solve $(x^2D^2 + 3xD + 1)\, y = \dfrac{1}{(1 - x)^2}$

Sol.: Let x^m be a solution of the equation

$$(x^2D^2 + 3xD + 1)\, y = 0$$

then $\quad [m(m-1) + 3m + 1]\, x^m = 0$

$$m(m-1) + 3m + 1 = 0$$

i.e., the auxiliary equation is

$$m^2 + 2m + 1 = (m+1)^2 = 0$$

$$\text{i.e.,} \quad m = -1, -1$$

The complementary function is $(c_1 + c_2 \log x)\, x^{-1}$

$$\text{The P.I.} = \frac{1}{(D_1+1)^2}(1-x)^{-2}$$

$$= \frac{1}{(D_1+1)(D_1+1)}[1-x]^{-2}$$

$$= \frac{1}{(D_1+1)} \cdot x^{-1} \int x^{-1+1}(1-x)^{-2}\, dx$$

[using formula (A) above]

$$= \frac{1}{(D_1+1)} x^{-1}(1-x)^{-1}$$

$$= x^{-1} \int x^{-1+1} \cdot x^{-1}(1-x)^{-1}\, dx$$

$$= x^{-1} \int \frac{dx}{x(1-x)}$$

$$= x^{-1} \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$

$$= x^{-1}[\log x - \log(1-x)] = \frac{1}{x} \log\left(\frac{x}{1-x}\right).$$

$\therefore$ The required complete solution of the given equation is

$$y = (c_1 + c_2 \log x)\frac{1}{x} + \frac{1}{x}\log\left(\frac{x}{1-x}\right)$$

Ex. 5: Solve $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$.

Sol: The auxiliary equation is $m(m-1) - 2m - 4 = 0$,

or $(m-4)(m+1) = 0 \Rightarrow m = 4, -1$

$\therefore$ The complementary function is $c_1 x^4 + \dfrac{c_2}{x}$

$$\text{P.I.} = \frac{1}{(D_1-4)(D_1+1)} x^4$$

$$= \frac{1}{5}\left(\frac{1}{D_1-4}x^4 - \frac{1}{D_1+1}x^4\right)$$

[Resolving into partial fractions]

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$$= \frac{1}{5}x^4 \int x^{-4-1}.\, x^4 dx - \frac{1}{5}x^{-1} \int x^{1-1+4}\, dx$$

$$= \frac{x^4}{5}\log x - \frac{x^4}{25}$$

The complete solution is $$y = c_1 x^4 + c_2 \frac{1}{x} + \frac{x^4}{5}\log x$$

$$\left(\frac{-x^4}{25} \text{ is included in } c_1 x^4\right)$$

20.3.2 Particular integral if $h(x) = x^m$

If $h(x)$ is of the form x^m, we derive a formula, to obtain the P.I. directly without going into the integrations followed in the previous article.

$$\text{Since } \left(x \frac{d}{dx}\right) x^m = x.m.x^{m-1} = mx^m$$

$$\left(x^2 \frac{d^2}{dx^2}\right) x^m = x^2 m.\,(m-1)x^{m-2} = m\,(m-1)x^m$$

....

$$\left(x^n \frac{d^n}{dx^n}\right) x^m = m\,(m-1) \,\ldots\ldots (m-n+1)\, x^m$$

$$\therefore f(D_1)x^m = f(m)\,.\,x^m$$

Operating both sides by $\frac{1}{f(D_1)}$, we obtain

$$x^m = \frac{1}{f(D_1)}\left[f(m)x^m\right]$$

$$= f(m)\,\frac{1}{f(D_1)}x^m$$

$$\therefore \frac{1}{f(D_1)}x^m = \frac{x^m}{f(m)} \text{ provided } f(m) \neq 0$$

If $f(m) = 0$, then m is a root of $f(D_1) = 0$

Then, let $f(D_1) = (D_1 - m)\,\phi(D_1)$. [where $\phi(m) \neq 0$]

$$\text{P.I.} = \frac{1}{(D_1 - m)\,\phi(D_1)}\,.\,x^m$$

$$= \frac{1}{\phi(m)}\cdot\frac{1}{(D_1 - m)}x^m$$

$$= \frac{1}{\phi(m)}\,x^m \log x$$

$$\therefore \text{If } f(m) = 0, \text{ the P.I.} = \frac{\log x\,.\,x^m}{\phi(m)}$$

If m is a k fold root of $f(D_1) = 0$,

then $f(D_1) = (D_1 - m)^k \phi_1(D_1)$ (where $\phi_1(m) \neq 0$)

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$$\therefore \text{ P.I.} = \frac{1}{(D_1-m)^k \phi_1(D_1)} \cdot x^m$$

$$= \frac{1}{\phi_1(m)} \cdot \left[\frac{1}{(D_1-m)^k} x^m\right]$$

$$= \frac{x^m (\log x)^k}{k!} \frac{1}{\phi_1(m)}.$$

Ex. 6: Solve $x^2 \dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 5y = x^5$.

Sol. $f(D_1) = D_1(D_1-1) + 7D_1 + 5$

Complementary function $= c_1 x^{-1} + c_2 x^{-5}$

Particular Integral $= \dfrac{1}{D^2 + 6D + 5} \cdot x^5 = \dfrac{x^5}{25 + 30 + 5} = \dfrac{x^5}{60}$

20.4 Second Order Linear Equation

The equation $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ (6)

where P, Q, R are functions of x, is the general form of a second order linear equation. If P and Q are constants, the equation can be solved by the methods of the preceding unit. Otherwise, no general method is known. Here we discuss some procedures which at times will yield a solution.

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Case Where One Solution is Known

Let $y = y_1$ be a known solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Let $y = y_1 v$ be the general solution of equation (6). That is, we have to find out the value of v.

Substituting $y = y_1 v$ in (6), we get

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx} = \frac{R}{y_1} \qquad (7)$$

$$\left(\because \frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = 0\right)$$

If we write $\dfrac{dv}{dx} = p$, the eqn. (7) reduces to

$$\frac{dp}{dx} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)p = \frac{R}{y_1} \qquad (8)$$

a first order linear equation in p. So the transformation $y = vy_1$ reduces the given second order equation into a first order equation. Solving the equation (8) for p,

$$p = \frac{dv}{dx} = c_1 \frac{e^{-\int P\,dx}}{y_1^2} + \frac{e^{-\int P\,dx}}{y_1^2}\int y_1 e^{\int P\,dx} R\,dx$$

Integrating, $v = c_2 + c_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + \int \frac{e^{-\int P\,dx}}{y_1^2} \int y_1 e^{\int P\,dx} R\,(dx)^2$

$$\therefore\ y = y_1 v = c_2 y_1 + c_1 y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + y_1 \int \frac{e^{-\int P\,dx}}{y_1^2} \int y_1 e^{\int P\,dx} R\,(dx)^2$$

is the general solution of the equation (6) as two arbitrary constants (equal to the order of the equation) are involved. Observe that

$y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$ is the second part of the complementary function and

$y_1 \int \frac{e^{-\int P\,dx}}{y_1^2} \int y_1 e^{\int P\,dx} R\,(dx)^2$ is the particular integral.

Note : If $y = e^x$, is a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$,

then $1 + P + Q = 0$, and if $y = x$ is a solution, $P + Qx = 0$.

Conversely, if $1 + P + Q = 0$ in the equation

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ i.e., the sum of the coefficients is zero then e^x is a solution and

$$\text{if } P + Qx = 0,\quad y = x \text{ is a solution}$$
$$\text{if } 1 - P + Q = 0,\quad y = e^{-x} \text{ is a solution}$$
$$\text{if } m^2 + mP + Q = 0,\quad y = e^{mx} \text{ is a solution.}$$

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Ex. 7: Solve the equation $xy'' + (1-x)\,y' - y = e^x$

Sol: From the above note, $y = e^x$ is a solution since the sum of the coefficients is zero.

Let the general solution of the given equation be $y = ve^x$. Substituting this in the given equation

$$x\frac{d^2v}{dx^2} + (1+x)\frac{dv}{dx} = 1$$

Put $\frac{dv}{dx} = p$, in this equation

Then $x\frac{dp}{dx} + (1 + x)\,p = 1$, a first order linear equation in p.

The solution of this equation is given by

$$p = \frac{dv}{dx} = c_1 \frac{e^{-x}}{x} + \frac{1}{x}$$

$$\text{Integrating, } v = \log x + c_1 \int x^{-1} e^{-x}\,dx + c_2$$

$\therefore$ The complete solution is $\quad y = v\,e^x.$

$$= e^x \log x + c_1 e^x \int x^{-1} e^{-x}\,dx + c_2 e^x$$

Change of Dependent variable

Under the transformation $y = u(x) . v(x)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2},$$

the equation (6) becomes

$$\frac{d^2v}{dx^2} + P_1\frac{dv}{dx} + Q_1 v = R_1 \qquad (9)$$

where $P_1 = P + \frac{2}{u}\frac{du}{dx}$

$$Q_1 = \frac{1}{u}\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right]$$

$$R_1 = \frac{R}{u},$$

This equation (9) is a linear equation in which v is the dependent variable.

a) If we choose u such that

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

then $Q_1 = 0$, and the equation (9) becomes

$$\frac{d^2v}{dx^2} + P_1\frac{dv}{dx} = R_1 \qquad (10)$$

The further substitution $\frac{dv}{dx} = p$, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$ reduces (10) to $\frac{dp}{dx} + P_1 p = R_1$ a linear equation of the first order.

b) If u is chosen so that $P_1 = \frac{2}{u}\frac{du}{dx} + P = 0$

$$\text{or } \frac{du}{u} = -\frac{1}{2}P(x)\,dx$$

$$\text{then } u = e^{-1/2\int P(x)\,dx}$$

$$\frac{du}{dx} = -\frac{1}{2}uP$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}u\frac{dP}{dx}$$

$$\text{so that } Q_1 = Q + \frac{P}{u}\frac{du}{dx} + \frac{1}{u}\frac{d^2u}{dx^2}$$

$$= Q + \frac{P}{u}\frac{du}{dx} + \frac{1}{u}\left[-\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}u\frac{dP}{dx}\right]$$

$$= Q + \frac{1}{2}P\frac{du}{dx} - \frac{1}{2}\frac{dP}{dx}$$

$$\therefore Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \quad \text{and } R_1 = \frac{R}{u} = Re^{1/2\int P\,dx} \right\} \qquad (11)$$

If $Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = A$, a constant, equation (9) becomes,

$$\frac{d^2v}{dx^2} + Av = \frac{R}{u} \qquad (12)$$

a linear equation with constant coefficients.

If $Q = \frac{A}{x^2}$, equation (9) becomes, $x^2\frac{d^2v}{dx^2} + Av = \frac{Rx^2}{u}$ a Cauchy equation, and the substitution $x = e^z$ will reduce it to one with constant coefficients.

Ex. 8: Solve $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$

Sol.: Comparing this equation with equation (6),

$$P = -\frac{2}{x},\ Q = 1 + \frac{2}{x^2},\ R = xe^x$$

$$\therefore\ Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 1 + \frac{2}{x^2} - \frac{1}{x^2} - \frac{1}{x^2} = 1$$

$$y_1 = e^{-1/2\int P\,dx} = e^{-1/2\int -2/x\,dx} = e^{\log x} = x$$

$$R_1 = Re^{1/2\int P\,dx} = xe^x x^{-1} = e^x$$

The transformation $y = xv$

$\frac{dy}{dx} = x\frac{dv}{dx} + v,\ \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$ reduces the given equation to

$\frac{d^2v}{dx^2} + v = e^x$, a linear equation with constant coefficients, whose complete solution is

$$v = \frac{y}{x} = c_1\cos x + c_2\sin x + \frac{e^x}{2}$$

$$\therefore\ y = vx = c_1x\cos x + c_2x\sin x + \frac{1}{2}xe^x$$

Ex. 9: Solve $\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 2x - 1$

Sol.: Here $P = -\frac{3}{x},\ Q = \frac{3}{x^2},\ R = 2x - 1$

$$u = e^{-1/2\int P\,dx} = e^{-1/2\int(-3/x)\,dx} = e^{3/2\log x} = x^{3/2}$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = \frac{3}{x^2} - \frac{3}{2x^2} - \frac{9}{4x^2} = -\frac{3}{4x^2}$$

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$$R_1 = \frac{R}{u} = (2x-1)\,x^{-3/2}$$

The transformation $y = uv = x^{3/2} v$ reduces the equation to

$$\frac{d^2v}{dx^2} - \frac{3}{4x^2}v = \frac{2x-1}{x^{3/2}} \text{ or}$$

$$x^2 \frac{d^2v}{dx^2} - \frac{3}{4}v = 2x^{3/2} - x^{1/2}, \text{ a Cauchy equation}$$

putting $x = e^z$, we have

$$\left(D_1{}^2 - D_1 - \frac{3}{4}\right) v = 2\,e^{3z/2} - e^{z/2}$$

The complementary function is $v = c_1 e^{-z/2} - c_2 e^{3z/2}$ and a particular Integral is

$$v = \frac{1}{\left[D_1{}^2 - D_1 - \frac{3}{4}\right]} \left[2\,e^{3z/2} - e^{z/2}\right]$$

$$= \frac{1}{\left[D_1 - \frac{3}{2}\right]} e^{3z/2} + e^{z/2}$$

$$= ze^{3z/2} + e^{z/2}$$

The complete solution is

$$v = \frac{y}{x^{3/2}} = c_1 x^{-1/2} + c_2 x^{3/2} + x^{3/2} \log x + x^{1/2}.$$

$$\text{and } y = c_1 x + c_2 x^3 + x^3 \log x + x^2.$$

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Change of Independent Variable

Let the transformation $z = \theta(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\,\frac{d^2z}{dx^2}$$

and (6) becomes

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(\frac{d^2z}{dx^2} + P\,\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R.$$

$$\frac{d^2y}{dz^2} + \frac{\left(\frac{d^2z}{dx^2} + P\,\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}\,\frac{dy}{dz} + \frac{Qy}{\left(\frac{dz}{dx}\right)^2} = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let $z = \theta(x)$ be chosen so that $\frac{dz}{dx} = \sqrt{\pm \frac{Q}{a^2}}$

the sign being that which makes $\frac{dz}{dx}$ real and a^2 being any positive constant. (one may consistently take $a^2 = 1$).

It now $\left[\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right] \Big/ \left(\frac{dz}{dx}\right)^2 = A$, a constant, then (10)

becomes $$\frac{d^2y}{dz^2} + A\frac{dy}{dz} \pm a^2y = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

a linear equation with constant coefficients.

Ex. 10 : Solve $\frac{d^2y}{dx^2} - (1+4e^x)\frac{dy}{dx} + 3e^{2x}y = e^{2(x+e^x)}$

Sol : Here $P = -(1+4e^x)$

$$Q = 3e^{2x}$$

$$\frac{dz}{dx} = \pm\sqrt{(Q/a^2)} = \left(\sqrt{\frac{3e^{2x}}{3}}\right) = e^x, \text{ by taking } a^2 = 3$$

$$\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{e^x - (1+4e^x)}{(e^x)^2} = -4 = A$$

The introduction of new independent variable $z = e^x$ leads to

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + a^2y = R\Big/\left(\frac{dz}{dx}\right)^2$$

or $$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = \frac{e^{2(x+e^x)}}{e^{2x}} = e^{2e^x} = e^{2z}$$

whose complete solution is

$$y = c_1e^z + c_2e^{3z} + \frac{1}{D^2 - 4D + 3}e^{2z}$$

$$= c_1e^z + c_2e^{3z} - e^{2z}$$

Replacing z by e^x, $$y = c_1e^{e^x} + c_2e^{3e^x} - e^{2e^x}$$

20.5 Summary

The general form of the linear equation with variable coefficients is discussed and its method of solution is given. The most general form of the linear equation, called the Legendre equation is also solved with the help of the transformation $ax + b = e^z$, so that the equation will be converted into equation with constant coefficients and hence can be solved with the known methods of solution. Depending on the non-homogeneous part of the equation, various methods for finding the particular integral have been discussed. The second order linear equation with variable coefficients is also dealt with–

20.6 Sample Examination Questions

I. Answer the following questions in detail.

(i) a) How do you solve the equation

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_ny = h(x)$$

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b) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

(ii) a) Solve the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

(where P, Q, R are functions of x), when one solution is known.

b) Solve $x^3 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$ if $y = x^3$ is *a* solution.

(iii) a) Explain the method of solving the general second order linear equation with variable coefficients, by changing the dependent variable.

b) Solve $(x \sin x + \cos x)\, y'' - x \cos x\, y' + y \cos x = x$

II *Solve the following equations*

(i) $(x^3D^3 + 6x^2D^2 + 8xD - 8)\, y = x^2$

(ii) $(x + a)^2 y'' - 4(x+a)\, y' + 6y = x$

(iii) $(x^2D^2 - 3xD + 4)\, y = x^m$

(iv) $(x^2D^2 + xD - 1)\, y = 0$, given that $y = x + \frac{1}{x}$ is a solution

(v) $y'' - 2xy' + (x^2 + 2)\, y = e^{(x^2 + 2x)/2}$

(vi) $(xD^2 - D + 4x^3)\, y = x^5$

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Answers

I. (i) b) $y = \frac{c_1}{x} + x\,[c_2\cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2 \log x}{x}$

(ii) b) $y = c_1x^3 + c_2x^{-3}$

(iii) b) $y = c_1x + c_2 \cos x - \sin x$

II. (i) $y = c_1x + \frac{1}{x^2}\,[c_2\cos(2\log x) + c_3 \sin(2\log x)] + \frac{x^2}{20}$

(ii) $y = c_1(x + a)^2 + c_2(x + a)^3 - \frac{3x + 2a}{6}$

(iii) $y = x^2\,[c_1 + c_2 \log x] + x^m/(m - 2)^4$

(iv) $y = \frac{c_1}{x} + c_2\left(x + \frac{1}{x}\right)$

(v) $y = e^{x^2/2}\left[c_1 \cos\sqrt{(3)}\, x + c_2 \sin \sqrt{(3)}\, x\right] + \frac{1}{4} e^{(x^2 + 2x)/2}$

(vi) $y = c_1 \sin(x^2 + c_2) + \frac{x^2}{4}$

Unit – 21 : Simultaneous Differential Equations

21.0 Contents

22.1 Aims and Objectives

After going through this unit, you will be able to solve a system of simultaneous equations.

21.2 Introduction

We have considered so far the differential equations involving only two variables, one of which is taken as the independent and the other as the dependent variable. In this unit, we consider equations involving more than two variables, only one of which is the independent variable. A set of such euqations involving the derivatives is called a system of simultaneous differential equations.

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The equations

$$\left.\begin{aligned} P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0 \\ P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0 \end{aligned}\right\} \qquad (1)$$

are called a pair of simultaneous diffrential equations of first order and first degree, in which x and y are the dependent variables and z is the independent variable and $P_1, Q_1, R_1, P_2, Q_2, R_2$ are functions of x, y and z.

These equations can be written as

$$\left.\begin{aligned} P_1\, dx + Q_1 dy + R_1 dz = 0 \\ P_2 dx + Q_2 dy + R_2 dz = 0 \end{aligned}\right\} \qquad (2)$$

The ratios of the differentials dx, dy, dz can be obtained as

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{P_2R_1 - P_1R_2} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

which is in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ (3)

where P, Q, R are functions of x, y and z.

We shall take (3) as the standard form for a pair of ordinary simulataneous differential equations of the first order and first degree.

The given equations are said to be completely solved when we get a solution of the form $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$, where u_1 and u_2 are two independent integrals (solutions) of the given equations. u_1 and u_2 are said to be independent integrals if u_1/u_2 is not merely a constant.

21.3 Geometrical Interpretation of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

From three dimensional geometry, it is known that the direction ratios of a tangent to a curve at any point (x, y, z) on it is proportional to the values of dx, dy, and dz at that point. Hence geometrically, the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ represent a system of curves in space, such that the direction ratios of the tangent to any of the curves at a point (x, y, z) on it are proportional to the values of P, Q and R at that point. If $u_1 = c_1, u_2 = c_2$ is the general solution of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, then it follows that the system of curves represented by the equations must be the curves of intersection of the two systems of surfaces given by the equations $u_1 = c_1$ and $u_2 = c_2$.

21.4 Methods for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Rule 1 :

Suppose that one of the variables is either absent or cancells out from any two fractions of the given equations. Then an integral can be obtained by the known methods. The same procedure can be repeated with another set of two fractions of the given equations.

Ex. 1 : Solve $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ (1)

Sol : Taking the first two equations,

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow \frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow x\,dx - y\,dy = 0 \quad (2)$$

Integrating $\frac{x^2}{2} - \frac{y^2}{2} = \frac{c_1}{2}$ (3)

Taking the first and the last terms of the equations, we get

$$\frac{dx}{yz} = \frac{dz}{yz} \Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

$$\Rightarrow x\,dx - z\,dz = 0 \quad (4)$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = \frac{c_2}{2} \quad (5)$$

Since $x^2 - y^2$ and $x^2 - z^2$ are independent, the required solution is given by (3) and (5) and generally represented by $\phi\,(x^2 - z^2,\ x^2 - y^2) = 0$

Ex. 2 : Solve $\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$

Sol : Taking the first two fractions, we get

$$x^2\,dx = y^2\,dy$$

Integrating $x^3 - y^3 = c_1$ (1)

Considering the first and the third fractions $x\,dx = z\,dz$ and so,

$$x^2 - z^2 = c_2 \quad (2)$$

The required solution of the equation is given by (1) and (2).

The general solution is $\phi(x^2 - z^2, x^3 - y^3) = 0$.

Rule II

Suppose one integral is known by using Rule I and suppose another integral cannot be obtained by using Rule I. Then the integral known to us is used to find another integral as shown in the following example,

Ex. 3 : Solve $\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4}$

Sol : Taking the first two fractors, we get

$$\frac{dx}{x} = -\frac{dy}{y}$$

Integrating, we have

$$\log x + \log y = c$$

$$\text{or} \quad xy = c_1 \qquad (1)$$

Using (1), the first and third fractions give

$$\frac{dx}{xz(z^2 + c_1)} = \frac{dz}{x^4}$$

$$\text{or} \quad x^3\,dx - (z^3 + c_1 z)\,dz = 0$$

Integrating $\frac{x^4}{4} - \left(\frac{z^4}{4} + \frac{1}{2} c_1 z^2\right) = \frac{1}{4} c_2$

$$\Rightarrow x^4 - z^4 - 2c_1 z^2 = c_2 \qquad (1a)$$

Substituting for c_1 from (1) in (1a), we get

$$x^4 - z^4 - 2xyz^2 = c_2 \qquad (2)$$

The general solution of the given equation is given by (1) and (2).

Ex. 4 : Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz - 2x)}$

Sol : Taking the first two fractions, we get

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \log x = \log y + \log c_1$$

$$\Rightarrow x = c_1 y \qquad (1)$$

Taking the second and third fractions and substituting (1) in the third fraction, we get

$$\frac{dy}{y^2} = \frac{dz}{c_1 y(yz - 2c_1 y)}$$

$$\text{or} \quad c_1 dy = \frac{dz}{z - 2c_1}$$

Integrating, $\quad c_1 y = \log(z - 2c_1) + \log c_2$

$$\Rightarrow \quad e^{c_1 y} = c_2(z - 2c_1)$$

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$$\Rightarrow \qquad e^x = c_2 \left[z - \frac{2x}{y}\right]$$

$$\Rightarrow \frac{y\,e^x}{yz - 2x} = c_2 \qquad (2)$$

(1) and (2) constitute the general solution of the given equation.

Rule III

Let P_1, Q_1, R_1 be functions of x, y, z. Then, by a well known principle of algebra, each fraction in (1) will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \qquad (2)$$

If the denominator of (2) is zero, then we know that the numerator of (2) is also zero. That is $P_1\,dx + Q_1\,dy + R_1\,dz = 0$, which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. Here P_1, Q_1, R_1 are called multipliers. As a special case, these can also be constants. Some times only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I or II as the case may be.

Ex. 5 : Solve the simultaneous equations

$$\frac{a\,dx}{(b-c)\,yz} = \frac{b\,dy}{(c-a)\,zx} = \frac{c\,dz}{(a-b)\,xy}$$

Sol : Choosing x, y and z as multipliers, each fraction

$$= \frac{ax\,dx + by\,dy + cz\,dz}{xyz\,[(b-c) + (c-a) + (a-b)]}$$

$$= \frac{ax\,dx + by\,dy + cz\,dz}{0}$$

$$\therefore \quad ax\,dx + by\,dy + cz\,dz = 0$$

Integrating, $\qquad ax^2 + by^2 + cz^2 = c_1 \qquad (1)$

Again, choosing ax, by and cz as multipliers, each fraction

$$\frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{xyz\,[a\,(b-c) + b\,(c-a) + c\,(a-b)]}$$

$$\therefore \quad a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$$

Integrating, $\qquad a^2x^2 + b^2y^2 + c^2z^2 = c_2 \qquad (2)$

The complete solution of the given equation is constituted by (1) and (2)

Ex. 6 : Solve $\dfrac{dx}{y^3x - 2x^4} = \dfrac{dy}{2y^4 - x^3y} = \dfrac{dz}{9z\,(x^3 - y^3)}$

Sol : Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction

$$= \frac{\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z}\right)}{(y^3 - 2x^3) + (2y^3 - x^3) + 3\,(x^3 - y^3)}$$

$$= \frac{\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z}\right)}{0}$$

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$$\text{or}\quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{3z} = 0 \;\Rightarrow\; \log x+\log y+\frac{1}{3}\log z = \log c_1$$

$$\text{or}\quad xyz^{1/3} = c_1 \qquad (1)$$

Now the first two fractions give

$$(2y^4-x^3y)\ dx = (y^3x-2x^4)\ dy$$

Dividing by $x^3 y^3$,

$$\left(\frac{2y}{x^3}-\frac{1}{y^2}\right) dx = \left(\frac{1}{x^2}-\frac{2x}{y^3}\right) dy$$

$$\text{or}\quad \left(\frac{1}{x^2}dy-\frac{2y}{x^3}dx\right)+\left(\frac{1}{y^2}dx-\frac{2x}{y^3}dy\right) = 0$$

$$\text{or}\quad d\left(\frac{y}{x^2}\right)+d\left(\frac{x}{y^2}\right) = 0$$

$$\text{Integrating}\qquad \frac{y}{x^2}+\frac{x}{y^2} = c_2 \qquad (2)$$

The required solution is given by (1) and (2).

Rule IV

Let P_1, Q_1, R_1 be functions of x, y, z and each fraction of the differential equation is equal to

$$\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R} \qquad (2)$$

Suppose the numerator of this is the exact differential of the denominator. Then this can be combined with a suitable fraction of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are chosen that the fraction

$$\frac{P_2dx+Q_2dy+R_2dz}{P_2P+Q_2Q+R_2R} \qquad (3)$$

is such that its numerator is exact differential of denominator. Fractions (2) and (3) are then combined to give an integral. This method may also be repeated in some problems to get another integral.

Ex. 7 : Find the integral curves of the simultaneous equations

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$$

Sol : Given $\dfrac{dx}{y^2(x-y)} = \dfrac{dy}{x^2(x-y)} = \dfrac{dz}{z(x^2+y^2)}$ (1)

Taking the first two fractions in (1), we get

$$x^2\ dx = -y^2\ dy$$

Integrating $x^3+y^3 = c_1$ (2)

Choosing $1, -1, 0$ as multipliers, each fraction in (1)

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x^2+y^2)(x-y)} \qquad (3)$$

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Takaing the third fraction of (1) and (3), we get

$$\frac{dz}{z\,(x^2+y^2)} = \frac{dx-dy}{(x-y)\,(x^2+y^2)}$$

$$\Rightarrow \quad \frac{dz}{z} = \frac{dx-dy}{x-y}$$

Integrating $\quad \log\,(x-y) - \log z = \log\,c_2$

$$\text{or} \quad \frac{(x-y)}{z} = c_2 \qquad (4)$$

(2) and (4) constitute the required solution of the given equation.

Ex. 8 : Solve $\dfrac{dx}{\cos\,(x+y)} = \dfrac{dy}{\sin\,(x+y)} = \dfrac{dz}{z}$ (1)

Sol : From (1) using 1, 1, 0 and 1, – 1, 0 as multipliers

$$\frac{dz}{z} = \frac{dx+dy}{\cos\,(x+y)+\sin\,(x+y)} = \frac{dx-dy}{\cos\,(x+y)-\sin\,(x+y)} \qquad (2)$$

Putting $x+y=t$ so that $dx+dy=dt$, the first two fractions are given as

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t}$$

$$\text{or} \quad \sqrt{2}\,\frac{dz}{z} = \operatorname{cosec}\,(t+\pi/4)\,dt$$

$$\left[\because \cos t+\sin t = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{2}}\sin t\right) = \sqrt{2}\sin\,(t+\pi/4)\right]$$

Integrating, $\quad \sqrt{2}\log z = \log\tan\frac{1}{2}(t+\pi/4) + \log c_1$

$$\text{or} \quad z^{\sqrt{2}} = c_1\tan\frac{1}{2}(t+\pi/4)$$

$$\text{or} \quad z^{\sqrt{2}}\cot\frac{1}{2}(t+\pi/4) = c_1 \qquad (3)$$

Now from the last two fractions of equation (2), we get

$$\frac{\cos\,(x+y)-\sin\,(x+y)}{\cos\,(x+y)+\sin\,(x+y)}\,(dx+dy) = dx-dy$$

$$\text{or} \quad \frac{\cos t-\sin t}{\cos t+\sin t}\,dt = dx-dy$$

Integrating,

$$\log\,(\cos t+\sin t) - \log c_2 = x-y$$

$$\text{or} \quad (\cos t+\sin t)/c_2 = e^{x-y}.$$

$$\text{or } [\cos\,(x+y)+\sin\,(x+y)]\,e^{(y-x)} = c_2 \qquad (4)$$

The complete solution is given by (3) and (4)

21.5 Simultaneous Linear Differential equations with constant coefficients

A pair of simultaneous linear differential equations can be written as

$$f_1(D)\,x + \phi_1(D)\,y = T_1 \qquad (1)$$

$$f_2(D)\,x + \phi_2(D)\,y = T_2 \qquad (2)$$

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where $f_1, f_2, \phi_1, \phi_2,$ are rational integral functions of D with constant coefficients, T_1, T_2 are explicit functions of t and $D = \frac{d}{dt}$. First we shall obtain an equation which contains only x and t.

Operating on both sides of (1) by $\phi_2(D)$ and on both sides of (2) by $\phi_1(D)$ and then subtracting we have

$$\left[f_1(D)\,\phi_2(D) - f_2(D)\phi_1(D)\right] x = \phi_2(D)\,T_1 - \phi_1(D)\,T_2 \qquad (3)$$

This is a linear equation with constant coefficients which can be solved. The general solution of this equation will contain independent arbitrary constants equal in number to the highest power of D in $f_1(D)\,\phi_2(D) - f_2(D)\,\phi_1(D)$.

Let the general solution of the equation (3) be

$$x = F(t, c_1, c_2 \ldots\ldots\ldots) \qquad (4)$$

Substituting this value of x in (1) or (2), the value of y can be obtained.

Alternative method of finding *y*

Operating on both sides of (1) by $f_2(D)$ and then subtracting, we have

$$\left[\phi_1(D)\,f_2(D) - f_1(D)\phi_2(D)\right] y = f_2(D)\,T_1 - f_1(D)\,T_2 \qquad (5)$$

This being a linear equation with constant coefficients can be solved.

Let the general solution be

$$y = F_1(t, c_1', c_2', \ldots\ldots) \qquad (6)$$

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The arbitrary constants $c_1, c_2 \ldots$ in (4) will in general be functions of the arbitrary constants $c_1', c_2', \ldots$ in (6).

The relation between the two sets of arbitrary constants will be obtained by substituting the values of x and y obtained from (4) and (6) in either of the equations (1) and (2) and then equating the coefficients of every different function of t on the two sides of the resulting identity.

Ex. 1 : Solve the simultaneous equations

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

$$\text{given that } x = 0 = y \text{ when } t = 0$$

Sol : Given equation can be written as

$$(D+5)\,x - 2y = t \qquad (1)$$

$$2x + (D+1)\,y = 0 \qquad (2)$$

Eliminate x, by multiplying (1) by (2) and operating on (2) by $D + 5$ and then subtracting, we get

$$[-4 - (D+5)(D+1)]\,y = 2t$$

$$\text{or } (D^2 + 6D + 9)\,y = -2t$$

The solution of this equation is given by

$$y = (c_1 + c_2 t)\,e^{-3t} - \frac{2t}{9} + \frac{4}{27} \qquad (3)$$

Now to find x, either eliminate y from (1) and (2) and solve the resulting equation o substitute the value of y in (2). Here, it is more convenient to adopt the latter method. From (3),

$$Dy = c_2 e^{-3t} + (c_1 + c_2 t)(-3)\ e^{-3t} - \frac{2}{9}$$

Substituting for y and Dy in (2), we get

$$x = \left[\left(c_1 - \frac{1}{2}c_2\right) + c_2 t\right] e^{-3t} + \frac{t}{9} + \frac{1}{27} \quad (4)$$

Hence (3) and (4) constitute the solutions of the given equations.

Since $x = y = 0$, when $t = 0$, the equations (3) and (4) give

$$0 = c_1 + \frac{4}{27}, c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0$$

$$\Rightarrow \quad c_1 = -\frac{4}{27}, c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1+6t)\ e^{-3t} + \frac{1}{27}(1+3t)$$

$$y = -\frac{2}{27}(2+3t)\ e^{-3t} + \frac{9}{27}(2-3t)$$

Ex. 2 : Solve the equations

$$2\frac{dx}{dt} + x + \frac{dy}{dt} = \cos t$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} + y = 0$$

Sol : The given equations are

$$(2D+1)x + Dy = \cos t \quad (1)$$

$$Dx + (2D+1)y = 0 \quad (2)$$

$$\text{where } D = \frac{d}{dt}$$

We eliminate x by operating with D on (1) and $(2D + 1)$ on (2)

$$D(2D+1)x + D^2 y = -\sin t$$

$$D(2D+1)x + (2D+1)^2 y = 0.$$

Subtracting the first equation from the second, we have

$$(3D^2 + 4D + 1)y = \sin t \quad (3)$$

Solving this equation, we get

$$y = Ae^{-t} + Be^{-t/3} - \frac{1}{10}(\sin t + 2\cos t)$$

Equation (3) is of order two and so x and y should contain only two arbitrary constants. Now to obtain x, we follow the alternative method stated above.

Eliminating y between equations (1) and (2), we get

$$(3D^2 + 4D + 1)\,x = -2 \sin t + \cos t$$

Solving this equation, we get

$$x = A_1 e^{-t} + B_1 e^{-t/3} + \frac{3 \cos t + 4 \sin t}{10}$$

From (2), we have

$$\frac{d}{dt}\left(A_1 e^{-t} + B_1 e^{-t/3} + \frac{3 \cos t + 4 \sin t}{10}\right)$$

$$+ 2\frac{d}{dt}\left(A_1 e^{-t} + B_1 e^{-t/3} - \frac{\sin t + 2 \cos t}{10}\right)$$

$$+ Ae^{-t} + Be^{-t/3} - \frac{\sin t + 2 \cos t}{10} = 0$$

Simplifying we get,

$$(-A_1 - A)\,e^{-t} + \left(-\frac{1}{3} B_1 + \frac{B}{3}\right) e^{-t/3} = 0$$

$$\Rightarrow \quad -A_1 - A = 0, -\frac{1}{3}B_1 + \frac{B}{3} = 0$$

$$\therefore \quad A_1 = -A, \; B_1 = B$$

21.6 Summary

Methods for solving systems of simultaneous differential equations have been discussed. When the simultaneous equations are linear with constant coefficients, the method of solution leads to solving of a linear differential equation with constant coefficients (of higher order than the given equations) and substituting back its solution in one of the given simultaneous equations, the other solution is obtained.

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21.7 Sample Examination Questions

I. *Answer the following questions in detail.*

(i) a) What is the standard form of a pair of ordinary simultaneous differential equations? Give their geometrical interpretation.

b) Solve $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$

(ii) a) How do you solve a pair of simultaneous differential equations with constant coefficients?

b) Solve $\dfrac{dx}{dt} + y = \sin t, \; \dfrac{dy}{dt} + x = \cos t$ given that

$$x = 2 \text{ and } y = 0, \text{ when } t = 0$$

II *Solve the following*

(i) $dx = dy = dz$

(ii) $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{3x^2 \sin (y + 2x)}$

(iii) $\frac{dx}{x - y} = \frac{dy}{y - x - z} = \frac{dz}{z}$

(iv) $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y) z}$

(v) $(5D + 4) y - (2D + 1) z = e^{-x}$

$(D + 8) y - 3z = 5e^{-x}$

(vi) $\frac{d^2y}{dt^2} = x, \ \frac{d^2x}{dt^2} = y$

Answers

I (i) (a) $\frac{x}{z} = c_1, \ \frac{y}{z} = c_2$

(ii) (b) $x = e^t + e^{-t}, \ y = e^{-t} - e^t + \sin t$

II (i) $x - y = c_1, \ x - z = c_2$

(ii) $2x + y = c_1, \ x^3 \sin (y + 2x) - z = c_2$

(iii) $x + y + z = c_1, \ (y - x - z) = c_2 z^2$

(iv) $x^{-1} - y^{-1} = c_1, \ x - y = c_2 z$

(v) $y = c_1 e^x + c_2 e^{-2x} + 2e^{-x}, \ z = 3 c_1 e^x + 2 c_2 e^{-2x} + 3e^{-x}$

(vi) $x = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t,$

$y = c_1 e^t - c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

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Unit – 22 : Condition of Integrability for $Pdx+Qdy+Rdz = 0$

22.0 Contents

22.1 Aims and Objectives

After going through this unit you must be able to check whether the given differential equation of the form $P\,dx + Q\,dy + R\,dz = 0$ is integrable or not.

22.2 Introduction

The equation $P\,dx + Q\,dy + R\,dz = 0$, (1) where P, Q, R are functions of x, y, z is called a total differential equation.

Equation (1) can be directly integrated if there exists a function $u\,(x, y, z)$ whose total differential du is equal to the L.H.S. of (1). In other cases (1) may or may not be integrable.

We now proceed to find the condition which P, Q, R must satisfy, so that (1) may be integrable.

22.3 Necessary and sufficient condition for integrability of differential equation P dx + Q dy + R dz = 0.

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The condition is necessary : Consider the differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

where P, Q, R are functions of x, y, z.

Let (1) have an integral $u\,(x, y, z) = c \qquad (2)$

Then total differential du must be equal to $P\,dx + Q\,dy + R\,dz$ or to it multiplied by a factor λ that is $(P\,dx + Q\,dy + R\,dz)\ \lambda\,(x, y, z)$.

$$\text{We know that} \quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\,. \qquad (3)$$

Since (2) is an integral of (1), P, Q, R must be proportional to

$\partial u/\partial x,\ \ \partial u/\partial y,\ \ \partial u/\partial z.$

i.e.

$$\frac{(\partial u/\partial x)}{P} = \frac{(\partial u/\partial y)}{Q} = \frac{(\partial u/\partial z)}{R} = \lambda\ (x, y, z)$$

$$\therefore\ \lambda P = \frac{\partial u}{\partial x}\ ,\ \lambda Q = \frac{\partial u}{\partial y}\ \ \lambda R = \frac{\partial u}{\partial z} \qquad (4)$$

From the first two equations of (4), we have

$$\frac{\partial}{\partial y}\,(\lambda P) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \text{ and } \frac{\partial}{\partial x}\,(\lambda Q) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\therefore\ \frac{\partial}{\partial y}\,(\lambda P) = \frac{\partial^2 u}{\partial y\,\partial x}\ ;\ \frac{\partial}{\partial x}\,(\lambda Q) = \frac{\partial^2 u}{\partial x\,\partial y}$$

$$\text{Hence } \frac{\partial}{\partial y}(\lambda P) = \frac{\partial}{\partial x}(\lambda Q)$$

$$\lambda \frac{\partial P}{\partial y} + P \frac{\partial \lambda}{\partial y} = \lambda \frac{\partial Q}{\partial x} + Q \frac{\partial \lambda}{\partial x}$$

$$\lambda \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = Q \frac{\partial \lambda}{\partial x} - P \frac{\partial \lambda}{\partial y} \tag{5}$$

Similarly

$$\lambda \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) = R \frac{\partial \lambda}{\partial y} - Q \frac{\partial \lambda}{\partial z} \tag{6}$$

$$\lambda \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = P \frac{\partial \lambda}{\partial z} - R \frac{\partial \lambda}{\partial x} \tag{7}$$

Multiplying (5), (6) and (7) by R, P and Q respectively and adding

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0 \tag{8}$$

This is therefore, the necessary condition for the integrability of the equation (1).

Condition is Sufficient :

Suppose that the coefficients P, Q, R of (1) satisfy the relation (8). We have to show that, an integral of (1) can be found, when relation (8) holds.

We can prove that, if we take $P_1 = \mu P$; $Q_1 = \mu Q$; $R_1 = \mu R$; where μ is any function of x, y, and z; the same condition (8) is satisfied by P_1, Q_1, R_1.

Now $P\,dx + Q\,dy$ may be regarded as a exact differential. If it is not so, then multiplying the equation (1) by integrating factor $\mu(x, y, z)$ we can make it so. Thus, there is no loss of generality in regarding $P\,dx + Q\,dy$ as an exact differential.

$$\text{For this the condition is } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{9}$$

$$\text{Let } V = \int P\,dx + Q\,dy \tag{10}$$

$$\text{then it follows that } P = \frac{\partial V}{\partial x}, \quad Q = \frac{\partial V}{\partial y} \tag{11}$$

$$\text{From (4) } \frac{\partial P}{\partial z} = \frac{\partial^2 V}{\partial z\, \partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z\, \partial y}$$

Using (9) and (11), (8) gives

$$\frac{\partial V}{\partial x} \left(\frac{\partial^2 V}{\partial z\, \partial y} - \frac{\partial R}{\partial y} \right) + \frac{\partial V}{\partial y} \left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z\, \partial x} \right) = 0$$

$$\frac{\partial V}{\partial x} \cdot \frac{\partial}{\partial y} \left(\frac{\partial V}{\partial z} - R \right) - \frac{\partial V}{\partial y} \cdot \frac{\partial}{\partial x} \left(-R + \frac{\partial V}{\partial z} \right) = 0$$

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$$\begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

This shows that a relation independent of x and y exists between V and $\left(\frac{\partial V}{\partial z} - R.\right)$, consequently $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V alone. That is we can take

$$\frac{\partial V}{\partial z} - R = \phi(z, V) \qquad (12)$$

Using (11) and (12), we have

$$P\,dx + Q\,dy + R\,dz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \left(\frac{\partial V}{\partial z} - \phi\right)dz$$

$$= \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \phi\,dz$$

$$= dV - \phi\,dz.$$

Thus (1) may be written as $dV - \phi\,dz = 0$ which is an equation in two variables. Hence its integration will give an integral of the form

$$F(V, z) = 0$$

Hence the condition (8) is sufficient. Thus (8) is the necessary and sufficient condition that (1) has an integral.

Note : The condition (8) can be remembered because P, Q, R, x, y, z appear in it in a regular cyclic order. Also (8) can be obtained by expanding the determinant in the following interms of the elements of its first row.

$$\begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0$$

When all the parts of a particular equation are exact differential, we need not apply the condition of integrability. In other cases the students are advised not to forget to verify the condition (8) for integrability of given total differential equation.

The necessary and sufficient conditions for the equation

$P\,dx + Q\,dy + R\,dz = 0$ to be exact are

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad (1)$$

Proof follows if we note that $(du = P\,dx + Q\,dy + R\,dz = 0; \lambda = 1)$

Note that when conditions (1) are satisfied the condition for integrability, namely

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (2)$$

is also satisfied, for each term of (2) vanishes identically.

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Examples

1. Verify that the equation

$(y-z)(y+z-2x)\,dx+(z-x)(z+x-2y)\,dy+(x-y)(x+y-2z)\,dz=0$ is exact.

Sol : The equation is of the form $P\,dx+Q\,dy+R\,dz=0$

Here $P=y^2-z^2-2x(y-z)$; $Q=z^2-x^2-2y(z-x)$

$R=x^2-y^2-2z(x-y)$

$$\left.\begin{aligned}\frac{\partial P}{\partial y}&=2y-2x; & \frac{\partial P}{\partial z}&=2x-2z\\ \frac{\partial Q}{\partial x}&=2y-2x; & \frac{\partial Q}{\partial z}&=2z-2y\\ \frac{\partial R}{\partial x}&=2x-2z; & \frac{\partial R}{\partial y}&=2z-2y\end{aligned}\right\} \qquad (1)$$

The condition of integrability is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial z}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (2)$$

From (1), we observe that

$$\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z},\quad \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

Thus each bracket of (2) vanish and hence the condition of integrability is satisfied.

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2. Verify that the conditions of integrability is satisfied by :

$$\left(2x^2+2xy+2xz^2+1\right)dx+dy+2z\,dz=0$$

Sol : Comparing the given equation with

$$P\,dx+Q\,dy+R\,dz=0 \qquad (1)$$

We have $P=2x^2+2xy+2xz^2+1$;

$Q=1$

$R=2z$

$$\left.\begin{aligned}\therefore\ \frac{\partial P}{\partial y}&=2x; & \frac{\partial P}{\partial z}&=4xz\\ \frac{\partial Q}{\partial z}&=0; & \frac{\partial Q}{\partial x}&=0\\ \frac{\partial R}{\partial x}&=0; & \frac{\partial R}{\partial y}&=0\end{aligned}\right\} \qquad (2)$$

The conditon of integrability is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (3)$$

Substituting the values from (2) in (3), we have

$P\ (0-0)+Q\ (0-4xz)+2z\ (2x-0)=0-4xz+4xz=0$

Hence the condition of integrability is satisfied.

3. Find $f(y)$ such that the total differential equation

$[(yz+z)/x]\,dx - z\,dy + f(y)\,dz = 0$ is integrable

Sol : Multiplying throughout by x the given equation is

$(yz+z)\,dx - xz\ dy + xf(y)\,dz = 0$ (1)

Comparing (1) with $Pdx+Qdy+Rdz=0$

we have $P=yz+z,\ Q=-xz,\ R=xf(y)$ (2)

Suppose that (1) is integrable so that the following condition of integrability is satisfied by it.

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (3)$$

Using (2) and denoting $\frac{df}{dy}$ by f', (3) gives

$(yz+z)(-x-xf')-xz(f-(y+1))+xf(z-(-z))=0$

$xz(1+y)f'=xzf$

$$\frac{df}{dy}=\frac{f}{1+y}$$

$$\therefore \quad \frac{df}{f}=\frac{dy}{1+y}\ , \text{ integrating}$$

$$\log f = \log(1+y)+\log k$$

$$f(y) = k(1+y) \text{ where } k \text{ is arbitrary constant.}$$

4. Find $f(z)$ such that $[(y^2+z^2-x^2)/2x]\,dx - ydy + f(z)\,dz = 0$ is integrable

Sol : Multiplying through out by $2x$, the given equation is

$(y^2+z^2-x^2)\,dx - 2xy\ dy + 2xf(z)\,dz = 0.$ (1)

Comparing (1) with $Pdx+Qdy+Rdz=0$, we have

$P=y^2+z^2-x^2,\ Q=-2xy,\ R=2x\,f(z).$ (2)

Suppose that (1) is integrable so that the following condition of integrability is satisfied by it.

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (3)$$

Using (2) and denoting $f(z)$ by f, (3) gives

by $(y^2+z^2-x^2)(0-0)-2xy\ (2f-2z)+2xf\ [(2y-(-2y)]=0$

$-4xy\,(f-z)+8xfy=0$

or $f=-z$. So the required value of $f(z)=-z$.

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5. Verify that the condition of integrability is satisfied by

$$(\cos x + e^x y)\, dx + (e^x + ze^y)\, dy + e^y\, dz = 0$$

Sol: Comparing this equation with

$$Pdx + Qdy + Rdz = 0$$

we have $P = \cos x + e^x y$, $Q = e^x + ze^y$, $R = e^y$

$$\left.\begin{aligned} \therefore\ \frac{\partial P}{\partial y} &= e^x; & \frac{\partial P}{\partial z} &= 0 \\ \frac{\partial Q}{\partial z} &= e^y; & \frac{\partial Q}{\partial x} &= e^x \\ \frac{\partial R}{\partial x} &= 0; & \frac{\partial R}{\partial y} &= e^y \end{aligned}\right\} \qquad (2)$$

The conditions of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (3)$$

Substituting the values of (2) in (3), we have

$$(\cos x - e^x y)\,(e^y - e^y) + (e^x + ze^y)\,(0 - 0) + e^y\,(e^{-x} - e^x) = 0$$

Hence the condition of integrability is satisfied.

22.4 Summary

The integrability of the total differential equation $Pdx + Qdy + Rdz = 0$ is checked through the condition of integrability given by

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

It was proved that this condition is necessary and sufficient for the integrability of the equation $Pdx + Qdy + Rdz = 0$.

22.5 Sample Examination Questions

I. Verify that the condition of integrability of the following total differential euqations.

1. $(yz + z^2)\, dx - xz\, dy + xy\, dz = 0$
2. $2(y + z)\, dx - (x + z)\, dy + (2y - x + z)\, dz = 0$
3. $yz\,(y + z)\, dx + zx\,(x + z)\, dy + xy\,(x + y)\, dz = 0$
4. $z^2 dx + (z^2 - 2yz)\, dy + (2y^2 - yz - xz)\, dz = 0$
5. $(2x^2 + 2xy + 2xz^2 + 1)\, dx + dy + 2z\, dz = 0$
6. $zydx + (x^2 - zx)\, dy + (x^2 z - xy)\, dz = 0$
7. $(1 + yz)\, dx + (z - x)\, xdy - (1 + xy)\, dz = 0$
8. $(2x^3 - z)\, z\, dx + 2x^2 yz\, dy + x(z + x)\, dz = 0$
9. $(yz - 1)\, dx + (z - x)\, x\, dy + (1 - xy)\, dz = 0$
10. Find $F(y)$ if $F(y)\, dx - zx\, dy - xy \log y\, dz = 0$ is integrable.

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Unit – 23 : Total Differential Equations

23.0 Contents

23.1 Aims and Objectives

After going through this unit you must be able to solve the total differential equations.

23.2 Introduction

An ordinary differential equation of the first order and first degree involving three variables is of the form

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \tag{1}$$

where P, Q, R are functions of x, y, z and x is the independent variable.

In terms of differentials, (1) can be written as

$$P\,dx + Q\,dy + R\,dz = 0 \tag{2}$$

which is integrable if and only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \tag{3}$$

The conditions for the equation (2) to be exact are

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x};\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \tag{4}$$

Note that when conditions (4) are satisfied the condition for intetegrability (3) is also satisfied.

23.3 Solution by inspection

Some times by re-arranging the terms of the given equation and or by dividing by a suitable function of x, y, z the equation thus obtained will contain several parts which are exact differentials. In such cases there is no need to verify the condition of integrability.

In other cases you must verify the condition of integrability (3). While verifying the condition of integrability, if the bracketed terms cancel in pairs then clearly conditions (4) are fulfilled and the equation is exact.

The following list will help to re-write the given equations.

1. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

2. $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

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3. $\dfrac{y\,dx - x\,dy}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$

4. $\dfrac{x\,dy + y\,dx}{xy} = d\,[\log xy]$

5. $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log\left(x^2 + y^2\right)\right]$

6. $\dfrac{2xy\,dy - y^2\,dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

7. $x\,dy + y\,dx = d\,(xy)$

8. $xy\,dz + xz\,dy + yz\,dx = d\,(xyz)$

9. $\int \dfrac{d[f(x,y,z)]}{f(x,y,z)} = \log f(x, y, z).$

10. $y^2\,dx + 2\,xy\,dy = d\left(y^2x\right)$

11. $2\,(x\,dx + y\,dy) = d\left(x^2 + y^2\right)$

12. $\dfrac{ye^x\,dx - e^x\,dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

Examples

1. $yz\,(1 + x)\,dx + zx\,(1 + y)\,dy + xy\,(1 + z)\,dz = 0$

Sol : Dividing by $x\,y\,z$, the given equation becomes

$$\left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} + 1\right)dy + \left(\frac{1}{z} + 1\right)dz = 0$$

Integrating, we get

$(\log x + x) + (\log y + y) + (\log z + z) = C$

$x + y + z + \log\,(xyz) = C.$

2. Verify that the equation

$(y - z)\,(y + z - 2x)\,dx + (z - x)\,(z + x - 2y)\,dy + (x - y)\,(x + y - 2z)\,dz = 0$ is exact and find the solution

Sol: The equation is of the form

$$P\,dx + Q\,dy + R\,dz = 0$$

Here $P = (y - z)\,(y + z - 2x),$

$Q = (z - x)\,(z + x - 2y),$

$R = (x - y)\,(x + y - 2z).$

$$\left.\begin{aligned}\frac{\partial P}{\partial y} &= 2y - 2x, \quad \frac{\partial Q}{\partial x} = 2y - 2x, \quad \frac{\partial Q}{\partial z} = 2z - 2y\\ \frac{\partial R}{\partial y} &= 2z - 2y, \quad \frac{\partial R}{\partial x} = 2x - 2z, \quad \frac{\partial P}{\partial z} = 2x - 2z\end{aligned}\right\}$$

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Hence $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \; \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \; \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

∴ The given equation must be exact.

Re-writing it, we get

$$\left(y^2\, dx + 2xy\, dy\right) - \left(z^2\, dx + 2zx\, dz\right) + \left(z^2\, dy + 2zy\, dz\right)$$

$$- \left(x^2\, dy + 2xy\, dx\right) + \left(x^2\, dz + 2zx\, dx\right) - \left(y^2\, dz + 2zy\, dy\right) = 0$$

or $d\left(y^2x\right) - d\left(z^2x\right) + d\left(z^2y\right) - d\left(x^2y\right) + d\left(x^2z\right) - d\left(y^2z\right) = 0$

Integrating, $y^2x - z^2x + z^2y - x^2y + x^2z - y^2z = C$

3. Solve (i) $x\, dy - y\, dx - 2x^2z\, dz = 0$

(ii) $zy\, dx = zx\, dy + y^2\, dz$

Sol : (i) Dividing by x^2, we have

$$\frac{x\, dy - y\, dx}{x^2} - 2z\, dz = 0$$

$$d\left[\frac{y}{x}\right] - d\left(z^2\right) = 0$$

Integrating $\frac{y}{x} - z^2 = C$

(ii) Dividing by y^2z and re-writing, we have

$$\frac{y\, dx - x\, dy}{y^2} = \frac{dz}{z}$$

$$d\left(\frac{x}{y}\right) = d\,(\log z). \text{ Integrating,}$$

$$\frac{x}{y} = \log z - \log c$$

$$\text{or } z = c\, e^{x/y}$$

4. Solve $\left(x^2y - y^3 - y^2z\right) dx + \left(xy^2 - x^2z - x^3\right) dy + \left(xy^2 + x^2y\right) dz = 0$

Sol : Dividing by $x^2 y^2$, the given equation becomes

$$\left(\frac{1}{y} - \frac{y}{x^2} - \frac{z}{x^2}\right) dx + \left(\frac{1}{x} - \frac{z}{y^2} - \frac{x}{y^2}\right) dy + \left(\frac{1}{x} + \frac{1}{y}\right) dz = 0$$

$$\therefore \frac{y\, dx - x\, dy}{y^2} + \frac{x\, dy - y\, dx}{x^2} + \frac{x\, dz - z\, dx}{x^2} + \frac{y\, dz - z\, dy}{y^2} = 0$$

$$d\left(\frac{x}{y}\right) + d\left(\frac{y}{x}\right) + d\left(\frac{z}{x}\right) + d\left(\frac{z}{y}\right) = 0$$

Integrating, $\frac{x}{y} + \frac{y}{x} + \frac{z}{x} + \frac{z}{y} = c$

i.e., $x^2 + y^2 + z\,(x + y) = cxy$

5. Solve $x^2\, dx^2 + y^2\, dy^2 - z^2\, dz^2 + 2xy\, dx\, dy = 0$

Sol: Re-writing, we have

$$(x\, dx + y\, dy)^2 - (z\, dz)^2 = 0$$

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$$(x\,dx + y\,dy + z\,dz)\ (x\,dx + y\,dy - z\,dz) = 0$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0 \qquad (1)$$

$$\text{or} \quad x\,dx + y\,dy - z\,dz = 0 \qquad (2)$$

Integrating, we have

$$\frac{1}{2}\left(x^2 + y^2 + z^2\right) = c_1 \text{ and } \frac{1}{2}\left(x^2 + y^2 - z^2\right) = c_2$$

i.e., $x^2 + y^2 + z^2 = c_1$ and $x^2 + y^2 - z^2 = c_2$

Exercise–I

Solve the following equations, by the method of inspection :

1. $(a-z)\,(y\,dx + x\,dy) + xy\,dz = 0$
2. $yz \log z\,dz - zx \log z.\,dy + xy\,dz = 0$
3. $\left(2x^3 - z\right) z\,dx + 2x^2 y\,z\,dy + x\,(z + x)\,dz = 0$
4. $\dfrac{(x\,dx + y\,dy + z\,dz)}{\left(x^2 + y^2 + z^2\right)^{1/2}} + \dfrac{(z\,dx - x\,dz)}{\left(x^2 + z^2\right)} + 3ax^2\,dx + 2\,by\,dy + c\,dz = 0$
5. $x\,dx - y\,dy - \sqrt{a^2 - x^2 - y^2}\,dz = 0$
6. $2yz\,dx - 3zx\,dy - 4xy\,dz = 0$
7. $(yz + 2x)\,dx + (zx - 2z)\,dy + (xy - 2y)\,dz = 0$
8. $x\,dx + z\,dy + (y + 2z)\,dz = 0$
9. $x\,dx + y\,dy + z\,dz = 0$
10. $(x - y)\,dx - x\,dy + z\,dz = 0$

Answers

1. $xy = c\,(a - z)$
2. $x \log z = cy$
3. $x^2 + y^2 + \log z + \dfrac{z}{x} = c.$
4. $\left(x^2 + y^2 + z^2\right)^{1/2} + \tan^{-1}(x/z) + ax^2 + by^2 + cz = k.$
5. $x^2 + y^2 + (z + c)^2 = a^2$
6. $x^2 = cy^3 z^4$
7. $xyz + x^2 - 2yz = c$
8. $\dfrac{x^2}{2} + yz + z^2 = c.$
9. $x^2 + y^2 + z^2 = c$
10. $x^2 - 2xy + z^2 = c$

23.4 Homogeneous Equation in x, y, z

Method 1 : i) First find P, Q, R and verify the condition for integrability i.e.,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

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ii) Calculate $Px + Qy + Rz$. If it is not equal to zero then

$\frac{1}{Px + Qy + Rz}$ is taken as integrating factor (I.F) of the given equation.

iii) Multiplying the given equation by integrating factor, say $\frac{1}{D}$, where D denote the denominator or the integrating factor, find $d(D)$, i.e., total differential of D. Now add and substract $d(D)$ from the numerator of the given equation after multiplying it with the integrating factor and then integrate. Several terms in the resulting equations will be exact differential and hence keep the list of rules while re-grouping the remaining terms.

Method 2: The fist method fails when $Px + Qy + Rz = 0$. In such cases we apply the following method which is applicable to homogeneous equation.

i) Verify the condition of integrability

ii) Put $x = zu$; $y = zv$ so that

$$dx = u\,dz + z\,du;\ dy = z\,dv + v\,dz$$

Substituting these in the given equation two cases may arise.

Case (i): If the coefficient of dz is zero, we shall have an equation in only two variables u and v. By regrouping properly, it can be easily integrated.

Case (ii): If the coefficient of $dz \neq 0$, then we shall be able to separate z from u and v. Thus the resulting equation will be of the form

$$\frac{f_1(u, v)\,du + f_2(u, v)\,dv}{f(u, v)} + \frac{dz}{z} = 0$$

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We now denote $f(u, v)$ by D and find $d(D)$. Now Add and subtract $d(D)$ from the numerator, write the equation in a proper form and then integrate. After integration u and v are replaced by $\frac{x}{z}$ and $\frac{y}{z}$ respectively so as to get the desired solution in x, y, z.

Note : When P, Q, R are homogenous functions of the same degree, $P\,dx + Q\,dy + R\,dz = 0$ is called a Homogeneous equation.

Examples

1 Solve $(yz + z^2)\,dx - xz\,dy + xy\,dz = 0$ (1)

Sol : Comparing the given equation with

$P\,dx + Q\,dy + R\,dz = 0$, we have

$P = yz + z^2,\ Q = -xz,\ R = xy$

The condition of integrability is satisfied.

Let $x = uz,\ y = vz$

$$\text{then}\quad dx = u\,dz + z\,du$$
$$dy = v\,dz + z\,dv.$$

Putting these values in equation (1), we get

$(vz^2 + z^2)(u\,dz + z\,du) - uz^2(v\,dz + z\,dv) + vu\,z^2\,dz = 0$

$(v + 1)\,z^3\,du - uz^3\,dv + (v + 1)\,uz^2\,dz = 0$

Dividing by $(v+1)\,u\,z^3$, we get

$$\frac{du}{u}-\frac{dv}{v+1}+\frac{dz}{z} = 0$$

Integrating, we get

$$\log u - \log (v+1) + \log z = \log c$$

$$\text{or } uz = c\,(v+1)$$

But $u = x/z$ and $v = y/z$

$$\therefore \frac{x}{z}z = c\left(\frac{y}{z}+1\right)$$

or $xz = c\,(y+z)$ which is the required solution.

2. Solve : $(y^2+yz)\,dx + (xz+z^2)\,dy + (x^2-xy)\,dz = 0$

Sol : As usual verify the condition on integrability. Since the given equation is homogeneou: we put

$x = zu$, $y = zv$ so that

$$dx = z\,du + u\,dz;\ dy = z\,dv + v\,dz, \qquad (1)$$

putting these values in the given equation, we get

$$(v^2+v)\,z^3\,du + (u+1)\,z^3\,dv + \left[u\left(v^2+v\right) + v\,(u+1) + u^2 - uv\right] z^2\,dz = 0$$

$$\therefore z^3\left[(v^2+v)\,du + (u+1)\,dv\right] + (u+1)\,(v^2+v)\,z^2\,dz = 0$$

Dividing by $z^3\,(u+1)\,(v^2+v)$ we get

$$\frac{du}{u+1}+\frac{dv}{v^2+v}+\frac{dz}{z} = 0.$$

Resolving into partial fractions,

$$\frac{du}{u+1}+\left[\frac{1}{v}-\frac{1}{v+1}\right]dv+\frac{dz}{z} = 0$$

Integrating

$$\log (u+1) + \log v - \log (v+1) + \log z = \log c.$$

$$\therefore (u+1)\,vz = c\,(v+1)$$

Subsituting $u = x/z$ and $v = y/z$ and simplifying, we get

$$(x+z)\,y = c\,(y+z)$$

which is the required solution.

23.5 General method of solution $Pdx + Qdy + Rdz = 0$ by taking one variable as constant

If the condition of integrability is satisfied, consider one of the variables say z, as constant so that $dz = 0$. Then integrate the equation $Pdx + Qdy = 0$. Replace the arbitary constant appearing in its integral by $\phi(z)$. Now differentiate the integral just obtained with respect x, y, z Finally, compare this result with the given differential euqtion to determine $\phi(z)$.

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Examples

1. Solve $3x^2\,dx + 3y^2\,dy - \left(x^3 + y^3 + e^{2z}\right) dz = 0.$

Sol : You can easily verify the condition of integrability. Treating z as constant so that $dz = 0$; then the equation becomes

$$3x^2\,dx + 3y^2\,dy = 0$$

$$x^3 + y^3 = f(z) \qquad (1)$$

where $f(z)$ is the constant of integration.

Differentiating (1), we get

$$3x^2\,dx + 3y^2\,dy - \frac{df}{dz}\,dz = 0 \qquad (2)$$

Comparing it with the given equation, we get

$$x^3 + y^3 + e^{2z} = \frac{df}{dz}$$

using (1), we have

$$f(z) + e^{2z} = \frac{df}{dz}$$

$$\frac{df}{dz} - f(z) = e^{2z} \qquad (3)$$

which is a linear differential equation

$\therefore$ I. F. $= e^{-\int dz} = e^{-z}$

Hence the solution of (3) is

$$fe^{-z} = \int e^{-z}\,e^{2z}\,dz + c$$

$$= e^{z} + c$$

$$\therefore f = e^{2z} + ce^{z}$$

Putting it in (1), we get

$$x^3 + y^3 = e^{2z} + ce^{z}$$

as the required solution.

2. Solve $(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0$

Sol : Here $P = y^2 + yz$, $Q = z^2 + zx$, $R = y^2 - xy$.

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y + yz)\ [2z + x - (2y - x)] + (z^2 + zx)\,[-y - y] + (y^2 - xy)\,(2y + z - z) = 0.$$

Hence the condition of integrability is satisfied.

Considering z as constant, the given equation becomes

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$(y^2 + zy)\, dx + (z^2 + zx)\, dy = 0$

or $\quad \dfrac{dx}{z\,(z+x)} + \dfrac{dy}{y\,(y+z)} = 0$

Integrating and noting that z is constant, we get

$$\frac{1}{z}\int \frac{dx}{z+x} + \frac{1}{z}\int \left(\frac{1}{y} - \frac{1}{y+z}\right) dy = \text{constant (w.r.t. } z)$$

or $\dfrac{1}{z}\,\{\log (z+x) + \log y - \log (y+z)\} = \text{constant}$

Let $\dfrac{y(z+x)}{y+z} = \phi(z)$ say

or $y\,(z+x) - (y+z)\,\phi(z) = 0 \qquad (1)$

Differentiating w.r. to x, y, z we obtained

$y\,(dz + dx) + (z+x)\,dy - (y+z)\,\phi'(z)\,dz - (dy + dz)\,\phi(z) = 0$

$y\,dx + [(z + x - \phi(z)]\,dy + [y - (y+z)\,\phi'(z) - \phi(z)\,]\,dz = 0 \qquad (2)$

Comparing (2) with the given differential equaiion, we get

$$\frac{y^2 + yz}{y} = \frac{z^2 + zx}{z + x - \phi(z)} = \frac{y^2 - xy}{y - (y+z)\,\phi'(z) - \phi(z)}$$

the relation

$$\frac{y^2 + yz}{y} = \frac{z^2 + zx}{z + x - \phi(z)}$$

reduces to (1) and therefore gives no information about $\phi(z)$.

Taking

$$\frac{y^2 + yz}{y} = \frac{y^2 - xy}{y - (y+z)\,\phi'(z) - \phi(z)} \qquad (3)$$

Using (1) in (3) and simplifying we get

$(y+z)^2\,\phi'(z) = 0$

i.e. $\phi(z) = \text{constant } (= c)$.

Hence the required solution is $y(z+x) = (y+z)\,c$.

3. Verify that the conditions of integrability are satisfied by

$(2x^2 + 2xy + 2xz^2 + 1)\; dx + dy + 2z\; dz = 0$ and solve it.

Sol : Verify the condition of integrability

Taking x as constant so that $dx = 0$, the equation becomes $dy + 2z\,dz = 0$.

(we have taken x as constant since the resulting equation (1) can be easily integrated).

Integrating (1), we get

$y + z^2 = f(x) \qquad (2)$

where $f(x)$ is an arbitrary function of x.

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Differentiating (2), we obtain

$$dy + 2z\,dz = \frac{df}{dx}\,dx \tag{3}$$

Comparing (3) with the given equation, we get

$$2x^2 + 2x\,(y + z^2) + 1 = -\frac{df}{dx}$$

using (2) we have

$$2x^2 + 2xf + 1 = -\frac{df}{dx}.$$

$$\text{or}\quad \frac{df}{dx} + 2xf = -(2x^2 + 1)$$

which is linear in x and f, and its solution is

$$f\,e^{\int 2x\,dx} = -\int (2x^2 + 1)\,e^{\int 2x\,dx}\,dx + c$$

$$\text{or}\qquad f e^{x^2} = -\int (2x^2 + 1)\,e^{x^2}\,dx + c$$

$$= -2\int x^2 e^{x^2}\,dx - \int e^{x^2}\,dx + c \tag{4}$$

$$\int 2x^2\,e^{x^2}\,dx = \int x\left(e^{x^2}\,2x\right)dx = \int x\,d\left(e^{x^2}\right)$$

$$= x e^{x^2} - \int e^{x^2}\,dx$$

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Putting this in (4), we get

$$\therefore\ f e^{x^2} = -x e^{x^2} + \int e^{x^2}\,dx - \int e^{x^2}\,dx + c = -\,x e^{x^2} + c$$

$$\therefore f(x) = c e^{-x^2} - x.$$

Putting this in (2), the required solution is

$$y + z^2 = c e^{-x^2} - x$$

$$\therefore x + y + z^2 = c e^{-x^2}$$

23.6 Summary

Methods for finding the solution of total differential equations are discussed. If the given differential equation is homogeneous in x, y, z, $\frac{1}{Px + Qy + Rz}$ is taken as the integrating factor provided $Px + Qy + Rz \neq 0$. If $Px + Qy + Rz = 0$, we substitute x by zu and y by zv and solve the equation.

In the general method of solution we consider one of the variables, say z, as constant so that $dz = 0$ and we integrate the equation $Pdx + Qdy = 0$ and put $\phi(z)$ as the arbitrary constant.

Now differentiate the integral obtained with respect to x, y, z and compare this result with the given differential equation to determine $\phi(z)$.

23.7 Sample Examination Questions

Solve :

1. $3y\,dx = 3x\,dy + y^2\,dy$
2. $(2x^2 - z)\,z\,dx + 2x^2yz\,dy + x(z + x)\,dz = 0$
3. $(yz - 1)\,dx + (z - x)\,x\,dy + (1 - xy)\,dz = 0$
4. $x\,dx + y\,dy - \sqrt{(a^2 - x^2 - y^2)}\;dz = 0$
5. $(x + z)^2\,dy + y^2\,(dx + y\,dz) = 0$
6. $(\cos x - e^x y)\,dx + (e^x + ze^y)\,dy + e^y\,dz = 0$
7. $(e^x y + e^z)\,dx + (e^y z + e^x)\,dy + (e^y - e^x y - e^y z)\,dz = 0$
8. $(2xz - yz)\,dx + (2yz - zx)\,dy - (x^2 - xy + y^2)\,dz = 0$
9. $(1 + yz)\,dx + (z - x)\,x\,dy - (1 + xy)\,dz = 0$
10. $(z + z^3)\cos x\,dx - (z + z^3)\,dy + (1 - z^2)(y - \sin x)\,dz = 0$

Answers

1. $x = y \log cz$

4. $x^2 + y^2 + (z + c)^2 = a^2$

5 $y(x + z) = c(x + y + z)$

6. $e^x y + e^y z + \sin x = c$

7. $ye^x + ze^y + xe^z = ce^x$

8. $x^2 - xy + y^2 = cz$

9. $1 + xy = c(z - x)$

10. $(\sin x - y)(z^2 + 1) = cz$

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BLOCK – 7 : PARTIAL DIFFERENTIAL EQUATIONS

Introduction

Several scientific and technological Phenomena are governed by a complex systems of linear and nonlinear partial differential equations. These partial differential equations have infinite number of solutions with varied physical conditions governing the problems. The task of mathematics usually rests upon the two factors :

i) given the physical aspects of the problem one has to form the partial differential equations,

ii) given the partial differential equations, to solve it and to select the solution describing the suitable conditions of the problem on hand.

In the end one has to examine, whether the solution exists at all with given conditions and if exists whether it is unique. If these two aspects viz: the existence and uniqueness of a solution of the partial differential equation describing the physical aspect con... are covered, the purpose of the scientists and technologists utilising these, is accomplished.

Unit – 24 : Formation of Partial Differential Equations

24.0 Contents

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24.1 Aims and Objectives

By the end of this unit you will be able to formulate a partial differential equation from a given relation in three variables, by eliminating the arbitrary constants or arbitrary functions involved in the given relation.

24.2 Introduction

A partial differential equation is one which involves partial derivates. Hence in such a differential equation there will be more than one independent variable. For instance

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2 \qquad (1)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \qquad (2)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \qquad (3)$$

are all partial differential equations. The equation (1) is a partial differential equation of first order in which z is (the dependent variable) considered as a function of two independent variables x and y. The equation (2) is a partial differential equation of the second order involving two independent variables x and t. The equation (3) is a partial differential equation of the second order involving three independent variables.

Order : The order of a partial differential equation is the order of the highest order differential coefficient occuring in it.

In what follows, z will be taken as the dependent variable and x, y the independent variables so that $z = f(x, y)$. We will use the following notation :

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x\, \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

24.3 Formation of partial differential equations

Partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

24.3.1 By eliminating arbitrary constants

Consider an equation $F(x, y, z, a, b) = 0$ (1)

where a and b denote artbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\frac{\partial F}{\partial x} + p\,\frac{\partial F}{\partial z} = 0 \qquad (2)$$

$$\frac{\partial F}{\partial y} + q\,\frac{\partial F}{\partial z} = 0 \qquad (3)$$

Now, eliminating the two arbitrary constants a and b from (1), (2) and (3) we get a partial differential equation of the first order of the form

$$f(x, y, z, p, q) = 0 \qquad (4)$$

Note : Equation (1) is said to be the primitive or complete solution of the first order differential equation (4).

If the number of constants to be eliminated is equal to the number of independent variables, the differential equation got after elimination of arbitrary constants will be of the first order. On the other hand, if the number of constants to be eliminated is more than the number of independent variables, the resulting partial differential equation will be of the second or higher orders.

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Examples

1. Form the partial differential equation by eliminating the arbitrary constants a and b from

$$z = (x^2 + a)(y^2 + b)$$

Sol :
$$z = (x^2 + a)(y^2 + b) \qquad (1)$$

Differentiating partially w.r. to x and y,

we get
$$\frac{\partial z}{\partial x} = 2x(y^2 + b)$$

and
$$\frac{\partial z}{\partial y} = 2y(x^2 + a)$$

$$\therefore\ y^2 + b = \frac{p}{2x}\ ;\ x^2 + a = \frac{q}{2y}.$$

Substituting these in (1) we get

$$z = \frac{p}{2x} \cdot \frac{q}{2y}.$$

or $4xyz = pq$ is the required partial differential equation.

2. Find a partial differential equation by eliminating a, b, c from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

Sol :
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \qquad (1)$$

Differentating (1) w.r. to x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \text{ or } c^2x+a^2z\frac{\partial z}{\partial x} = 0 \qquad (2)$$

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \text{ or } c^2y+b^2z\frac{\partial z}{\partial x} = 0 \qquad (3)$$

Now differentiating (2) w. r. to x and (3) w. r. to y,

we get $$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \qquad (4)$$

$$c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0 \qquad (5)$$

from (2) $$c^2 = -\frac{a^2z}{x}\frac{\partial z}{\partial x} \qquad (6)$$

Substituting this value of c^2 in (4) and dividing by a^2

we obtain $$\frac{-z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

or $$zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \qquad (7)$$

Similarly (3) and (5) give rise to

$$yz\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \qquad (8)$$

Thus (7) and (8) are the required equations.

3. Form the partial differential equation by eliminating the arbitrary constants from

$$(x-a)^2+(y-b)^2+z^2 = 1$$

Sol :
$$(x-a)^2+(y-b)^2+z^2 = 1 \qquad (1)$$

differentiating (1) partially with respect to x, we get

$$2(x-a)+2z\frac{\partial z}{\partial x} = 0 \text{ or } (x-a)+zp = 0 \qquad (2)$$

differentiating (1) partially with respect to y we have

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$$2(y-b) + 2z \frac{\partial z}{\partial y} = 0 \text{ or } (y-b) + zq = 0 \quad (3)$$

Eliminate a and b from (1), (2) and (3).

from (2) $\quad x - a = -zp$

from (3) $\quad y - b = -zq$

Substuting these values in (1), we get

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

i.e., $$z^2 (p^2 + q^2 + 1) = 1$$

This is the required partial differential equation.

4. Construct the partial differential equation by eliminating a, b and c from

$$z = a(x+y) + b(x-y) + ab\,t + c$$

Sol :

$$z = a(x+y) + b(x-y) + ab\,t + c \quad (1)$$

Differentiating (1) partially w.r. to x, y and t in turn, we get

$$\frac{\partial z}{\partial x} = a+b, \quad \frac{\partial z}{\partial y} = a-b, \quad \frac{\partial z}{\partial t} = ab \quad (2)$$

Since $\quad (a+b)^2 - (a-b)^2 = 4ab$

we get $$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(\frac{\partial z}{\partial t}\right)$$

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Assignments – 1

Obtain the partial differential equations by eliminating the arbitrary constants a and b in the following cases :

1. $z = ax + by + ab$
2. $z = ax + by + a^2 + b^2$
3. $z = ax^2 + by^2$
4. $z = xy + y\sqrt{(x^2 - a^2)} + b$
5. $z = a e^{-b^2 t} \cos bx$
6. $v = a e^{-bx} \sin(2b^2kt - bx)$
7. $z = (x-a)^2 + (y-b)^2$
8. $az + b = a^2x + y$
9. $ax^2 + by^2 + z^2 = 1$
10. $2z = (ax + y)^2 + b$

Answers

1. $z = px + qy + pq$
2. $z = px + qy + p^2 + q^2$

3. $z\,px + qy = 2z$

4. $px + qy = pq$

5. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial t}$

6. $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial v}{\partial t}$

7. $4z = p^2 + q^2$

8. $pq = 1$

9. $z\,(px + qy) = z^2 - 1$

10. $px + qy = q^2$

24.3.2 Elimination of arbitrary functions

Formation of partial differential equation by the elimination of arbitary function ϕ from the equation $\phi\,(u, v) = 0$ (1)

where u and v are functions of x, y and z.

We treat z as dependent variable and x and y as independent variabes so that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial y}{\partial x} = 0, \quad \frac{\partial x}{\partial y} = 0 \tag{2}$$

Differentiating (1) partially with respect to x and y we get,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\,p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\,p\right) = 0 \tag{3}$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\,q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\,q\right) = 0 \tag{4}$$

Eliminating $\dfrac{\partial \phi}{\partial u}$, $\dfrac{\partial \phi}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}\,p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}\,p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}\,q & \dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z}\,q \end{vmatrix} = 0$$

i.e., $$\left(\frac{\partial u}{\partial x} + p\,\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\,q\right) - \left(\frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\,\frac{\partial v}{\partial z}\right) = 0$$

which on simplification, gives a partial differential equation of the form

$$Pp + Qq = R \tag{5}$$

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where $$P = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z}$$

$$Q = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x}$$

$$R = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y}$$

The relation $\phi(u, v) = 0$ is a solution of (5), whatever the function ϕ may be.

Thus we obtain a linear partial differential equation of first order and of first degree in p and q.

Note : If the given equation in x, y, z contains two arbitrary functions, then, in general, their elimination gives rise to equations of higher order.

Examples

1. Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x + y + z, x^2 + y^2 - z^2) = 0$

Sol : Given

$$\phi(x + y + z, x^2 + y^2 - z^2) = 0 \qquad (1)$$

Let $u = x + y + z$; $v = x^2 + y^2 - z^2$ (2)

Then (1) becomes $\phi(u, v) = 0$ (3)

Differentiating (3) partially w.r. to x we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\,\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\,\frac{\partial v}{\partial z}\right) = 0 \qquad (4)$$

From (2), we obtain

$$\left.\begin{array}{lll} \frac{\partial u}{\partial x} = 1 ; & \frac{\partial u}{\partial z} = 1 ; & \frac{\partial v}{\partial x} = 2x \\ \frac{\partial v}{\partial z} = -2z ; & \frac{\partial u}{\partial y} = 1 ; & \frac{\partial v}{\partial y} = 2y \end{array}\right\} \qquad (5)$$

Substituting these values in (4), we get

$$\frac{\partial \phi}{\partial u}(1 + p) + \frac{\partial \phi}{\partial v}(2x - 2pz) = 0 \qquad (6)$$

Again differentiating (3) partially w.r. to y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\,\frac{\partial v}{\partial z}\right) = 0 \qquad (7)$$

Substituting the values of (5) in (7), we get

$$\frac{\partial \phi}{\partial u}(1 + q) + \frac{\partial \phi}{\partial v}(2y - 2zq) = 0 \qquad (8)$$

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Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (4) and (8), we get

$$\begin{vmatrix} (1+p) & 2x - 2pz \\ (1+q) & 2y - 2zq \end{vmatrix} = 0 \qquad (9)$$

or $(1+p)(2y-2zq) - (1+q)(2x-2pz) = 0$

Simplifying, this gives the equation

$$(y+z)p - (x+z)q = (x-y).$$

This is the desired partial differential equation of first order.

2. Eliminate the arbitrary functions

$$f \text{ and } F \text{ from } y = f(x-at) + F(x-at)$$

Sol : Now $y = f(x-at) + F(x-at)$ (1)

Differentiating partially w.r. to t, we have

$$\frac{\partial y}{\partial t} = -af' + aF'$$

Differentiating again w.r. to t, we get

$$\frac{\partial^2 y}{\partial t^2} = a^2 (f'' + F'') \qquad (2)$$

Differentiating (1) partially w.r. to x, we get

$$\frac{\partial^2 y}{\partial x^2} = f'' + F'' \qquad (3)$$

From equations (2) and (3), we get the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (4)$$

of the second order.

Note : Equation (4) is known as one-dimensional wave equation.

3. Eliminate the arbitrary function from $z = f(x^2 + y^2)$.

Sol : Now, $z = f(x^2 + y^2)$ (1)

Differentiating partially w. r. to x and y in turn we get

$$p = 2x\, f'(x^2 + y^2) \qquad (2)$$

$$q = 2y\, f'(x^2 + y^2) \qquad (3)$$

Eliminating $f'(x^2 + y^2)$ from (2) and (3), we get $py = qx$ (4)

4. Form a partial differential equation by eliminating the arbitrary function from the equation

$$x + y + z = f(x^2 + y^2 + z^2)$$

Sol : $x + y + z = f(x^2 + y^2 + z^2)$ (1)

Differentiating (1) partially w. r. to x and y in turn we get

$$1 + p = f'(x^2 + y^2 + z^2)\,(2x + 2zp) \qquad (2)$$

$$1 + q = f'(x^2 + y^2 + z^2)\,(2y + 2zq) \qquad (3)$$

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Determining $f'(x^2+y^2+z^2)$ from (2) and (3) and equating the values, we eliminate f and obtain

$$\frac{1+p}{2x+2zp} = \frac{1+q}{2y+2zq}$$

which on simplification takes the form

$$(y-z)\,p + (z-x)\,q = (x-y)$$

This is the desired partial differential equation.

5. Form the partial differential equation by eliminating the arbitrary functions from the relation

$$z = f(2x+3y) + g(2x+y)$$

Sol : Now $z = f(2x+3y) + g(2x+y)$ (1)

Partially differentiating (1) w.r. to x we get

$$\frac{\partial z}{\partial x} = f'(2x+3y)\,2 + g'(2x+y)\,2 \qquad (2)$$

Here $f'(2x+3y)$ means the derivative of $f(2x+3y)$ with respect to its argument $(2x+3y)$.

Again partially differentiating (1) w.r. to y, we get

$$\frac{\partial z}{\partial y} = f'(2x+3y)\,3 + g'(2x+y) \qquad (3)$$

Partially differentiating (2) w.r. to x,

$$\frac{\partial^2 z}{\partial x^2} = 4f''(2x+3y) + 4g''(2x+y) \qquad (4)$$

Partially differentiating (3) w.r. to y,

$$\frac{\partial^2 z}{\partial y^2} = 9f''(2x+3y) + g''(2x+y) \qquad (5)$$

Partially differentiating (2) w.r. to y,

$$\frac{\partial^2 z}{\partial y \partial x} = 6f''(2x+3y) + 2g''(2x+y) \qquad (6)$$

We can eliminate the two arbitrary functions f'' and g'' from the three equations (4); (5); and (6). On solving (4) and (6), we get

$$f''(2x+3y) = \frac{1}{4}\frac{\partial^2 z}{\partial y \partial x} - \frac{1}{8}\frac{\partial^2 z}{\partial x^2}$$

and

$$g''(2x+y) = \frac{3}{8}\frac{\partial^2 z}{\partial x^2} - \frac{1}{4}\frac{\partial^2 z}{\partial y \partial x}$$

Substituting these in (5) and simplifying, we get

$$3\frac{\partial^2 z}{\partial x^2} - 8\frac{\partial^2 z}{\partial y \partial x} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

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6. Form a partial differential equation by eliminating the arbitrary function ϕ from $xyz = \phi(x+y+z)$.

Sol. Now, $$xyz = \phi(x+y+z) \quad (1)$$

Partially differentiating (1) w.r. to x, y in turn, we get

$$xyp + yz\,1 + zx\,0 = \phi'(x+y+z)(1+0+p)$$

$$pxy + yz = (1+p)\,\phi'(x+y+z) \quad (2)$$

$$xy.q + yz\,.\,0 + zx\,.\,1 = \phi'(x+y+z)(0+1+q)$$

$$qxy + zx = (1+q)\,\phi'(x+y+z) \quad (3)$$

Obtaining the value of $\phi'(x+y+z)$ from (2) and (3) and equating them, we get

$$\frac{pxy + yz}{1+p} = \frac{qxy + zx}{1+q}$$

which on simplification, takes the form

$$px(y-z) + qy(z-x) = z(x-y)$$

This is the desired partial differential equation.

24.4 Summary

Partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

If the number of constants to be eliminated in the given relation, is equal to the number of independent variables, the differential equation got after elimination of arbitrary constants will be of the first order. On the other hand if the number of constants to be eliminated is more than the number of independent variables, the resulting partial differential equation will be of the second or higher orders.

If the given equation in x, y, z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

24.5 Sample Examination Questions

Form the partial differential equations by eliminating the arbitrary functions from the follwong cases :

1. $f(xy + z^2, x+y+z)$
2. $z = e^{ay}\phi(x+ay)$
3. $z = f_1(x+y) + f_2(x-y)$
4. $z = y\,f(x) + x\,g(y)$
5. $z = f(2x+y) + g(3x-y)$
6. $z = xy + f(x^2+y^2)$
7. $z = f\left(\frac{xy}{z}\right)$
8. $z = x + y + f(xy)$

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9. $z = f(x + ay) + \phi(x - ay)$

10. $f(x^2 + y^2, z - xy) = 0$

11. $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$

12. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Answers

1. $p(x - 2z) + q(2z - y) = y - x$

2. $q = ap + lz$

3. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$

4. $xy \frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$

5. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\, \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

6. $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2$

7. $px = qy$

8. $px - qy = x - y$

9. $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

10. $py - qx = y^2 - x^2$

11. $x^2 p + (2z^2 - xy)q = xz$

12. $px^2 + qy = 2y^2$.

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Unit – 25 : Solution of Lagranges and Other standard Equations

25.0 Contents

25.1 Aims and Objectives.

After goint through this unit, you will be able to solve the partial differential equations of first order and other partial differentials equations of standard types.

25.2 Introduction

A solution of a partial differential equation in a region R of the space of independent variables is a function of the variables appearing in the equation and satisfying the partial differential equation. The general solution of a partial differential equation contains arbitrary constants or arbitrary functions (or both). In general a partial differential equation may have a large number of different solutions. For example Laplaces equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ has the functions $x^2 - y^2$; $e^x \sin y$; $\log\left(x^2 + y^2\right)$, $\cosh x \cos y$ as its solutions. The unique solution of a partial differential equation corresponding to a physical problem, will satsify, besides the partial differential equation certain other prescribed conditions on the boundary of the region R and/or at an initial time. The conditions given at the boundary of the region are known as boundary conditions. The conditions given at the initial time $t = t_0$ are termed as initial conditions. Any linear combination solutions of a linear homogeneous partial differential equation gives another solution.

Solution of partial differential equations

There are two types of solutions, one containing arbitrary constants and the other containing arbitrary functions. Both these types may occur as solutions for the same partial differential equation.

Complete Integral : A solution which contains the maximum possible number of arbitrary constants is called a complete integral

Particular Integral : A Solution obtained by giving particular values to the arbitrary constants in a complete integral is called particular integral.

General Integral : A solution which contains the maximum possible number of arbitrary functions is called a general integral.

Consider the equations

$$z = ax + by \quad (1)$$

$$z = xf\left(\frac{x}{y}\right) \quad (2)$$

where a and b are arbitrary constants and f an arbitrary function.

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By eliminating the arbitrary constants a and b from (1) and the arbitrary function from (2), we obtain the same partial differential equation

$$px + qy = z \qquad (3)$$

Therefore, equations (1) and (2) are both solutions of (3).

Singular Integral:

Let $f(x, y, z, p, q) = 0$, (1)

is the partial differential equation whose Complete integral is

$$\phi(x, y, z, a, b) = 0 \qquad (2)$$

where a and b are constants.

Differentiating (2) partially w.r.to a and b

we obtain $\dfrac{\partial \phi}{\partial a} = 0$ (3)

and $\dfrac{\partial \phi}{\partial b} = 0.$ (4)

The eliminant of a and b from (2), (3) and (4), when it exists is called the singular integral of (1). The singular integral is not contained in the complete integral, whereas the particular integral is obtained from the complete integral.

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The general Integral

In the complete integral (2), assume that one of the constants is a function of the other.

That is, $b = f(a)$

Then (2) becomes

$$\phi[x, y, z, a, f(a)] = 0 \qquad (5)$$

Differentiating (2), partially w.r.to a

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot f'(a) = 0 \qquad (6)$$

The eliminant of 'a' between the two equations (5) and (6), when it exists is called the general integral of (1).

5.3 Lagranges linear Equation

A linear partial differential equation of the first order known as Lagrange's linear equation is of the form

$$Pp + Qq = R \qquad (1)$$

where P, Q and R are functions of x, y, z. We have seen that by eliminating the arbitrary function from the relation

$$\phi(u, v) = 0$$

where u, v are functions of x, y, z we get partial differential equation of the form

$$Pp + Qq = R$$

where

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

Hence $\phi(u, v) = 0$ is the general solution of (1), ϕ being any arbitrary function.

Now suppose $u = a$ and $v = b$ where a and b are constants.

$$\therefore du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

From these equations by the method of cross multiplication, we get

$$\frac{dx}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}}$$

i.e.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (3)$$

The solutions of equations (3) are $u = a$ and $v = b$. Therefore $\phi(u, v) = 0$ is the general solution of (1), where $u = a$ and $v = b$ are the solutions of (3).

Working procedure : To solve the equations

$$Pp + Qq = R$$

i) Form the auxiliary simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ii) Solve the auxiliary simultaneous equations giving two solutions $u = a$ and $v = b$.

Then write down the solution as $\phi(u, v) = 0$ or $u = f(v)$ or $v = F(u)$ where the function is arbitrary.

25.4 Solution of the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Method of Grouping

Suppose one of the variables is absent from two terms in the above equations. i.e., say, let z be absent from P and Q. Then the solution of $\frac{dx}{P} = \frac{dy}{Q}$ (a differential equation connecting x and y) gives a relation between x and y which is a part of the general solution. Let this be $u = a$.

Similarly we can obtain another relation $v = b$ which is also a part of the solution. Then $\phi(u, v) = 0$ is the general solution.

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Example 1 : Find the general solution of the equation

$$xzp + yzq = xy$$

Sol :

$$xzp + yzq = xy \qquad (1)$$

The subsidiary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

Taking the first two ratios

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\log x = \log y + \log a$$

$$\therefore \quad \frac{x}{y} = a \qquad (\text{or } x = ay)$$

From the last two ratios

$$\frac{dy}{z} = \frac{dz}{x}$$

or $\dfrac{dy}{z} = \dfrac{dz}{ay}$ by substituting $x = ay$ from the first solution.

$$\text{or} \quad ay\,dy = z\,dz$$

Integrating, we get

$$\frac{ay^2}{2} = \frac{z^2}{2} + \text{constant}$$

$$\frac{x}{y} \cdot \frac{y^2}{2} = \frac{z^2}{2} + \text{constant}$$

$$\therefore \quad xy - z^2 = b \quad (\text{constant})$$

Hence the general solution is

$$\phi\left(\frac{x}{y}, xy - z^2\right) = 0$$

Note : One integral known to us is used to find another integral. Also observe, in the second integral, the constant of integration of first integral is removed.

Ex. 2 : Obtain the general solution of the equation

$$\tan x.p + \tan y.q = \tan z.$$

Sol : Here $P = \tan x$; $Q = \tan y$ and $R = \tan z$. Lagranges subsidiary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two ratios, we have

$$\therefore \quad \cot x\,dx = \cot y\,dy$$

Integrating, $\log \sin x = \log \sin y + \log a$

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$$\therefore \quad \frac{\sin x}{\sin y} = a$$

Similarly from the last two ratios, we get

$$\sin y = b \sin z$$

$$\frac{\sin y}{\sin z} = b$$

Hence the general solution required is

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0,$$

where ϕ is arbitrary.

25.4.1 Solution of the subsidiary equation by the method of multipliers

We know

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{l\,P + m\,Q + n\,R} = \frac{l'\,dx + m'\,dy + n'\,dz}{l'P + m'Q + n'R}$$

where the two sets of multipliers $l, m, n\,;\ l', m', n'$ may be constants or variables in x, y, z.

Choosing l, m, n such that $lP + mQ + nR = 0$

we have $\qquad du = l\,dx + m\,dy + n\,dz = 0 \qquad (1)$

If $l\,dx + m\,dy + n\,dz$ is a perfect differential of some function say $u\,(x, y, z)$ then $du = 0$ by (1). Hence integrating (1) we get $u = a$ as one solution.

Hence $\qquad l'dx + m'dy + n'dz = 0$

This yields another solution $\qquad v = b$

$\therefore$ The general solution is $\phi\,(u, v) = 0$ or $u = f(v)$.

Examples

1. Solve $\quad (mz - ny)p + (nx - lz)\,q = ly - mx$

Sol : $\qquad (mz - ny)p + (nx - lz)\,q = ly - mx \qquad (1)$

This is in Lagranges linear equation form

$$Pp + Qq = R$$

where P, Q and R are functions x, y, z.

$\therefore$ Lagranges subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \qquad (2)$$

Using x, y, z as one set of multipliers, we get each of the ratio in (2)

$$= \frac{x\,dx + y\,dy + z\,dz}{x\,(mz - ny) + y\,(nx - lz) + z\,(ly - mx)}$$

Since the denominator is zero

$$x\,dx + y\,dy + z\,dz = 0 \qquad (3)$$

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Integrating (3) we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \text{constant.}$$

Similarly using l, m, n as the second set of multipliers, we get each of the ratio in (2)

$$= \frac{l\,dx + m\,dy + n\,dz}{l\,(mz - ny) + m\,(nx - lz) + n\,(ly - mx)}$$

$$\Rightarrow \; l\,dx + m\,dy + n\,dz = 0$$

Hence $lx + my + nz = b$, (b const of integration).

Hence the general integral is

$$\phi\,(x^2 + y^2 + z^2,\; lx + my + nz) = 0$$

2. Find the general solution of

$$x\,(z^2 - y^2)p + y\,(x^2 - z^2)\,q = z\,(y^2 - x^2)$$

Sol : The Lagranges subsidiary equations are

$$\frac{dx}{x\,(z^2 - y^2)} = \frac{dy}{y\,(x^2 - z^2)} = \frac{dz}{z\,(y^2 - z^2)} \qquad (1)$$

Taking x, y, z as one set of multipliers

each of ratio in (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2\,(z^2 - y^2) + y^2\,(x^2 - z^2) + z^2\,(y^2 - x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

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Hence $x\,dx + y\,dy + z\,dz = 0$. Integrating this

we get $x^2 + y^2 + z^2 = \text{const } (= a)$

Again choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as the second set of multipliers

we get each of ratio (1)

$$\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0.$$

Integrating

$$\log x + \log y + \log z = \text{const.}$$

$$\therefore \quad x\,y\,z = b.$$

Hence the general integral is

$$\phi\left(x^2 + y^2 + z^2,\; x\,y\,z\right) = 0$$

3. Solve $\quad x\,(y - z)p + y\,(z - x)q = z\,(x - y)$

Sol : Lagranges subsidiary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \qquad (1)$$

Using 1, 1, 1 as one set of multipliers, we get

cach of ratio in (1)

$$= \frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{dx + dy + dz}{0}$$

Hence $dx + dy + dz = 0.$

Integrating this we get $x + y + z = \text{const} \; (= a)$

Again choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as the second of multipliers, we get

each of Ratio in (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y - z + z - x + x - y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Hence $\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$

Integrating this

$$\log x + \log y + \log z = \text{Constant}$$

$$\text{or} \quad \log xyz = \log b \text{ Constant}$$

$$\therefore \quad xyz = b$$

Hence the general solution is

$$\phi\,(x + y + z,\, xyz) = 0.$$

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4. Solve $(x^2 - y^2 - z^2)\,p + 2\,xy\,q = 2xz$

Sol : The Lagranges subsidiary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \qquad (1)$$

From the last two fractions, we have $\frac{dy}{y} = \frac{dz}{z}$

which on integration gives

$$\log y = \log z + \log a$$

$$\therefore \quad \frac{y}{z} = x. \qquad (2)$$

Using the multipliers x, y, z we have from (1)

each fraction

$$= \frac{x\,dx + y\,dy + z\,dz}{x\,(x^2 + y^2 + z^2)} \qquad (3)$$

Therefore from (3) and (1), we have

$$\frac{x\,dx + y\,dy + z\,dz}{x\,(x^2 + y^2 + z^2)} = \frac{dz}{2\,xz}$$

which on integration gives

$$\log\,(x^2 + y^2 + z^2) = \log z + \log b$$

$$\therefore \quad \frac{x^2 + y^2 + z^2}{z} = b \qquad (4)$$

Hence from (2) and (4), the required solution is

$$\phi\left(\frac{y}{z}\,,\ \frac{x^2 + y^2 + z^2}{z}\right) = 0$$

or $$\frac{x^2 + y^2 + z^2}{z} = f\left(\frac{y}{z}\right)$$

i.e., $$x^2 + y^2 + z^2 = z\,f\left(\frac{y}{z}\right)$$

5. Solve $$y^2p - xyq = x\,(z - 2y)$$

Sol : The Lagranges equations are

$$\frac{dx}{y^2} = \frac{dy}{-\,xy} = \frac{dz}{x\,(z - 2y)} \qquad (1)$$

Taking the last two fractions of (1) and rewriting we get

$$\frac{dz}{dy} = \frac{-\,(z - 2y)}{y}$$

i.e., $$\frac{dz}{dy} + \frac{1}{y}\,z = 2 \qquad (2)$$

which is linear in z and y.

$$\therefore \quad \text{Integrating Factor} = exp\left[\int \frac{1}{y}\,dy\right] = e^{\log y} = y$$

Hence the solution of (2) is

$$z.\,y = \int 2y\,dy + a$$

where a is a constant of integration.

$$\therefore \quad zy - y^2 = a \qquad (3)$$

Again taking the first two fractions of (1) and rewriting, we get

$$2x\,dx + 2y\,dy = 0$$

Integrating, we get $$x^2 + y^2 = b \qquad (4)$$

where b is the constant of integration.

Hence the general solution is

$$\phi\,(x^2 + y^2\,,\ zy - y^2) = 0$$

where ϕ is an arbitrary function.

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6. Solve $(x-y)p + (x+y)q = 2xz$

Sol : The lagranges equations are

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \qquad (1)$$

Taking the first two fractions in (1), we get homogeneous equation

$$\frac{dy}{dx} = \frac{x+y}{x-y} \qquad (2)$$

$$\text{put } y = vx \text{ i.e., } \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

Hence from (2) and (3) $v + x\dfrac{dv}{dx} = \dfrac{1+v}{1-v}$

$$\text{or} \quad 2\frac{dx}{x} = \frac{2(1-v)}{1+v^2}\,dv$$

which on integration and simplification gives

$$x^2 + y^2 = a\,e^{2\tan^{-1}(y/x)}$$

where a is the constant of integration

$$\text{i.e.,} \qquad \frac{x^2+y^2}{e^{2\tan^{-1}(y/x)}} = a \qquad (4)$$

Now, choosing 1, 1, $-\frac{1}{z}$ as multipliers,

each fraction of (1)

$$= \frac{dx + dy - \frac{dz}{z}}{(x-y)+(x+y)-2x} = \frac{dx + dy - \frac{dz}{z}}{0}$$

$$\therefore \quad dx + dy - \frac{dz}{z} = 0$$

Integrating, we get

$$x + y - \log z = b \qquad (5)$$

where b is the constant of integration.

Hence the general integral from (4) and (5) is

$$\phi\left[(x^2+y^2)\,e^{-2\tan^{-1}(y/x)}\right] = x + y - \log z$$

$$\text{or} \quad \phi\left[(x^2+y^2)\,e^{-2\tan^{-1}(y/x)},\ x + y - \log z\right] = 0$$

Assignment - I

Solve :

1. $xp + yq = z$
2. $x^2p + y^2q = z^2$
3. $xyp + y^2q = zxy - 2x^2$
4. $px + qy = xyz^2(x^2 - y^2)$

5. $z(xp - yq) = y^2 - x^2$

6. $(x + 2z)p + (4zx - y)q = 2x^2 + y$

7. $(y - zx)p + (x + yz)q = x^2 + y^2$

8. $yp - xq = 2x - 3y$

9. $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

10. $px(x + y) = qy(x + y) - (x - y)(2x + 2y + z)$

Answers

1. $\phi\left(\frac{x}{z}, \frac{y}{z}\right) = 0$

2. $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$

3. $\phi\left(\frac{x}{y}, x - \log(z - 2x^2)/y^2\right) = 0$

4. $\phi\left(x^2 - y^2, \frac{x^2 - y^2}{x^2} + \frac{2}{z}\right) = 0$

5. $\phi(xy, x^2 + y^2 + z^2) = 0$

6. $\phi(xy - z^2, x^2 - y - z) = 0$

7. $\phi(x^2 - y^2 + z^2, xy - z) = 0$

8. $\phi(x^2 + y^2, 3x + 2y + z) = 0$

9. $\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0$

10. $\phi[(x + y)(x + y + z), xy] = 0$

25.5 Methods to solve the first order partial differential equations

The partial differential equation of the first order can be written as

$$F(x, y, z, p, q) = 0$$

$$\text{where} \quad p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}.$$

We shall consider some standard forms of such equations and solve them by special methods.

Type I : $$F(p, q) = 0 \qquad (1)$$

i.e., the equations contain p and q only.

Suppose that $z = ax + by + c$

is a solution of the equation $$F(p, q) = 0 \qquad (2)$$

Then $$p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = b$$

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Substituting these in the given equation, we get

$$F(a,b) = 0. \quad (3)$$

Solving for b from (3), we get $b = \phi(a)$, say, then $z = ax + \phi(a)\,y + c$ is the complete integral of the given equation, since it contains two arbitrary constants.

Singular integral is got by eliminating a and c from

$$z = ax + \phi(a)\,y + c$$

$$0 = x + \phi'(a)\,y$$

$$0 = 1$$

The last equation being absurd, there is no singular integral for the given partial differential equation. To find the general integral put $c = f(a)$, f being arbitrary. Then

$$z = ax + y\phi(a) + f(a) \quad (4)$$

Differentiating partially w. r. to 'a' we get

$$0 = x + y\phi'(a) + f'(a) \quad (5)$$

Eliminating a between (4) and (5) we get the general solution.

Examples

1. Solve $\sqrt{p} + \sqrt{q} = 1$

Sol : This is of the form $F(p, q) = 0$

Hence the complete integral is

$$z = ax + by + c$$

$$\text{where } \sqrt{a} + \sqrt{b} = 1$$

$$\therefore \; b = \left(1 - \sqrt{a}\right)^2$$

Differentiating partially w. r. to c, we find that there is no singular solution

$$\text{Taking } c = f(a) \text{ where } f \text{ is arbitrary}$$

$$z = ax + \left(1 - \sqrt{a}\right)^2 y + f(a) \quad (2)$$

Differentiating (2) partially w.r. to a we get

$$x + 2y\left(1 - \sqrt{a}\right)\left(\frac{-1}{2\sqrt{a}}\right) + f'(a) = 0 \quad (3)$$

Eliminating a between (2) and (3), we get the general integral.

2. Solve $p + q = pq$

Sol : The equation is of the form $F(p, q) = 0$, hence its solution is given by

$$z = ax + by + c$$

$$\text{where } a + b = ab$$

$$\therefore \; b = \frac{a}{a-1}$$

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Hence the complete integral is

$$z = ax + \frac{ay}{a-1} + c.$$

We observe that singular integral does not exist. The general integral is obtained by taking $c = \phi(a)$, where ϕ denotes an arbitrary function, and then eliminating a between

$$z = ax + \frac{ay}{a-y} + \phi(a) \text{ and}$$

$$0 = x - \frac{y^2}{(a-y)^2} + \phi'(a)$$

3. Solve $pq + p + q = 0$

Sol : The equation is of the form $F(p, q) = 0$, hence its solution is given by

$$z = ax + by + c$$

Then $$\frac{\partial z}{\partial x} = p = a, \ \frac{\partial z}{\partial y} = q = b$$

Substituting p and q in (1)

$$ab + a + b = 0$$

$$\therefore \ b = -\frac{a}{a+1}$$

Hence $z = ax - \dfrac{ay}{a+1} + c$ is the complete solution

The general solution is the eliminant of a between

$$z = ax - \frac{ay}{(a+1)} + \phi(a)$$

and $$0 = x - \frac{y}{(a+1)^2} + \phi'(a)$$

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Type II Clairauts Type

Suppose the equation is of the form

$$z = px + qy + f(p, q). \qquad (1)$$

We can easily prove that

$$z = ax + by + f(a, b) \qquad (2)$$

is the complete solution of (1) where a, b are arbitrary constants.

Differentiating (2) partially w. r. to a and b, we get

$$x + \frac{\partial f}{\partial a} = 0 \qquad (3)$$

$$y + \frac{\partial f}{\partial b} = 0 \qquad (4)$$

By eliminating a and b from (2), (3) and (4) we get the singular integral of (1).

Taking $b = \phi(a)$, (2) becomes

$$z = ax + \phi(a)\, y + f[a, \phi(a)] \qquad (5)$$

Differentiating partially w.r.to a we get

$$0 = x + y\phi'(a) + f'(a) \quad (6)$$

Eliminating a between (5) and (6) we get the general integral of (1).

Examples :

1. Solve $z = px + qy + pq$

Sol : Now, $z = px + qy + pq$ (1)

This is of the form

$z = px + 2y + f(p,q)$.

Hence the complete integral is

$$z = ax + by + ab \quad (2)$$

where a and b are arbitrary constants.

Differentiating (2) partially w. r. to a and b in turn and equating the results to zero we have

$$x + b = 0 \quad (3)$$

and $y + a = 0$ (4)

From (2), (3), (4) eliminate a and b.

We have $b = -x$ and $a = -y$ and putting in (1), we get

$$z = -xy - xy + xy = -xy$$

Hence the singular integral is

$$z = -xy.$$

To get the general integral put $b = f(a)$ where f is arbitrary. Then

$$z = ax + yf(a) + af(a) \quad (5)$$

Differentiating (5) partially w.r. to a and equating the result to zero, we have

$$0 = x + yf'(a) + f(a) + af'(a) \quad (6)$$

The eliminant of a between (5) and (6) is the general integral.

2. Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$

Sol : This is of the form

$$z = px + qy + f(p, q).$$

Hence the complete integral is

$$z = ax + by + \sqrt{1 + a^2 + b^2} \quad (1)$$

where a and b are arbitrary constants.

Singular and general integrals are found out in the usual manner. Hence to find the singular integral, differentiating partially w. r. to a and b, we have

$$x + \frac{a}{\sqrt{(1 + a^2 + b^2)}} = 0 \quad (2)$$

and $$y + \frac{b}{\sqrt{(1 + a^2 + b^2)}} = 0 \quad (3)$$

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Eliminating a and b, the singular integral is

$$x^2 + y^2 + z^2 = 1$$

To find the general integral, assume $b = f(a)$ where f is arbitrary.

$$\text{Then} \quad z = ax + yf(a) + \sqrt{1 + a^2 + f^2(a)}$$

Differentiate partially with respect to a and eliminate a between the two equations.

Type III : (a) : $F(z, p, q) = 0$

i. e., equations not containing x and y explicitly.
As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

Now, $z = f(u)$, $u = x + ay$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot 1 = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} \cdot a = a \frac{dz}{du}$$

Substituting these values of p and q in $F(z, p, q) = 0$

we get $F\left(z \frac{dz}{du}, a \frac{dz}{du}\right) = 0$ which is an ordinary differential equation of the first order.

Solving for $\frac{dz}{du}$, we obtain $\frac{dz}{du} = \phi(z, a)$ say

$$\therefore \frac{dz}{\phi(z, a)} = du$$

Integrating this

we get $$\int \frac{dz}{\phi(z, a)} = u + c$$

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i.e., $f(z, a) = x + a y + c$. This is the complete integral. Singular and general integrals are found out as usual.

(b) Suppose that the given equation is of the form

$$F(x, p, q) = 0 \qquad (1)$$

Since z is a function of x and y,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Assume that $q = a$. Then equation (1) becomes $F(x, p, a) = 0$

Solving for p, we obtain $\quad p = \phi(x, a)$

$$\therefore dz = \phi(x, a)\, dx + a\, dy$$

$$\text{Integrating} \quad z = \int \phi(x, a)\, dx + ay + c \qquad (2)$$

(2) is complete integral of (1) since it contains two arbitary constants a and c.

(c) If the given equation is of the form $F(y, p, q) = 0$

Assume $p = a$ and proceed as before.

The complete integral will be of the form

$$z = ax + \int f(y, a)\, dy + c$$

Examples

1. Solve $p(1 + q) = qz$

Sol.: Assume $u = x + ay$

$$z = f(u) = f(x + ay)$$

Then $$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}.$$

Substituting these values in the given equation we get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du}$$

$$\therefore a\frac{dz}{du} = az - 1$$

$$\frac{a\, dz}{az - 1} = du. \text{ Integrating,}$$

$\log(az - 1) = x + ay + c$, which is the required complete integral.

Singular and general integrals are found out as usual.

2. Solve $q^2 = z^2 p^2 (1 - p^2)$

Sol.: Assume $z = f(x + ay)$

$= f(u)$ as a tentative solution.

Then $$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

Substituting these values in the given equation, we get,

$$a^2 = z^2\left\{1 - \left(\frac{dz}{du}\right)^2\right\}$$

$$\therefore \frac{dz}{du} = \frac{\sqrt{(z^2 - a^2)}}{z}$$

$$\frac{z\, dz}{\sqrt{(z^2 - a^2)}} = du. \text{ Integrating,}$$

$$\sqrt{(z^2 - a^2)} = u + b$$

$$= x + ay + b.$$

$$\therefore z^2 = (x + ay + b)^2 + a^2 \text{ is the complete integral.} \quad (1)$$

Differentiating with respect to a and b partially

$$0 = 2a + 2y(x + ay + b) \quad (2)$$

$$0 = 2(x + ay + b) \quad (3)$$

$a = 0$ from (2) and (3). Eliminating a and b, $z = 0$ is the singular solution.

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Putting $b = f(a)$ where f is arbitrary

$$\sqrt{(z^2 - a^2)} = x + ay + f(a).$$

Differentiating with respect to a partially,

$$-\frac{a}{\sqrt{(z^2 - a^2)}} = f'(a) + y$$

3. Solve $p = 2qx$.

Sol.: Let $q = a$. Then $p = 2ax$

$$\text{But } dz = p\,dx + q\,dy$$

Integrating this, we get $\quad z = ax^2 + ay + c \qquad (1)$

Equation (1) is the complete integral of the given equation.

Differentiating partially w.r.t. c, we get $1 = 0$. Hence there is no singular integral. General integral can be found out in the usual way.

Type IV : *Separable equations :* We say that a first order partial differential equation is separable if it can be written as

$$f(x, p) = \phi(y, q).$$

$$\text{Let } f(x, p) = \phi(y, q) = a \qquad (1)$$

where a is an arbitrary constant.

Solving for p and q, we get $p = f_1(x, a);\ q = \phi_1(y, a) \qquad (2)$

$$\text{But } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= p\,dx + q\,dy$$

$$= f_1(x, a)\,dx + \phi_1(y, a)\,dy$$

$$\therefore z = \int f_1(x, a)\,dx + \int \phi_1(y, a)\,dy \qquad (3)$$

Now (3) contains two arbitrary constants and hence it is the complete integral.

The singular and general integrals are found out as usual.

Examples

1. Solve $p^2 + q^2 = x + y$

Sol. The equation is separable.

i.e., $\quad p^2 - x = -q^2 + y = a$ (say)

Thus $p^2 = a + x$ and hence $p = \sqrt{(a + x)}$

$a^2 = y - a$ and hence $q = \sqrt{(y - a)}$

But $dz = p\,dx + q\,dy = \sqrt{(a + x)}\,dx + \sqrt{(y - a)}\,dy$

Therefore on integrating, we get

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$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b \qquad (2)$$

This is the complete integral of the given equation.

Differentiating (2) partially w.r.t. b we find that there is no singular integral since we get $1 = 0$ which is absurd.

2. Solve $p^2 y\,(1+x^2) = q\,x^2$

Sol.: This equation is separable and can be written as

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \text{ (say)}$$

$$\text{Hence } p = \frac{x\sqrt{a}}{\sqrt{1+x^2}} \text{ and } q = ay$$

$$\text{But } dz = p\,dx + q\,dy$$

$$= \frac{x\sqrt{a}}{\sqrt{1+x^2}}\,dx + ay\,dy$$

Integrating, we get $\quad z = \sqrt{a\,(1+x^2)} + \frac{1}{2}ay^2 + b$

This is the complete integral where a and b are arbitrary constants. Differentiating partially w.r.t. b, we find that there is no singular integral.

25.6 Summary

We have solved here various standard types of partial differential equations. General information about complete integral, particular integral and general integral and singular integral was given. Lagrange's linear equation is solved. Solution of subsidiary equations by the method of grouping and by the method of multipliers has been considered. The type of equations $F(p, q) = 0$ and the Clairaut's equation $z = px + qy + f(p, q)$ have been solved. The standard types $F(z, p, q) = 0$, $F(x, p, q) = 0$, $F(y, p, q) = 0$ and $F(x, p) = \phi(y, q)$ are dealt with.

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25.7 Sample Examination Questions

Solve the following equations.

1. $p^3 - q^3 = 0$
2. $p^2 + q^2 = npq$
3. $q^2 - 3q + p = 2$
4. $z = xq + yp + p^2 - q^2$
5. $z = px + qy - 4p^2q^2$
6. $z^2(p^2 + q^2 + 1) = 1$
7. $q^2 = (1-p^2)z^2p^2$
8. $\sqrt{p} + \sqrt{q} = \sqrt{x}$
9. $pq = xy$
10. $p + q = \sin x + \sin y$

11. $yp + xp + pq = 0$

12. $q - p + x - y = 0$

Answers–II

1. $z = a(x+y) + c$
2. $ax + \frac{ay}{2}\left[n \pm \sqrt{(n^2-4)}\right] + b$
3. $z = (2 + 3b - b^2)x + by + c$
4. $z = ax + by + a^2 - b^2$
5. $z = ax + by - 4a^2 b^2$
6. $(a^2+1)(1-z^2) = (x + ay + b)^2$
7. $z^2 = (x + ay + b)^2 + a^2$
8. $z = a^2(x+y) - \frac{x^2}{2} - \frac{4a}{3}x^{3/2} + b$
9. $2z = ax^2 + \frac{y^2}{a} + b$
10. $z = a(x-y) - (\cos x + \cos y) + b$
11. $2z = ay^2 - \frac{ax^2}{a+1} + b$

25.8 References

1. Elementary Differential equations — E. Rain Ville
2. An introduction to ordinary differential equations — E.A. Coddington
3. Differential Equations — S.L. Ross
4. Avakalana Samikaranalu — Telugu Academy M - 12
5. Ordinary differential Equations — W. Kalpan
6. Ordinary and Partial Differential Equations — M.D. Rai Singhania, R.S. Agarwal

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BLOCK–8 : DETERMINANTS AND MATRIX THEORY

Introduction

James Joseph Sylvester (1850) was the first to write down matrices as rectangular arrangements of numbers from which determinants could be found. Rowan Hamilton (1853) and Arthur Cayley (1858) used matrices to express systems of equations in a brief form. Hermann Grassmann (1862) and Leopold Kronecker (1866) developed the idea of elementary transformations in matrices. David Hilbert (1904) introduced the matrices of infinite order in connection with the theory of integral equations. The development of matrices over the years as a useful and important tool extended to many fields of pure and applied mathematics like the algebraic and differential equations, astronomy, quantum mechanics, aerodynamics, relativity and nuclear physics. They are also used in the interpretation of results in economics, sociology, statistics, education and psychology. Being of such use in so many diverse fields, a study of the fundamental ideas of determinants and matrices is attempted in this Block.

Unit – 26 : Determinants

26.0 Contents

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26.1 Aims and Objectives

After going through this unit, you must be able to (i) Evaluate the given determinant, (ii) Solve a given system of equations using Cramer's rule, (iii) Test the consistency of a given system of equations.

26.2 Introduction

A determinant is one which determines anything or leads to determine anything. A square arrangement of real or complex numbers determines another number on expanding the arrangement according to certain well defined guidelines is called a determinant. Detrminants were first used to solve linear equations and test their consistency. Determinants were first used to solve linear equations and test their consistency. That is why, some text book writers first write down equations to define determinants. We shall however follow the general definition as follows.

Let $a, b, c, d, \ldots\ldots\ldots\ldots k$ be n letters and

$1, 2, 3, 4, \ldots\ldots\ldots\ldots n$ be n suffixes. Associating these, consider the term

$a_1 b_2 c_3 d_4$ ....k_n called the 'natural order'. This is taken positive. Keeping the letters fixed interchange any two suffixes or keeping the suffixes fixed interchange any two letters. Both are indentical processes (Ex: $a_2 b_1 \equiv b_1 a_2$). In any case we get an "inversion" (change) from the natural order. The new term derived bears the negative sign if it has one inversion.

26.3 Rule of signs

If a term derived as above contains an even number of inversions then it is positive and if it has an odd number of inversions then it is negative.

Definition: The natural order term $a_1 b_2 c_3$ ......k_n gives $n!$ terms on interchanging the n–suffixes. Giving proper sign to each term as earlier stated and adding them all, the sum of the $n!$ terms, is called a determinant of nth order. This is written as

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & d_2 & \cdots & k_2 \\ a_3 & b_3 & c_3 & d_3 & \cdots & k_3 \\ . & . & . & . & & . \\ . & . & . & . & & . \\ . & . & . & . & & . \\ a_n & b_n & c_n & d_n & \cdots & k_n \end{vmatrix}$$

$$= (a_1 b_2 c_3 d_4 \ldots\ldots\ldots\ldots k_n)$$

The horizontal lines of Δ are called rows and the vertical lines are called columns. The leading diagonal of the determinant $a_1 b_2 c_3 d_4$ ....k_n is formed by taking one and only one element in each row and each column. All other terms arise out of inversions in the leading diagonal and therefore their formation also satisfies the same condition.

The sign of any term :

Ex.1: What is the sign of the terms (1) $a_4 b_2 c_1 d_3$ and (2) $a_2 b_3 c_4 d_1$ in

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Sol.: Count the number of inversions in the givem terms. (p, q) is an inversion if $p > q$.

$a_4 b_2 c_1 d_3 \Rightarrow (4, 2), (4, 1), (4, 3), (2, 1)$; four inversions.

The term is positive.

$a_2 b_3 c_4 d_1 \Rightarrow (2, 1), (3, 1), (4, 1)$; three inversions. The term is negative.

Ex. 2 :

In $\Delta = \begin{vmatrix} a & b & c & \underline{d} \\ \underline{d} & a & b & c \\ c & \underline{d} & a & b \\ b & c & \underline{d} & a \end{vmatrix}$ find the sign of the term (1) d^4 (2) $a^2 bd$

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Sol.: Imagine

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 & \underline{d_1} \\ \underline{a_2} & b_2 & c_2 & d_2 \\ a_3 & \underline{b_3} & c_3 & d_3 \\ a_4 & b_4 & \underline{c_4} & d_4 \end{vmatrix}$$

d^4 is formed in the given determinant Δ taking one 'd' from each line (row or column) as shown underlined in Δ. This corresponds to the term $d_1 a_2 b_3 c_4$ as in Δ'.

$d_1 a_2 b_3 c_4 \Rightarrow (d, a), (d, b), (d, c)$; Three inversions

$\therefore$ d^4 is – ve

Similarly the sign of $a^2 b d$ in Δ is same as that

$a_1 b_2 d_3 c_4 \Rightarrow (d, c)$ one inversion $\therefore$ $a^2 b d$ is – ve.

26.4 Minor of determinants

From any given determinant we can obtain a new determinant by omitting the row and column through any element a_r. This determinant so obtained is called the minor of that element a_r and is denoted by Δa_r. The minor of the element b_s is Δb_s and so on.

If m rows and m columns are omitted in a determinant the remaining elements form a determinant called the mth minor or a minor of the mth order.

26.5 The development of determinants

If $\Delta a_1, \Delta a_2, \Delta a_3, \ldots\ldots, \Delta a_n$ are the minors of $a_1, a_2, a_3, \ldots\ldots, a_n$, then the determinant

$$\Delta = (a_1\ b_2\ c_3\ d_4 \ldots\ldots k_n)$$
$$= a_1 \Delta a_1 - a_2 \Delta a_2 + a_3 \Delta a_3 - \ldots\ldots + (-1)^{n-1} a_n \Delta a_n$$

Proof : Each term of Δ contains one and only one element from each row and each column. Therefore Δ can be taken as an expression of homogeneous terms coming along with the elements $a_1\ a_2 \ldots a_n$ of the first column.

If a_1 is taken from the first line, then no other element occurs from that line in any term of Δ. Therefore, a_1 comes with elements from other lines which form Δa_1.

$\therefore a_1 \Delta a_1 = a_1 (b_2 c_3 d_4 \ldots\ldots k_n)$, +ve since no inversions exist.

Similarly $a_2 \Delta a_2 = a_2 (b_1 c_3 d_4 \ldots\ldots k_n)$, – ve since one inversion (2, 1) exist.

$a_3 \Delta a_3 = a_3 (b_1 c_2 d_4 \ldots\ldots k_n)$, +ve since there are (3, 1), (3, 2) two inversions and so on.

Hence $\Delta = a_1 \Delta a_1 - a_2 \Delta a_2 + a_3 \Delta a_3 - \ldots\ldots + (-1)^{n-1} a_n \Delta a_n$

Note 1. a_3 occurs in the 3rd row and the 1st column. Therefore, $a_3 \Delta a_3$ bears the sign $(-1)^{3+1}$ i.e., +ve;

If an element belongs to the i th row and the j th column, then the sign of the minor is $(-1)^{i+j}$ in the expansion of Δ.

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2. We could similarly expand Δ in terms of the r th row as

$$\Delta = (-1)^{r+1}\left[a_r \Delta a_r - b_r \Delta b_r + c_r \Delta c_r - \ldots\ldots\ldots\ldots\right]$$

26.6 Co-factors

The co-factor of any element of a determinant is the minor of that element along with its sign.

Ex. 1: Co-factor of $a_r = A_r = (-1)^{r+1}\Delta a_r$, since a_r lies in the r th row and 1st column; the sign of the minor is $(-1)^{r+1}$.

2. Co-factor of $c_s = C_s = (-1)^{s+3}\Delta c_s$ etc. Hence we can write the expansion of Δ given in note (2) as

$\Delta = a_r A_r + b_r B_r + c_r C_r + \ldots + k_r K_r$ in terms of the co-factors of the elements of the r th row.

i.e., $\Delta = \sum a_r A_r$ (1)

Thus $\Delta = d_1 D_1 + d_2 D_2 + d_3 D_3 + \ldots\ldots + d_n D_n$ in terms of the fourth column. If $p_1, p_2, \ldots\ldots, p_n$ are the elements of the s th column then

$$\Delta = p_1 P_1 + p_2 P_2 + \ldots\ldots\ldots + p_n P_n$$

$$= \sum_{i=1}^{n} p_i P_i$$

from (1) and (2) we conclude

"If the elements of a line are multiplied by their co-factors and summed up, then the sum equals the value of the determinant".

Theorem 1: If the elements of a row or column are multiplied by the corresponding co-factors of the elements of other rows or columns and summed up then their sum is zero.

Proof: We know $\Delta = a_r A_r + b_r B_r + c_r C_r + \ldots\ldots + k_r K_r$. In this we can write $a_s, b_s, c_s, \ldots\ldots\ldots k_s$ instead of $a_r, b_r, c_r, \ldots.\ k_r$ when the r th and the s th lines are identical. We will see that $\Delta = 0$ when two rows or columns become identical.

$$\therefore\ a_s A_r + b_s B_r + c_s C_r + \ldots\ldots + k_s K_r = 0$$

i.e., $\sum a_s A_r = 0,\ \sum a_r A_s = 0,$

$\sum v_i P_i = 0$ etc., hold good.

Theorem 2: The value of a determinant is not changed if rows are written as columns and the columns as rows.

Proof: Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & \ldots\ldots & k_1 \\ a_2 & b_2 & c_2 & d_2 & \ldots\ldots & k_2 \\ a_3 & b_3 & c_3 & d_3 & \ldots\ldots & k_3 \end{vmatrix}$$

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$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ b_1 & b_2 & b_3 & \dots & b_n \\ c_1 & c_2 & c_3 & \dots & c_n \\ d_1 & d_2 & d_3 & \dots & d_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ k_1 & k_2 & k_3 & \dots & k_n \end{vmatrix}$$ rows of Δ are columns of Δ' to show; $\Delta = \Delta'$.

Δ and Δ' have the same leading diagonal which generates the same terms on causing inversions in suffixes or letters. An interchange of any two rows of Δ causes an inversion in the suffixes of the leading diagonal, while the same causes in Δ' an inversion in the letters of the leading diagonal. Therefore, if any term is derived from the leading diagonal of Δ by x inversions of suffixes, then the same term can be derived in Δ' from the leading diagonal by x inversions of letters, since in each case x interchanges of rows are required to get the given term to the position of the leading diagonal. This can be understood by considering any example.

$\therefore$ The sign of each term of Δ is unchanged when it appears as a term of Δ'. Hence $\Delta = \Delta'$.

Theorem 3: The interchange of any two rows or columns of a determinant, changes the sign of the determinant.

Proof : Interchange of any two rows of $\Delta \Rightarrow$ An inversion in the suffixes of the leading diagonal and hence an extra inversion is caused in each term, changing the sign of each term. Interchange of any two columns of $\Delta \Rightarrow$ An inversion in the letters of the leading diagonal and hence an extra inversion in each term is caused changing the sign of each term.

In both cases Δ has changed in sign.

Theorem 4: A determinant vanishes when two of its lines (rows or columns) are identical.

Proof : Interchanging the identical lines, Δ becomes $-\Delta$, by Theorem 3. But Δ should not change at all, since only identical lines are interchanged

$$\therefore \Delta = -\Delta, \Delta + \Delta = 0 \text{ or } 2\Delta = 0$$

$$\therefore \Delta = 0$$

Theorem 5: If the elements of any line (row or column) of a determinant is multiplied by a constant λ then the value of the detrminant is multiplied by λ.

Proof: Let $\Delta = a_r A_r + b_r B_r + \dots\dots + k_r K_r$

If each element of the r th row is multiplied by λ

then $\Delta_n = (\lambda a_r) A_r + (\lambda a_r) B_r + \dots\dots + (\lambda k_r) K_r = \lambda [\Delta]$

Thus if the elements of a line have a factor λ then λ is a factor of that determinant.

Ex. :

$$\begin{vmatrix} a & b & c \\ 2c & 2a & 2b \\ b & c & a \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

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Theorem 6 : If the elements of any line (row or column) of a determinant are sums, then the determinant equals a sum of determinants.

Proof : Let $\Delta = \Sigma a_r\ A_r$. If the r^{th} line elements are $a_r + \alpha_r - \alpha_r'$, etc.

$$\text{Then } \Delta' = \Sigma (a_r + \alpha_r - \alpha_r') A_r$$

$$= \Sigma a_r\ A_r + \Sigma \alpha_r A_r - \Sigma \alpha_r'\ A_r$$

$$= \Delta + \Delta_1 + (-\Delta_2)$$

Ex. :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3 & b_3. & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 7 : A determinant is not changed in value if to the elements of a row (or column), some multiples of corresponding elements of other rows (or columns) are added.

Proof : Let $\Delta = \Sigma a_r A_r$.

To the elements of the r^{th} row add λ times those of the sth row and μ times those of the i^{th} row and so on. Now the elements of the r^{th} row become.

$a_r + \lambda a_s + \mu a_i + \ldots,\ b_r + \lambda b_s + \mu b_i + \ldots, \ldots$

If Δ' is the determinant so obtained, then

$$\Delta' = \Sigma (a_r + \lambda a_s + \mu a_i + \ldots) A_r$$

$$= \Sigma a_r A_r + \lambda \Sigma a_s A_r + \mu \Sigma a_i A_r + \ldots$$

$$= \Delta + \lambda.\ 0 + \mu.\ 0 + \ldots \text{ (using Theorem 1)}$$

Hence $\Delta' = \Delta$.

Theorem 8 : If the elements of any line of a determinant are sums of corresponding elements of other lines, then the determinant vanishes.

Proof : Let the elements of the rth line be sums of the ith and jth line elements i. e., $a_r = a_i + a_j$ etc.,

$$\therefore\ \Delta = \Sigma a_r A_r = \Sigma (a_i + a_j) A_r$$

$$= \Sigma a_i A_r + \Sigma a_j A_r = 0 + 0$$

Ex. :

$$\begin{vmatrix} a & b & 2a+3b \\ p & q & 2p+3q \\ c & d & 2c+3d \end{vmatrix} = 0$$

Note : Theorem 7 is most useful in simplifying and evaluating the determinants. But we can also use remainder theorem, some times, in evaluating the determinants as follows.

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Use of remainder theorem

If $(x-a)$ is a factor of $f(x)$, then $f(a)=0$. If determinant Δ is considered as $f(x)$ and on putting $x=a$ in Δ, two of its lines become identical then $f(a)=0$ and $(x-a)$ is a factor of Δ. To test if $(a-b)$ is a factor of Δ, put $a=b$ in Δ. If two lines of Δ become identical then $(a-b)$ is a factor of Δ. If three lines become identical then $(a-b)^2$ is a factor of Δ and so on.

To test if $a+b+c$ is a factor, we put $a+b+c=0$ i.e., $a+b=-c,\ b+c=-a, c+a=-b$ in Δ and see if we get $\Delta=0$.

Similarly to test 'a' as a factor of Δ, we put 'a' = 0 in Δ and see if two lines become identical.

Examples :

1. Evaluate $\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Sol : In terms of 1st row,

$$\Delta = a\begin{vmatrix} a & b \\ c & a \end{vmatrix} - b\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c\begin{vmatrix} c & a \\ b & c \end{vmatrix}$$

$$= a(a^2-bc)-b(ca-b^2)+c(c^2-ab)$$

$$= a^3+b^3+c^3-3abc.$$

2. Evaluate $\Delta = \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$

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Sol : In terms of 1st row :

$$\Delta = 1\begin{vmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 0\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$$

$$+ 2\begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} - 0\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} \quad (2)$$

The first det $= 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 2(0-1)-(4-0) = -6$

and the second det $= (-2)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$ in terms of 1st row.

$$= -4(2-0) = -8$$

$\therefore\ \Delta = -6-8 = -14.$

3. Evaluate $\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

Sol : Using theorem 7 and taking $r_1 - r_2, r_2 - r_3$

i.e., 1st row – 2nd row, 2nd row – 3rd row, we get

$$\Delta = \begin{vmatrix} 25 & 15 & 21 \\ 21 & 27 & 17 \\ 219 & 198 & 181 \end{vmatrix}, \text{ now } r_3 - 9r_1 \Rightarrow$$

$$\Delta = \begin{vmatrix} 25 & 15 & 21 \\ 21 & 27 & 17 \\ -6 & 63 & -8 \end{vmatrix}$$

Take out 3 from col (2)
Now Col (1) – col (3) i.e.
$c_1 - c_3 \Rightarrow$

$$\Delta = 3\begin{vmatrix} 4 & 5 & 21 \\ 4 & 9 & 17 \\ 2 & 21 & -8 \end{vmatrix} \text{ Take out 2 from } c_1$$

$$\therefore \Delta = 6\begin{vmatrix} 2 & 5 & 21 \\ 2 & 9 & 17 \\ 1 & 21 & -8 \end{vmatrix} \; r_1 - r_2, r_2 - 2r_3 \Rightarrow$$

$$\Delta = 6\begin{vmatrix} 0 & -4 & 4 \\ 0 & -33 & 33 \\ 1 & 21 & -8 \end{vmatrix} \Rightarrow$$

$$\Delta = 6 \times 4 \times 33\begin{vmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 21 & -8 \end{vmatrix} = 0 \quad \because r_1 \equiv r_2$$

4. Show that

$$\Delta = \begin{vmatrix} 1+a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & 1+a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & 1+a_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & 1+a_n \end{vmatrix}$$

$$= 1 + a_1 + a_2 + \dots + a_n.$$

Sol : Take $c_1 + c_2 + c_3 + \dots + c_n$ and then get the common factor $1 + a_1 + a_2 + a_3 + \dots + a_n$ from c_1.

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$$\therefore\ \Delta = (1+a_1+a_2+\dots+a_n)\begin{vmatrix} 1 & a_2 & a_3 & \dots & a_n \\ 1 & 1+a_2 & a_3 & \dots & a_n \\ 1 & a_2 & 1+a_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_2 & a_3 & \dots & 1+a_n \end{vmatrix}$$

Now take $r_2 - r_1, r_3 - r_1, \dots, r_n - r_1$

$$\therefore\ \Delta = (1+a_1+a_2+\dots+a_n)\begin{vmatrix} 1 & a_2 & a_3 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$= \quad (1+a_1+a_2+\dots+a_n)$ on expanding in terms of c_1

5. Show that

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c)$$

Sol : Put $b+c=-a,\ \ c+a=-b,\ \ a+b=-c$

$$\therefore\ \Delta = \begin{vmatrix} a & -a & a^2 \\ b & -b & b^2 \\ c & -c & c^2 \end{vmatrix},\ c_1 \equiv c_2 \text{ on taking out } -1 \text{ from } c_2.$$

$\therefore\ \Delta = 0$ and $a+b+c$ is a factor of Δ. Put $a=b$, then $r_1 \equiv r_2$

$\therefore\ (a-b)$ is a factor of Δ. By symmetry or similarity, $b-c,\ \ c-a$ are also factors of Δ.

In Δ the leading diagonal $a.\,(c+a).\,c^2 \Rightarrow \Delta$ is of 4th degree terms in a, b, c. We have four factors $(a+b+c)(a-b)(b-c)(c-a)$ upto 4^{th} degree.

$\therefore$ There can only be a numerical factor k so that

$\Delta \equiv k(a+b+c)(a-b)(b-c)(c-a)$.

Put $a=0, b=1, c=2$ in this to find k

$\therefore\ \Delta = k(3)(-1)(-1)(2) = 6k$

But putting $a=0,\ \ b=1,\ \ c=2,$

$$\Delta = \begin{vmatrix} 0 & 3 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{vmatrix} = -6 \text{ on expansion in terms of } r_1.$$

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$\therefore\ 6k = -6,\ k = -1$ and

$$\Delta = -(a+b+c)(a-b)(b-c)(c-a).$$

6. Prove that

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix} = (ax - by + cz)^2$$

Sol : [In this determinant, the element of the i^{th} row and j^{th} column = – the element of the jth row and ith column. i.e. $a_{ij} = -a_{ji}$. Such a determinant is called skew - symmetric. A skew - symmetric determinant of even order is a perfect square and that of an odd order vanishes.]

Multiplying c_2 by 'a'

$$\Delta = \frac{1}{a} \begin{vmatrix} 0 & ax & y & z \\ -x & 0 & c & b \\ -y & ac & 0 & a \\ -z & -ab & -a & 0 \end{vmatrix} \text{ Now } c_2 - b.c_3 + c.c_4 \Rightarrow$$

$$\Delta = \frac{1}{a} \begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0 & c & b \\ -y & 0 & 0 & c \\ -z & 0 & -a & 0 \end{vmatrix}$$

Expand in terms of c_2

$$\Delta = \frac{-(ax - by + cz)}{a} \begin{vmatrix} -x & c & b \\ -y & 0 & a \\ -z & -a & 0 \end{vmatrix}$$

Again expand in terms of c_2

$$\Delta = \frac{-(az - by + cz)}{a} [-c\,(az) + a\,(-ax + by)]$$

$$= (ax - by + cz)\ (ax - by + cz)$$

7. Show that

$$\Delta = \begin{vmatrix} 1 & a & a^2 & 0 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix} = 1 + a^4 + a^8$$

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Sol : $r_1+r_2+r_3+r_4 \Rightarrow$

$$\Delta = \begin{vmatrix} 1+a+a^2 & 1+a+a^2 & 1+a+a^2 & 1+a+a^2 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix}$$

$$= (1+a+a^2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & a & a^2 \\ a^2 & 0 & 1 & a \\ a & a^2 & 0 & 1 \end{vmatrix}$$

$c_1-c_2,\ c_2-c_3,\ c_3-c_4 \Rightarrow$

$$\Delta = (1+a+a^2)\begin{vmatrix} 0 & 0 & 0 & 1 \\ -1 & 1-a & a-a^2 & a^2 \\ a^2 & -1 & 1-a & a \\ a-a^2 & a^2 & -1 & 1 \end{vmatrix} \text{ Expand in terms of } r_1$$

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$$\Delta = -1(1+a+a^2)\begin{vmatrix} -1 & 1-a & a-a^2 \\ a^2 & -1 & 1-a \\ a-a^2 & a^2 & -1 \end{vmatrix}$$

Take out -1 from c_1 and c_3 and then $r_1+r_3 \Rightarrow$

$$\Delta = -(1+a+a^2)\begin{vmatrix} 1-a+a^2 & 1-a+a^2 & 1-a+a^2 \\ -a^2 & -1 & a-1 \\ a^2-a & a^2 & 1 \end{vmatrix}$$

$$= -(1+a+a^2)(1-a+a^2)\begin{vmatrix} 1 & 1 & 1 \\ -a^2 & -1 & a-1 \\ a^2-a & a^2 & 1 \end{vmatrix} \text{ now } c_1-c_2,\ c_2-c_3 \Rightarrow$$

$$\Delta = -(a^4+a^2+1)\begin{vmatrix} 0 & 0 & 1 \\ 1-a^2 & -a & a-1 \\ -a & a^2-1 & 1 \end{vmatrix}$$

$$= -(a^4+a^2+1)\ [(1-a^2)\ (a^2-1)-a^2] = (a^4+a^2+1)\ (a^4-a^2+1) = a^8+a^4+1$$

8. Show that

$$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix} = 1+a^2+b^2+c^2+d^2$$

Sol : Taking out 'a' from r_1, b from r_2, c from r_3 and d from r_4,

$$\Delta = abcd \begin{vmatrix} a+\frac{1}{a} & b & c & d \\ a & b+\frac{1}{b} & c & d \\ a & b & c+\frac{1}{c} & d \\ a & b & c & d+\frac{1}{d} \end{vmatrix}$$

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Multiplying c_1 by 'a', c_2 by b, c_3 by c and c_4 by d,

$$\Delta = \begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

Now $c_1+c_2+c_3+c_4 \Rightarrow$

$$\Delta = (1+a^2+b^2+c^2+d^2) \begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 1 & b^2+1 & c^2 & d^2 \\ 1 & b^2 & c^2+1 & d^2 \\ 1 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

$r_1-r_2,\ r_2-r_3,\ r_3-r_4 \Rightarrow$

$$\Delta = (1+a^2+b^2+c^2+d^2)\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & b^2 & c^2 & 1+d^2 \end{vmatrix}$$

Expanding in terms of c_1,

$$\Delta = -(1+a^2+b^2+c^2+d^2)\begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -(1+a^2+b^2+c^2+d^2)\,(-1) = (1+a^2+b^2+c^2+d^2)$$

9. If $a+b+c=2s$ prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$$

Sol : Put $s-a=\alpha,\ \ s-b=\beta,\ \ s-c=\gamma$

$\therefore\ \alpha+\beta+\gamma = 3s-(a+b+c)=s,\ \ \beta+\gamma = 2s-b-c=a$

$\gamma+\alpha=b,\ \ \alpha+\beta=c$

To show that

$$\Delta = \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma+\alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2 \end{vmatrix} = 2\alpha\beta\gamma(\alpha+\beta+\gamma)^3$$

put $\alpha+\beta=-\gamma,\ \ \beta+\gamma=-\alpha,\ \gamma+\alpha=-\beta$

$$\therefore\ \Delta = \begin{vmatrix} \alpha^2 & \alpha^2 & \alpha^2 \\ \beta^2 & \beta^2 & \beta^2 \\ \gamma^2 & \gamma^2 & \gamma^2 \end{vmatrix} = 0,\ \because c_1 \equiv c_2 \equiv c_3 \qquad \therefore (\alpha+\beta+\gamma)^2 \text{ is a factor of } \Delta.$$

put $\alpha = 0$

$$\therefore\ \Delta = \begin{vmatrix} (\beta+\gamma)^2 & 0 & 0 \\ \beta^2 & \gamma^2 & \beta^2 \\ \gamma^2 & \gamma^2 & \beta^2 \end{vmatrix} = 0,\ \therefore\ \alpha \text{ is a factor of } \Delta.$$

Similarly β and γ are factors,

Δ is of 6^{th} degree in α, β, γ and is cyclically symmetric in α, β, γ. We have found a 5^{th} degree factor $\alpha\beta\gamma(\alpha+\beta+\gamma)^2$ for Δ.

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$\therefore$ We must have the 1st degree factor $k(\alpha+\beta+\gamma)$.

$$\Delta \equiv k\ \alpha\beta\gamma(\alpha+\beta+\gamma)^3. \quad \text{Put } \alpha=\gamma=1,\ \beta=-1$$

$$\therefore\ \Delta = k(-1)(1) = -k$$

$$\text{But}\quad \Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2 \text{ on expansion.}$$

$$\therefore\ \Delta = -k = -2, \text{ or } k = 2$$

$$\Delta = 2.\ \alpha\beta\gamma(\alpha+\beta+\gamma)^3.$$

$$= 2\ (s-a)(s-b)(s-c)s^3.$$

SAQ 1 : Solve

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$$

26.7 Product Rule

To multiply two determinants of the same order Δ_1 & Δ_2 :

Take the 1st row of Δ_1 and multiply the corresponding elements of the 1st column of Δ_2 and write down the sum of products as the first element of a row or column of the product determinant Δ_3. With the 1st row of Δ_1, repeat the same operation on other columns of Δ_2. We now have one row or column of Δ_3.

Repeating the same process with all rows of Δ_1 on all columns of Δ_2, we get all the rows or columns of Δ_3.

The above is called "row by column" rule and the proof will be known in matrix theory, i.e., in the next few units. We can also infer that since determinants do not change when rows are written as columns and the columns as rows we can multiply the determinants by the "Column by row" or "row by row" or "column by column" rules.

Examples

1. Evaluate $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$ and deduce that

$$\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} \text{ is a perfect square.}$$

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Sol.
$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 2\,abc \text{ on expansion.}$$

$$\begin{vmatrix} 0 & c & b \\ \longrightarrow & & \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \begin{vmatrix} 0 & c & b \\ c\downarrow & 0 & a \\ b & a & 0 \end{vmatrix} = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix}$$

(1) (2) (Δ)

[rows of (1) x columns of (2) ⇒ rows of Δ]

$\therefore \Delta = (2\,abc)\;(2\,abc) = 4\,a^2 b^2 c^2$

2. Show that

$$\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} \quad \begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix}$$

$$= \lambda^3\left(\lambda^2 + a^2 + b^2 + c^2\right)^3$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = \lambda\left(\lambda^2 + a^2 + b^2 + c^2\right)$$

$\therefore a^2 + \lambda^2 =$ co-factor λ in Δ in r_1.

$ab + c\lambda =$ co-factor of c in Δ, in r_1 and so on. Let Δ be the 2nd determinant given, whose elements are the above respective co-factors.

Then symbolically

$$\Delta\,\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1+b_1B_1+c_1C_1 & 0 & 0 \\ 0 & a_2A_2+b_2B_2+c_2C_2 & 0 \\ 0 & 0 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

on using the 'row by row' multiplication rule

 and $\Sigma a_r A_r = \Delta$ and $\Delta a_r A_s = 0$.

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$$\therefore \Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$$\therefore \Delta' = \Delta^2 = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}^2 = \lambda^2\left(\lambda^2 + a^2 + b^2 + c^2\right)^2$$

on expansion of Δ

26.8 Solution of a system of equations - Cramer's Rule

consider the equations

$$a_1x + b_1y + c_1z = d_1 \quad \dots \quad (1)$$

$$a_2x + b_2y + c_2z = d_2 \quad \dots \quad (2)$$

$$a_3x + b_3y + c_3z = d_3 \quad \dots \quad (3)$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let A_1, B_1, C_1 etc be

the co-factors of a_1, b_1, c_1 etc

To solve (1), (2), (3), Take (1) A_1 + (2) A_2 + (3) A_3

This gives

$$x(\Sigma a_1 A_1) + y(\Sigma b_1 A_1) + z(\Sigma c_1 A_1) = \Sigma d_1 A_1$$

i.e. $x\Delta + y \cdot 0 + z \cdot 0 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

i.e. $x\Delta = (d_1 b_2 c_3)$

$\therefore$ when $\Delta \neq 0$, $\quad x = \dfrac{(d_1 b_2 c_3)}{\Delta}$ and

similarly $\quad y = \dfrac{(a_1 d_2 c_3)}{\Delta}, \quad z = \dfrac{(a_1 b_2 d_3)}{\Delta}$

Note : In general if $x_1, x_2, x_3, \dots, x_n$ are the unknowns

then $\quad x_1 = \dfrac{(d_1 b_2 c_3 \dots k_n)}{(a_1 b_2 c_3 \dots k_n)}$

$$x_2 = \frac{(a_1 d_2 c_3 \dots k_n)}{(a_1 b_2 c_3 \dots k_n)}$$

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$$x_3 = \frac{(a_1\, b_2\, d_3 \dots k_n)}{(a_1\, b_2\, c_3 \dots k_n)}$$

and so on. This is known as Cramer's rule, valied when $\Delta \neq 0$.

Examples

1. Solve the equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 4y + z &= 7 \\ 3x + 2y + 9z &= 14 \text{ using Cramer's rule.} \end{aligned}$$

Sol :

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = 34 - 30 - 24 = -20$$

$$\therefore (a_1 b_2 c_3) = -20$$

$$(d_1 b_2 c_3) = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} \text{ on writing } d_1\, d_2\, d_3 \text{ for } a_1 a_2 a_3$$

$$= -20$$

$$\therefore x = \frac{(d_1\, b_2\, c_3)}{(a_1\, b_2\, c_3)} = \frac{-20}{-20} = 1$$

$$(a_1 d_2 c_3) = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = -20$$

$$\therefore y = \frac{(a_1\, d_2\, c_3)}{(a_1\, b_2\, c_3)} = \frac{-20}{-20} = 1$$

Similarly $z = 1$

SAQ 2 : Solve

$$\begin{aligned} ax + by + cz &= k \\ a^2x + b^2y + c^2z &= k^2 \\ a^3x + b^3y + c^3z &= k^3 \end{aligned}$$

26.9 Consistency of Equations

The system of equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \\ a_3x + b_3y + c_3 &= 0 \end{aligned}$$ are said to be consistent if

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$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

2. If $(f^2 - bc)\, x + (ch - fg)\, y + (bg - hf)\, z = 0$

$(ch - fg)x + (g^2 - ca)\, y + (af - gh)\, z = 0$

$(bg - hf)x + (af - gh)\, y + (h^2 - ab)\, z = 0,$

show that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

Sol: Eliminating x, y, z from the equations,

we have $$\begin{vmatrix} f^2-bc & ch-fg & bg-hf \\ ch-fg & g^2-ca & af-gh \\ bg-hf & af-gh & h^2-ab \end{vmatrix} = 0 \qquad \ldots (1)$$

Consider $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

In Δ co-factor 'a' $= A = bc - f^2$ etc.

$$\therefore\ (1) \Rightarrow \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0 = \Delta'$$

But $\Delta \,.\, \Delta' = \Delta^3$ $\qquad \therefore\ \Delta' = \Delta^2$

$\therefore\ \Delta' = 0 \Rightarrow \Delta = 0 \ \Rightarrow\ abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

26.10 Summary

The value of a given determinant is obtained by multiplying the elements of a line by their respective co-factors and summing them up.

While evaluating the determinant we use the row or column operations. If two rows or columns are identical then the determinant vanishes. The value of a determinant is not altered if we interchange the rows and columns. The interchange of any two rows or columns of a determinant changes the sign of the determinant.

We use Cramer's rule to solve a given system of equations, provided the coefficient determinant does not vanish. A system of homogeneous equations are said to be consistent if the coefficient determinant vanishes.

26.11 Sample Examination Questions

1. Evaluate

$$\begin{vmatrix} 67 & 9 & 21 \\ 9 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

2. Evaluate

$$\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix}$$

3. If ω is imaginary cube root of unity, show that

$$\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} = 3$$

4. Evaluate

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

5. Show that

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x-a)^3 (x+3a)$$

6. Show that

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$

7. Show that

$$\begin{vmatrix} 0 & -c & b & -l \\ c & 0 & -a & -m \\ -b & a & 0 & -n \\ x & y & z & 0 \end{vmatrix} = (ax+by+cz)\ (al+bm+cn)$$

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8. Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

9. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

10. If $S_r = \alpha_r + \beta_r + \gamma_r$ show that

$$\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$$

by using the product rule.

11. Show that

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3+b^3+c^3-3abc)^2$$

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12. Solve

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

13. Shlow that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

Where A_1, B_1, C_1, etc., are co-factors of $a_1, b_1, c_1, \ldots$ in the determinant on the left.

14. Solve, using Cramer's rule :

$3x+5y-7z=13$

$4x+y-12z=6$

$2x+9y-3z=20$

15. Find λ so that $h^2 \neq ab$ and

$$ax + hy + g = 0$$
$$hx + by + f = 0$$
$$gx + fy + c = \lambda$$

16. Solve :

$$\begin{aligned} x + y + z &= 1 \\ ax + by + cz &= k \\ a^2x + b^2y + c^2z &= k^2 \end{aligned}$$

Answers

(1) -43; (2) 9 ; (4) 0 ; (12) $x = 4$;

(14) $x = 1, \quad y = 2, \quad z = 0$;

(15) $\lambda = (abc + 2fgh - af^2 - bg^2 - ch^2)/(ab - h^2)$

(16) $x = \dfrac{(k-b)(c-k)}{(a-b)(c-a)}$ etc.

26.12 Answers to Self Assessment Questions

SAQ 1 : $c_2 - c_1, \ c_3 - c_1 \Rightarrow$

$$\Delta = \begin{vmatrix} x+2 & x+1 & x+1 \\ 2x+3 & x+1 & x+1 \\ 3x+5 & 2x+3 & 5x+9 \end{vmatrix} = 0$$

$$r_2 - r_1, \ r_3 - r_2 \Rightarrow \quad \Delta = \begin{vmatrix} x+2 & x+1 & x+1 \\ x+1 & 0 & 0 \\ x+2 & x+2 & 4(x+2) \end{vmatrix} = 0$$

expanding interms of the 2nd row

$$-(x+1)\ [(x+1)\,4\,(x+2) - (x+2)(x+1)] = 0$$

giving $x = -1, \ -2$ and 2 as solution.

SAQ 2 : $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = abc\,(a-b)(b-c)(c-a)$

$$\Delta_1 = \begin{vmatrix} k & b & c \\ k^2 & b^2 & c^2 \\ k^3 & b^3 & c^3 \end{vmatrix} = kbc\,(k-b)(b-c)(c-k)$$

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$$\Delta_2 = \begin{vmatrix} a & k & c \\ a^2 & k^2 & c^2 \\ a^3 & k^3 & c^3 \end{vmatrix} = kca\ (a-k)(k-c)(c-a)$$

$$\Delta_3 = \begin{vmatrix} a & b & k \\ a^2 & b^2 & k^2 \\ a^3 & b^3 & k^3 \end{vmatrix} = kba\ (a-b)(b-k)(k-a)$$

$$x = \frac{\Delta_1}{\Delta} = \frac{k(k-b)(c-k)}{a(a-b)(c-a)}\ ;\ y = \frac{\Delta_1}{\Delta} = \frac{k(a-k)(k-c)}{b(a-b)(b-c)}$$

$$z = \frac{\Delta_3}{\Delta} = \frac{k(b-k)(k-a)}{c(b-c)(c-a)}$$

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Unit – 27 : Types and Properties of Matrices

27.0 Contents

27.1 Aims and Objectives

After going through this unit, you will be able to acquaint yourself with various types of matrices and various properties of matrices.

27.2 Introduction

Let $(F, +, .)$ be a field of real or complex numbers or be any arbitrary field.

A matrix is a rectangular arrangement within brackets of members of a field $(F, +, .)$ in rows and columns, the rows being in order and each row consisting of an ordered set of the members of the field.

If a matrix contains m rows in order, each row containing an ordered set of n members of a field, then it is called an $m \times n$ matrix over the field. (m by n matrix).

Each member in the matrix is an 'element' and an $m \times n$ matrix contains $m\,n$ elements. $A, B, C,$ ..... capital letters denote matrices and small letters a_{11}, b_{23}, ... denote the elements.

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Notation :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

A is an $m \times n$ matrix.

The element in the 2nd row and 3rd column = a_{23}

i.e. $(2, 3)^{th}$ element = a_{23}

a_{ij} denotes the element of the i^{th} row and j^{th} column.

There are m rows, so, $1 \le i \le m$.

There are n columns, so, $1 \le j \le n$.

We can represent $A = [a_{ij}]_{m \times n}$

The diagonal elements of A are a_{11}, a_{22}, a_{33}, ... a_{mn}

Elements above the diagonal are a_{ij} $(i < j)$ i.e.,

a_{13}, a_{24}, a_{34}, ...,

The elements below the diagonal are a_{ij} $(i > j)$ i.e. a_{32}, a_{21}, ...

Particular cases and names of matrices

(1) A matrix having only one row, is a row matrix or a row vector

$A = [a_1, a_2, a_3 \ldots\ldots a_n]$ is a $1 \times n$ row matrix.

(2) A matrix having only one column, is a column matrix or column vector.

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix}$$ is an $m \times 1$ column matrix

For convenience we also denote the above column as

$A = \{a_1, a_2, a_3, \ldots\ldots\ldots a_m\}$

(3) If every element of a matrix is zero, then it is a 'null matrix $\mathbf{0}$'

Ex. : $$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is a null matrix of order 2×4

(4) If the number of rows and columns of a matrix are equal, then it is a 'square matrix'

In a square matrix, the diagonal elements are given by a_{ij}, $i = j$ i.e., the leading diagonal. Also a_{ij}, a_{ji} are called conjugate elements.

(5) If all the elements of a square matrix are zeros except those in the leading diagonal, then it is a diagonal matrix.

(6) If all the elements of a square matrix are zeros and those in the diagonal are each the same constant $\neq 0$, then it is a scalar matrix.

i.e., $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = k \neq 0$ when $i = j \Rightarrow [a_{ij}]$ is a scalar matrix.

(7) If all the elements of a square matrix are zeros except those in the diagonal which are each unity, then it is a unit matrix.

i.e., $a_{ij} = 0$, $i \neq j$ and $a_{ij} = 1$, $i = j \Rightarrow [a_{ij}]$ is a unit matrix.

It denotes an $n \times n$ unit matrix.

(8) In a square matrix, if every element above or below the leading diagonal is zero, then it is a triangular matrix

$a_{ij} = 0$, $i > j$ or $a_{ij} = 0$, $i < j \Rightarrow [a_{ij}]$ is a triangular matrix.

$a_{ij} = 0$, $i > j \Rightarrow$ all elements below the leading diagonal are zeros. In this case the matrix is called an upper triangular matrix.

$a_{ij} = 0$, $i < j \Rightarrow$ all elements above the diagonal are zeros. In this case the matrix is a lower triangular matrix.

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27.3 Algebra of matrices

(1) Multiplication of a matrix by a constant

To multiply a matrix by a scalar (a constant magnitude) we multiply all the elements of the matrix by that scalar.

$$A = [a_{ij}] \Rightarrow kA = [k\,a_{ij}] \text{ and}$$

$$kA = [k\,a_{ij}] = [a_{ij}k] = [a_{ij}]\,k = Ak$$

showing that multiplication of a matrix, by a constant is commutative.

when $k = -1$, we get $-A = [-a_{ij}]$.

Note : $\lambda(\mu A) = \mu(\lambda A) = \lambda\mu A$ from the definition where λ, μ are scalars.

Ex. : $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & -3 \end{bmatrix}$, $kA = \begin{bmatrix} 2k & 3k & -k \\ k & -2k & -3k \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & -3 & 1 \\ -1 & 2 & 3 \end{bmatrix}$

(2) Equality of matrices

Two matrices of the same order are equal iff the corresponding elements of them are same.

$$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{m\times n}$$

$$A = B \Leftrightarrow a_{ij} = b_{ij} \quad \forall_i, \forall_j$$

The equality of matrices is an equivalence relation on a set of matrices of the same order because

(1) $A = A$ (reflexive)

(2) $A = B \Leftrightarrow B = A$ (symmetric)

(3) $A = B,\ B = C \Rightarrow A = C$ (transitive)

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(3) Addition of matrices

Two matrices of the 'same order' can be added by adding the corresponding elements of both.

$$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{m\times n} \Rightarrow A + B = [a_{ij} + b_{ij}]_{m\times n}$$

Ex. : $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2-2 & 3+3 \\ -3+3 & 2-2 \end{bmatrix}$

$$A - B = A + (-B) = [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}] \text{ and}$$

in particular $A + (-A) = A - A = [a_{ij} - a_{ij}] = 0$

$\therefore$ $-A$ is the additive inverse of A.

Laws of addition :

(1) $A + B = B + A$, Commutative law

(2) $A + (B + C) = (A + B) + C$, Associative law

(3) $k(A + B) = kA + kB$, Distributive law

(4) $A + 0 = A$, Additive indentity

(5) $A + (-A) = 0$, Additive inverse

(6) $(\lambda + \mu)A = \lambda A + \mu A$, Distribution of scalars

(7) $A + B = A + C \Leftrightarrow B = C$, Cancellation law

All the above laws can be easily verified, using the definition of addition of matrices. For instance,

(Proof of 7) : let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ all $m \times n$ matrices

$$A + B = A + C \Leftrightarrow [a_{ij} + b_{ij}]_{m\times n} = [a_{ij} + c_{ij}]_{m\times n}$$

$$\Leftrightarrow a_{ij} + b_{ij} = a_{ij} + c_{ij} \qquad 1 \le i \le m$$

$$\Leftrightarrow b_{ij} = c_{ij} \qquad 1 \le i \le n$$

$$\Leftrightarrow B = C$$

(4) (a) Multiplication of matrices

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ then

$$AB = [c_{ij}]_{m\times p} \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

i.e., the $(i, j)^{th}$ elements of AB is got by multiplying the corresponding elements of i th row of A and those of the j^{th} column of B and adding the products.

This requires that the number of columns of A = the number of rows of B.

$$A = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & & a_{in} \\ \multicolumn{4}{c}{\longrightarrow} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{m\times n}$$

$$B = \begin{bmatrix} \cdots & \cdots & \downarrow b_{1j} & \cdots & \cdots \\ \cdots & \cdots & b_{2j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & b_{nj} & \cdots & \cdots \end{bmatrix}_{n\times p}$$

$AB = [c_{ij}]_{m\times p}$ where

$$c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots + a_{in}\, b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik}\, b_{kj} \text{ as defined.}$$

Ex. : $A = \begin{bmatrix} 1 & 2 \\ \multicolumn{2}{c}{\longrightarrow} \\ -2 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} \downarrow -4 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix}_{2\times 3} \Rightarrow$

$$AB = \begin{bmatrix} 1(-4)+2(2) & 1(2)+2(-1) & 1(0)+2(2) \\ -2(-4)+1(2) & -2(2)+1(-1) & -2(0)+1(2) \\ 3(-4)+0(2) & 3(2)+0(-1) & 3(0)+0(2) \end{bmatrix}_{3\times 3}$$

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Take the 1st row of A, multiply successive columns of B. This forms the first row of AB. Then take the 2nd row of A, repeat the process to get the 2nd row of AB. Then take the 3rd row of A, multiply columns of B as explained, to get the 3rd row of AB.

We will also find BA

$$B = \begin{bmatrix} -4 & 2 & 0 \\ & \longrightarrow & \\ 2 & -1 & 2 \end{bmatrix}_{2x3}, \quad A = \begin{bmatrix} \downarrow & 1 & 2 \\ & -2 & 1 \\ & 3 & 0 \end{bmatrix}_{3x2}$$

$$BA = \begin{bmatrix} -4(1)+2(-2)+0(3) & -4(2)+2(1)+0(0) \\ 2(1)+(-1)(-2)+2(3) & 2(2)+(-1)(1)+2(0) \end{bmatrix}_{2x2}$$

from this example we infer that

(1) AB and BA can both be defined if they are of orders $m \times n$ and $n \times m$ type.

(2) AB and BA need not be equal even if both exists. i.e., matrix multiplication is not commutative in general.

Note : $AB \Rightarrow A$ premultiplies, B the post factor

$BA \Rightarrow B$ premultiplies, A the post factor

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(4) (b) Matrix multiplication is (1) Associative (2) Distributive

(1) To show $A(BC) = (AB)C$ when the products are defined.

Proof : Let $A = [a_{ij}]_{mxn}$ $B = [b_{ij}]_{nxp}$ $C = [c_{ij}]_{pxq}$

$$\therefore \quad BC = [d_{ij}]_{nxq}, \; d_{ij} = \sum_{k=1}^{p} b_{ik} \; c_{kj} \tag{1}$$

$$A(BC) = [e_{ij}]_{mxq}, \; e_{ij} = \sum_{r=1}^{n} a_{ir} \; d_{rj}$$

$$\text{or } e_{ij} = \sum_{r=1}^{n} a_{ir} \left(\sum_{k=1}^{p} b_{rk} \; c_{kj} \right) \qquad \text{using (1)}$$

$$= \sum_{k=1}^{p} \left(\sum_{r=1}^{n} a_{ir} \; b_{rk} \right) c_{kj} \tag{2}$$

$$\text{But } AB = [u_{ij}]_{mxp}, \; u_{ij} = \sum_{r=1}^{n} a_{ir} \; b_{rj}$$

$$\therefore \quad u_{ik} = \sum_{r=1}^{n} a_{ir} \; b_{rk}$$

$\therefore$ from (2) we have $A(BC) = [e_{ij}]_{mxq}$ where

$$e_{ij} = \sum_{k=1}^{p} u_{ik} \; c_{kj} \tag{3}$$

$(AB)\, C = [u_{ij}]_{m\times p}\ [c_{ij}]_{p\times q} = [v_{ij}]_{m\times q}$ where

$$v_{ij} = \sum_{k=1}^{p} u_{ik}\, c_{kj} = e_{ij} \quad \text{from (3)}$$

$\therefore \quad A\,(BC) = [e_{ij}]_{m\times q} = [v_{ij}]_{m\times q} = (AB)\, C$

(2) To show $A\,(B + C) = AB + AC$ when AB, AC and $B + C$ are defined.

Proof : Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$, $C = [c_{ij}]_{n\times p}$

$$\therefore \quad B + C = [b_{ij} + c_{ij}]_{n\times p} = [u_{ij}]_{n\times p} \qquad (1)$$

$$\therefore \quad A\,(B + C) = [d_{ij}]_{m\times p}, \quad d_{ij} = \sum_{k=1}^{n} a_{ik}\ u_{kj}$$

$$d_{ij} = \sum_{k=1}^{n} a_{ik}\ (b_{kj} + c_{kj}) \qquad \text{using (1)}$$

$$= \sum_{k=1}^{n} a_{ik}\ b_{kj} + \sum_{k=1}^{n} a_{ik}\ c_{kj}$$

$$= v_{ij} + w_{ij} \qquad (2)$$

$\therefore \quad A\,(B + C) = [v_{ij} + w_{ij}]_{m\times p} = [v_{ij}]_{m\times p} + [w_{ij}]_{m\times p}$ where

$$v_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj},$$

$$w_{ij} = \sum_{k=1}^{n} a_{ik}\ c_{kj} \quad \text{from (2)}$$

$\therefore \quad [v_{ij}]_{m\times p} = AB, \quad [w_{ij}]_{m\times p} = AC$

$A\,(B+C) = AB + AC$

Note : Similarly we could prove $(B + C)\, A = BA + CA$, where BA, CA and $B + C$ are defined.

(4) c. If A is a square matrix and I the unit matrix of the same order $n \times n$, then

$AI = A = IA$ and I is multiplicative identity

Proof : Let $A = [c_{ij}]_{n\times n}$, $I = [b_{ij}]_{n\times n}$, $b_{ij} = 0, \ i \neq j$

$b_{ij} = 1, \ i = j$

$$\therefore \quad AI = [c_{ij}]_{n\times n}, \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj}$$

$$c_{ij} = a_{i1}\ b_{1j} + a_{i2}\ b_{2j} + \ldots\ldots + a_{ij}\ b_{jj} + \ldots\ldots + a_{in}\ b_{nj}$$

$$= a_{i1}\,.\,0 + a_{i2}\,.\,0 + \ldots\ldots + a_{ij}\,.1 + \ldots\ldots + a_{in}\,.\,0$$

$$= a_{ij}$$

$\therefore \quad AI = [a_{ij}]_{n\times n} = A$. Similarly $IA = A$

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$\therefore \quad AI = [a_{ij}]_{n \times n} = A$. Similarly $IA = A$

(4) d. Positive integral powers of matrix A :

We define $A^0 = I_n$ where A is an $n \times n$ square matrix

$A^1 = A, A^2 = AA$ is defined for $n \times n$ square matrix A

$A^3 = (AA)A = A(AA) = A^2 A = AA^2$ etc.

We can have $A^m . A^n = A^{m+n}$ and

$(A^m)^n = A^{mn}$ by induction

(4) e. In matrices $AB = \mathbf{0} \not\Rightarrow A = \mathbf{0}$ or $B = \mathbf{0}$ both $= \mathbf{0}$.

Ex. : $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}, \text{ but } A \neq \mathbf{0} \quad B \neq \mathbf{0}$$

$\therefore \quad A \neq \mathbf{0}, \quad B \neq \mathbf{0}$ may give $AB = \mathbf{0}$

In this case A, B are called proper zero divisors.

(4) f. If $AB = -BA$, we say A and B are anti-commutative

Ex. : $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\text{Then } AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $BA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Thus $AB = -BA$ and

A, B anti-commute with each other

(4) g. If A and B are $n \times n$ square matrices, the following results are easily obtained :

(1) $(A+B)^2 = A^2 + AB + BA + B^2$

(2) $(A-B)^2 = A^2 - AB - BA + B^2$

(3) $(A+B)(A-B) = A^2 - AB + BA - B^2$

(4) $(A-B)(A+B) = A^2 + AB - BA - B^2$

(5) $(A+B)^3 = A^3 + ABA + A^2B + AB^2 + BA^2 + B^2A + BAB + B^3$

27.4 Types of metrices

(a) *Transpose of a matrix :* If the rows of a matrix are written as columns and the columns as rows, then the resulting matrix is called the transpose of the given matrix.

A' denotes the transpose of matrix A

$A = [a_{ij}]_{mxn} \Rightarrow A' = [b_{ij}]_{nxm}, b_{ij} = a_{ji}$ i.e.,

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the $(i, j)^{th}$ element of $A = (j, i)^{th}$ element of A' as i and j suffixes get interchanged, since the rows become columns and vice versa.

We can at once prove the following results :

(1) $(A')' = A$ (2) $(A+B)' = A' + B'$ (3) $(AB)' = B'A'$

Proof : (1) $(A')' = A$

$$A = [a_{ij}]_{m\times n}, \quad A' = [b_{ij}]_{n\times m}$$

$$\text{where } b_{ij} = a_{ji} \qquad (1)$$

$$(A')' = [c_{ij}]_{m\times n}, \quad c_{ij} = b_{ji} = a_{ij} \quad \text{from (1)}$$

$$\therefore \quad (A')' = [a_{ij}]_{m\times n} = A$$

(2) $(A+B)' = A' + B'$

Proof : Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

$$A' = [c_{ij}]_{n\times m} \text{ where } c_{ij} = a_{ji} \qquad (1)$$

$$B' = [d_{ij}]_{n\times m} \text{ where } d_{ij} = b_{ji} \qquad (2)$$

$$A + B = [u_{ij}]_{m\times n}, \quad u_{ij} = a_{ij} + b_{ij} \qquad (3)$$

$$\therefore \quad (A+B)' = [v_{ij}]_{n\times m}, \quad v_{ij} = u_{ji} = a_{ji} + b_{ji} \quad \text{from (3)}$$

$$= c_{ij} + d_{ij} \quad \text{from (1), (2)}$$

$$\therefore \quad (A+B)' = [c_{ij} + d_{ij}]_{n\times m} = [c_{ij}]_{n\times m} + [d_{ij}]_{n\times m}$$

$$= A' + B'$$

(3) $(A\ B)' = B'\ A'$

Proof : Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$

$$\therefore \quad AB = [c_{ij}]_{m\times p}, \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj} \qquad (1)$$

$$\therefore \quad (A\,B)' = [d_{ij}]_{p\times m}, \quad d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}\ b_{ki} \quad \text{from (1)}$$

$$B' = [u_{ij}]_{p\times n}, \quad u_{ij} = b_{ji} \quad \text{and}$$

$$A' = [v_{ij}]_{n\times m}, \quad v_{ij} = a_{ji} \qquad (2)$$

$$\therefore B'A' = [w_{ij}]_{p\times m}, \quad w_{ij} = \sum_{k=1}^{n} u_{ik}\ v_{kj}$$

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$$= \sum_{k=1}^{n} b_{ki} \; a_{jk} \quad \text{from results} \quad (2)$$

$$= \sum_{k=1}^{n} a_{jk} \, b_{ki} = d_{ij} = (AB)'$$

$$\therefore B'A' = \left[d_{ij}\right]_{p \times m} = (AB)'$$

Note : (1) we can extend this result as

$(ABC)' = \{A(BC)\}' = (BC)' A' = C' B' A'$ and so on.

$(ABC \,.......H)' = H' \,........C' B' A'$

Note : (2) $(I)' = I$.

Symmetric and skew-symmetric (anti-symmetric) matrices

Def : A square matrix A is symmetric when $A = A'$

$A = \left[a_{ij}\right], \; A' = \left[a_{ji}\right]$

$\therefore A = A' \Leftrightarrow a_{ij} = a_{ji}$ is the condition for A to be symmetric.

Def : A square matrix A is skew symmetric when $A = -A'$

$\therefore$ As above $A = -A' \Leftrightarrow a_{ij} = -a_{ji}$ is the condition for A to be skew symmetric.

In such a matrix $i = j \Rightarrow a_{ii} = -a_{ii}$ i.e $a_{ii} = 0$.

The diagonal elements of a skew symmetric matrix are zeros.

Result (1): If A is any matrix, then AA' and $A'A$ are symmetric :

Proof : $(AA')' = (A')' A' = AA'$, $\therefore AA'$ is symmetric

$(A'A)' = A' (A')' = A'A$, $\therefore A'A$ is symmetric.

(2) If A and B are symmetric, then AB, if defined, is symmetric if and only if $AB = BA$

Proof : $A' = A, B' = B$ since A, B are symmetric.

$\therefore \; (AB)' = B'A' = BA = AB \Leftrightarrow AB = BA$

$\therefore \; AB$ is symmetric if $AB = BA$.

(3) If A and B are symmetric, then $A + B$ is symmetric when defined.

Proof : $A' = A, B' = B$ since A, B are symmetric

$\therefore (A + B)' = A' + B' = A + B$

$\therefore \; A + B$ is symmetric.

(4) If A and B are skew symmetric then $A + B$ is skew sysmmetric, if defined.

Proof : $(A + B)' = A' + B' = -A - B$,

$(\because A' = -A, B' = -B$ as A, B are skew symmetric)

$\therefore (A + B)' = -(A + B)$,

$\therefore A + B$ is skew symmetric.

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(5) Every square matrix can be uniquely expressed as the sum of a symmetric and skew symmetric matrix.

Proof : Let A be a square matrix, A' its transpose.

$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A') = P + Q$, taking

$P = \frac{1}{2}(A+A')$, $Q = \frac{1}{2}(A-A')$

$(A+A')' = A' + (A')' = A' + A = A + A'$,

$\therefore A + A'$ is symmetric.

$(A-A')' = A' - (A')' = A' - A = -(A-A')$

$\therefore A - A'$ is skew symmetric.

$\therefore A = P + Q$ where P is symmetric and Q skew symmetric.

Uniqueness : Let $A = R + S$, if possible where R is symmetric and S is skew symmetric.

$\therefore A' = (R+S)' = R' + S' = R - S$,

Now $A = R + S$ and $A' = R - S$ give,

$\frac{A+A'}{2} = R$ and $\frac{A-A'}{2} = S$, but $\frac{A+A'}{2} = P$ and $\frac{A-A'}{2} = Q$.

$\therefore P = R$ and $Q = S$

$\therefore A = P + Q$ is a unique expression.

Conjugate and transposed conjugate :

Let $A = [a_{ij}]_{m\times n}$, $\overline{a}_{ij}$ is the complex conjugate of the number a_{ij},

then $\overline{A} = [\overline{a_{ij}}]_{m\times n}$ is conjugate of the matrix A

$(\overline{A})' = [\overline{a_{ji}}]_{n\times m}$ is transposed conjugate of A, called some times as "tranjugate". is denoted as A^* (read A star).

We understand that $(\overline{A})' = (\overline{A'})$, since conjuagate of the transpose is transpose of the conjugate. Also $(A^*)^* = A$, since transpose of the transpose and conjugate of the conjugate are involved.

Let $A = [a_{ij}]_{m\times n}$, $A^* = [\overline{a_{ji}}]_{n\times m} = [b_{ij}]_{n\times m}$

$\therefore (A^*)^* = [\overline{b_{ji}}]_{m\times n}$. But $b_{ij} = \overline{a_{ji}}$, $b_{ji} = \overline{a_{ij}}$ and $\overline{b_{ji}} = a_{ij}$

$\therefore (A^*)^* = [a_{ij}]_{m\times n} = A$

Hermitian and Skew Hermitian matrices

Matrix A is Hermitian iff $A^* = A$ and skew Hermitian iff $A^* = -A$

$A^* = [\overline{a_{ji}}]_{n\times m}$ when $A = [a_{ij}]_{m\times n}$

$\therefore A$ is Hermitian or skew Hermitian according as

$\overline{a_{ji}} = \pm a_{ij}, \forall i$ and $\forall j \in [1, n]$.

In a Skew Hermitian matrix, the diagonal elements are given by

$\overline{a_{ii}} = -a_{ii}$ i.e., $a_{ii} + \overline{a_{ii}} = 0$

$(x+iy)+(x-iy) = 0 \Rightarrow a_{ii}$ is purely imaginary or $a_{ii} = 0$

$\therefore$ The diagonal elements of a Skew Hermitian matrix are purely imaginary or zeros.

Result (1) $\overline{AB} = \overline{A}\ \overline{B}$ where the multiplication is defined

Proof: Let $[a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$

$$\therefore AB = [c_{ij}]_{m\times p},\ c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$\overline{A} = \left[\overline{a_{ij}}\right]_{m\times n},\quad \overline{B} = \left[\overline{b_{ij}}\right]_{n\times p}$$

$$\therefore \overline{A}\,.\,\overline{B} = [d_{ij}]_{m\times p},\ d_{ij} = \sum_{k=1}^{n} \overline{a_{ik}}\ \overline{b_{kj}} \qquad (1)$$

$$\overline{AB} = \left[\overline{c_{ij}}\right]_{m\times p} \text{ where } \overline{c_{ij}} = \overline{\sum_{k=1}^{n} a_{ik}\, b_{kj}}$$

$$\therefore \overline{c_{ij}} = \sum_{k=1}^{n} \overline{a_{ik}\, b_{kj}} \quad [\because \text{conjugate of a sum} = \text{sum of conjugates}]$$

$$= \sum_{k=1}^{n} \overline{a_{ik}}\ \overline{b_{kj}} \quad \left[\because \overline{z_1 z_2} = \overline{z_1}\,.\,\overline{z_2}\right]$$

$\therefore \overline{c_{ij}} = d_{ij}$ using (1) and

$$\therefore \overline{AB} = [d_{ij}]_{m\times p} = \overline{A}\,.\,\overline{B}$$

Result (2) $(kA)^* = \overline{k}A^*$, where k is any constant.

Proof: $(kA)^* = (\overline{kA})' = \left[\overline{k\,a_{ij}}\right]' \quad \because A = [a_{ij}]$

$$= \left[\overline{k\,a_{ji}}\right] = \overline{k}\ \overline{A}' = \overline{k}A^*$$

Result (3) A is Hermitian $\Rightarrow iA$ is skew Hermitian

A is skew Hermitian $\Rightarrow iA$ is Hermitian,

Proof: Case (1) A Hermitian $\Rightarrow A^* = A$, data.

$(iA)^* = \overline{i}A^* = -iA^* = -iA$

$\therefore iA$ is skew Hermitian

Case (2) A skew Hermitian $\Rightarrow A^* = -A$, data

Now $(iA)^* = \overline{i}A^* = -i(-A) = iA$

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Proof : $(AB)^* = (\overline{AB})' = \overline{B'A'} = \overline{B'}\ \overline{A'} = B^* A^*$

we can genrealise this result as

$(ABC \ldots\ldots H)^* = H^* \ldots\ldots C^* B^* A^*$

Result (5) A is Hermitian $\Rightarrow A^n$ is Hermitian, $n \in Z_+$

(Positive integral powers of Hermitian matrices are Hermitians)

Proof : $A^* = A$, data

$(A^n)^* = (AAA \ldots\ldots A)^* = A^* \ldots\ldots A^* A^* A^*$

$= A \ldots\ldots AAA = A^n$

$\therefore A^n$ is Hermitian.

Result (6) If A is Hermitian or Skew Hermitian

then $B^* AB$ is Hermitian or Skew Hermitian.

Proof : Case (1) Let A be Skew Hermitian, $A^* = -A$.

Now $(B^* AB)^* = B^* A^* (B^*)^* = B^* A^* B = B^* (-A)B$

$= -(B^* AB)$

$\therefore B^* AB$ is skew Hermitian.

Case (2) Let A be Hermitian, proof similar to case (1)

Result (7) Every square matrix can be uniquely represented as the sum of a Hermitian and a Skew Hermitian matrices.

Proof : $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$

$= P + Q$, where $P = \frac{1}{2}(A + A^*)$ and

$Q = \frac{1}{2}(A - A^*)$

$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

$\therefore (A + A^*)$ is Hermitian.

$(A - A^*)^* = A^* - (A^*)^* = A^* - A = -(A - A^*)$,

$\therefore A - A^*$ is Skew Hermitian

$\therefore A = P + Q$ Where P is Hermitian and Q is Skew Hermitian.

Uniqueness : Let $A = R + S$ as above, if possible

$\therefore A^* = R^* + S^* = R - S \qquad \because R^* = R$ and $S^* = -S$

$A = R + S$ and $A^* = R - S$ give

$R = \frac{A + A^*}{2}, \quad S = \frac{A - A^*}{2}$

i.e., $R = P,\ S = Q$

$\therefore A = P + Q$ is a unique expression.

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i.e., $R = P,\ S = Q$

$\therefore\ A = P + Q$ is a unique expression.

Result (8) Every square matrix A can be uniquely expressed as $P + iQ$ where P and Q are Hermitians.

Proof :
$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$
$$= \frac{1}{2}(A + A^*) + i\,\frac{1}{2i}\ (A - A^*)$$
$$= P + iQ\ ;\quad P = \frac{1}{2}(A + A^*)\ \text{ and}$$
$$Q = \frac{1}{2i}\ (A - A^*)$$

$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

$\therefore A + A^*$ is Hermitian. $\therefore P$ is Hermitian

$$Q^* = \left\{ \frac{1}{2i}(A - A^*) \right\}^* = \left(\overline{\frac{1}{2i}}\right)\ (A - A^*)^*$$
$$= \frac{1}{2\,\overline{i}}(A^* - A) = -\ \frac{1}{2i}\ (A^* - A) = Q$$

$\therefore Q$ is Hermitian.

$\therefore A = P + iQ$ as required.

Let $A = R + iS$ as above. If possible, R, S are Hermitians

$A^* = (R + iS)^* = R^* + \overline{i}\,S^* = R - iS$

$$\therefore\quad \frac{A + A^*}{2} = R\ ,\ \frac{A - A^*}{2i} = S$$

$\therefore R = P$ and $S = Q$

$\therefore\ A = P + iQ$ is a unique expression.

Unitary matrices

A square matrix A is unitary iff $A^* A = I = AA^*$.

When A is a real, $\overline{A} = A$ and $A^* = \overline{(A)}\,' = A'$

$\therefore$ When A is defined over (R , + , .),

A is unitary $\Leftrightarrow A' A = I = A A'$.

Orthogonal matrices

A square matrix A is orthogonal iff

$$A' A = I = A A'$$

Result (1) If A, B are orthogonal matrices, the AB and BA are orthogonal when defined.

Proof : $AA' = I = A'A,\ \ BB' = I = B'B$, (data)

$$(AB)(AB)' = (AB)(B'A')$$
$$= A(BB')\ A' = AIA' = AA' = I$$

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Result (2) If A is orthogonal then A' and A^{-1} are orthogonal.

Proof : $AA' = I = A'A$ (data)

$$A'(A')' = A'A = I = AA' = (A')'A'$$

$\therefore A'$ is orthogonal.

$(A^{-1})(A^{-1})' = (A'A)^{-1} = (I)^{-1} = I$

$\therefore A^{-1}$ is orthogonal. (Ref : Next chapter, results on A^{-1})

Result (3) When A is orthogonal $|A| = \pm 1$

Proof : A determines a deteminant $|A|$

But in determinants $|A| = |A'|$,

$\therefore |AA'| = |A|\,|A'| = |A|^2 = |I| = 1$

$$\therefore |A| = \pm 1$$

Note (1) A is a proper matrix when $|A| = 1$

(2) $A' = A^{-1} \Leftrightarrow AA' = I = A'A$; since

$AA^{-1} = I = A^{-1}A$; (ref. next unit)

$\therefore A' = A^{-1} \Leftrightarrow A$ is orthogonal.

Involutary matrix

A square matrix A is involutary when $A^2 = I$

i.e., $(A - I)(A + I) = \mathbf{0}$

Nilpotent matrices

If A is a square matrix so that $A^n = \mathbf{0}$, where n is a positive integer, then A is nilpotent.

If n is the least positive integer for which $A^n = \mathbf{0}$, then A is nilpotent and of index n.

Idempotent matrices

If a square matrix A is such that $A^2 = A$, then A is called an idempotent matrix.

If $A^2 = A$, and further $A' = A$, then A is idempotent and symmetric.

Def : Matrix A is singular iff $|A| = 0$.

Result (1) Every non-singular idempotent matrix is an identity matrix.

Proof : A is non-singular, $|A| \neq 0$.

Then A^{-1} exists and $A^{-1}A = I = AA^{-1}$ (Ref : next Unit)

A is idempotent

$$\therefore AA = A^2 = A$$

$\therefore A^{-1}(AA) = A^{-1}A = I,$

$(A^{-1}A)A = I$, i.e., $IA = I$ and $A = I$

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Result (2) If A, B are idempotent, then AB is idempotent if $AB = BA$.

Proof : $A^2 = A,\ B^2 = B,\ AB = BA = 0$ (data)

$$(AB)^2 = (AB)(AB) = A(BA)B$$

$$= A(AB)B = (AA)(BB) = A^2B^2 = AB$$

$\therefore AB$ is idempotent, if $AB = BA$.

Result (3) If A and B are idempotent, then $A + B$ will be idempotent iff $AB = BA = \mathbf{0}$.

Proof : Case (1) $A^2 = A, B^2 = B,\ AB = BA = \mathbf{0}$ (data)

$$(A+B)^2 = A^2 + AB + BA + B^2$$

$$= A^2 + B^2 \text{ if } AB = BA = \mathbf{0}$$

$$= A + B, \qquad \therefore A + B \text{ is idempotent}$$

Case (2) Let $A + B, A, B$ be idempotent.

$$\therefore (A+B)^2 = A + B = A^2 + B^2$$

$$\text{But } (A+B)^2 = A^2 + AB + BA + B^2$$

$\therefore$ We have $AB + BA = \mathbf{0},\ AB = -BA$

$$A^2B = A(AB) = A(-BA) = (-AB)A = (BA)A = BA^2$$

But $A^2 = A$

$$\therefore A^2B = BA^2 \Rightarrow AB = BA$$

$$AB = BA = \mathbf{0} \qquad \because AB + BA = \mathbf{0}$$

Examples

1. If $\begin{bmatrix} x+3 & x+2y \\ z-1 & 4\omega-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2\omega \end{bmatrix}$ find the values of x, y, z, ω

Sol : From the definition of equality of matrices

$$x + 3 = 0,\ x + 2y = -7,\ z - 1 = 3,\ 4\omega - 6 = 2\omega$$

i.e., $x = -3,\ y = -2,\ z = 4,\ \omega = 3$

2. If $A = \begin{bmatrix} 0 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 4 & 1 \end{bmatrix}$

find the matrix equivalent to $2A - 3B$.

Sol :

$$2A = \begin{bmatrix} 0 & 4 & 6 \\ 6 & 10 & 14 \end{bmatrix}$$

$$3B = \begin{bmatrix} 3 & 9 & 21 \\ 9 & 12 & 3 \end{bmatrix}$$

$$\therefore 2A - 3B = \begin{bmatrix} 0-3 & 4-9 & 6-21 \\ 6-9 & 10-12 & 14-3 \end{bmatrix} = \begin{bmatrix} -3 & -5 & -5 \\ -3 & -2 & 11 \end{bmatrix}$$

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3. Find the product AB where

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

Sol : A is of 3 x 3, B is of 3 x 3 orders

$\therefore AB$ is also of 3 x 3 order

$$AB = \begin{bmatrix} 0.a^2+c.ab-b.ac & 0.ab+c.b^2-b(bc) & 0.ac+c.bc-b.c^2 \\ -c.a^2+0.ab+a.ac & -c.ab+0.b^2+a(bc) & -c.ac+0.bc+a\,c^2 \\ b.a^2-a.ab+0.ac & b.ab-a.b^2+0.bc & b.ac-a.bc+0.c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

Note : $AB = \mathbf{0}$, but $A \neq \mathbf{0}$, $B \neq \mathbf{0}$ in the above.

4. Find the product AB where

$$A = \begin{bmatrix} 4 & 2 & -1 & 2 \\ 3 & -7 & 1 & -8 \\ 2 & 4 & -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ 1 & 5 \\ 3 & 1 \end{bmatrix}$$

Sol : A is of 3 x 4 and B is of 4 x 2 orders.

$\therefore AB$ is of 3 x 2 order.

AB takes the form $\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$

$c_{11} = 4(2) + 2(-1) + (-1)(1) + 2(3) = 8 - 6 - 1 + 6 = 7$

$c_{12} = 4(3) + 2(0) + (-1)5 + 2.1 = 12 + 0 - 5 + 2 = 9$

$c_{21} \Rightarrow$ 2nd row of A x 1st column of $B = 4$

$c_{22} \Rightarrow$ 2nd row of A x 2nd column of $B = 6$

$c_{31} = -8$, $c_{32} = -8$

$$\therefore AB = \begin{bmatrix} 7 & 9 \\ 4 & 6 \\ -8 & -8 \end{bmatrix}$$

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5. Show that $\begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ is Skew Hermitian

Sol.: Let A be the given matrix; we will prove $A^* = -A$.

$$\overline{A} = \begin{bmatrix} -i & 3-2i & -2+i \\ -3-2i & 0 & 3+4i \\ 2+i & -3+4i & 2i \end{bmatrix}$$

$$A^* = \left(\overline{A}\right) = \begin{bmatrix} -i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix} = -A$$

$\therefore$ A is Skew Hermitian.

6. Show that the matrix

$$A = \begin{bmatrix} 1/6 & -2/6 & 1/6 \\ -2/6 & 4/6 & -2/6 \\ 1/6 & -2/6 & 1/6 \end{bmatrix}$$ is Symmetric and idempotent.

Sol.: To show $A^2 = A$ where A is the given matrix clearly $A' = A$,

$\therefore$ A is Symmetric

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$$A^2 = A.A = \frac{1}{36}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 6 & -12 & 6 \\ -12 & 24 & -12 \\ 6 & -12 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$\therefore$ $A^2 = A$ and A is idempotent.

7. Show that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent and find its index.

Sol. : To find n so that $A^n = \mathbf{0}$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ A^n = \mathbf{0} \Rightarrow A$ is nilpotent.

The least positive integral power n for which $A^n = \mathbf{0}$ is given by $n = 3$.

$\therefore$ Index of $A = 3$.

8. Show that $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutary.

Sol.: To show $A^2 = \mathrm{I}$

$$A^2 = A\,.\,A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$$

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$A^2 = \mathrm{I}$, $\therefore\ A$ is involutary.

9. If (l_r, m_r, n_r), r = 1, 2, 3 be three orthonormal unit vectors, then show that

$$\mathrm{A} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \text{ is orthogonal}$$

Sol :

Let $\hat{\mathbf{a}} = (l_1, m_1, n_1)$, $\hat{\mathbf{b}} = (l_2, m_2, n_2)$, $= \hat{\mathbf{c}}\ (l_3, m_3, n_3)$

be the vectors referred to a rectangular *OXYZ*

Taking $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ as axes of reference at O,

$$\left.\begin{array}{l} \text{unit vector } \mathbf{i} \text{ along } \overline{OX} = (l_1, l_2, l_3) \\ \text{unit vector } \mathbf{j} \text{ along } \overline{OY} = (m_1, m_2, m_3) \\ \text{unit vector } \mathbf{k} \text{ along } \overline{OZ} = (n_1, n_2, n_3) \end{array}\right\} \begin{array}{l} \text{Ref:} \\ \text{co-ordinate} \\ \text{Geometry} \end{array}$$

$|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{c}}| = |\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$ and $\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} = \hat{\mathbf{b}}\,.\,\hat{\mathbf{c}} = \hat{\mathbf{c}}\,.\,\hat{\mathbf{a}} = \mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k}. = \mathbf{k}.\mathbf{i} = 0$

$l_1^2 + m_1^2 + n_1^2 = 1,\ l_1^2 + l_2^2 + l_3^2 = 1$ etc.

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0, \; l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$ etc

$$\text{Now } AA' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1,$$

$\therefore A$ is orthogonal, we could also show that $A'A = I$ as above.

10. If A is a real Skew Symmetric matrix so that $A^2 + I = 0$, show that A is orthogonal and is of even order.

Sol : A is Skew Symmetric, $\therefore = A' = -A$

$$A^2 + I = 0 \Rightarrow AA = -I \Rightarrow AA' = I.$$

$\therefore A$ is orthogonal.

$$|AA'| = |A|\,|A'| = |I| = 1 \text{ and } |A| = |A'|$$

$$\therefore |A|^2 = 1, \; |A| \neq 0$$

$\therefore$ $|A|$ is not of odd order,

$\because$ a Skew Symmetric determinant of odd order vanishes.

$\therefore$ $|A|$ and hence A must be of even order.

Aliter : $A' = -A \quad \therefore \; |A'| = |-A| = (-1)^n \; |A|$

on taking out -1 from each of n rows of $|A|$.

$$\text{But } A' = -A$$

$\therefore$ $|A'| = (-1)^n |A|$ and $|A| \neq 0 \Rightarrow n$ is even.

SAQ 1 : Express $\begin{bmatrix} 6 & 8 & -5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrices.

SAQ 2 : Show that $\frac{1}{2}\begin{bmatrix} 1+i & -(1-i) \\ 1+i & 1-i \end{bmatrix}$ is unitary.

27.5 Summary

A matrix is a rectangular arrangement within brackets of members of a field in rows and columns, the rows being in order and each row consisting of an ordered set of members of the field.

Two matrices of the same order are equal iff the corresponding elements of them are the same.

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AB and BA need not be equal even if both exists. i.e., the matrix multiplication is not commutative.

If the rows and columns of a matrix (A) are interchanged, the resultant matrix (A') is called the tranpose of the original matrix.

If $A' = A$, then we say that A is symmetric and if $A' = -A$, A is called skew symmetric. Every square matrix can be uniquely expressed as the sum of a symmetric and skew symmetric matrices. If the transposed conjugate of a matrix is same as A; then A is called Hermitian matrix.

i.e, $\overline{A'} = A$ or $A^* = A$ and

if $A^* = -A$, A is called Skew Hermitian.

The diagonal elements of a Skew–Hermitian matrix are purely imaginary or zeros.

Every square matrix can be uniquely expressed as the sum of a Hermitian and Skew-Hermitian matrices.

A square matrix A is said to be orthogonal if $A' = A$.

A square matrix A is involutary when $A^2 = I$ and if a square matrix A is such that $A^n = 0$, n is a positive integer, then A is nilpotent and if $A^2 = A$, then A is called an indempotent matrix.

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27.6 Sample Examination Questions

(1) If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$

Evaluate $3A - 4B$.

(2) If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

show that $AB \neq BA$ by finding the products AB, BA.

(3) In question (2) above, verify that $(AB)' = B' A'$

(4) If $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$

find AB. Does BA exist?

(5) Show that

$$[x\ y\ z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy\right]$$

(6) If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, find the matrix given by $(A - 2I) . (A - 3I)$.

(7) If A and B are Symmetric matrices of the same order, show that $AB + BA$ is Symmetric and that $AB - BA$ is Skew Symmetric.

(8) Express $\begin{bmatrix} 4 & 6 & 8 \\ 2 & 3 & 4 \\ 2 & -2 & 2 \end{bmatrix}$ as the sum of a Symmetric and a Skew Symmetric matrix and check your result.

(9) If $B = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix}$, show that (1) iB is Hermitian (2) $\overline{B}$ is Skew Hermitian.

(10) If A and B are Hermitian so that AB and BA are defined, prove that

(1) $AB + BA$ is Hermitian,

(2) $AB - BA$ is Skew Hermitian

(11) Show that $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal

(12) If A is a square matrix and $A - \frac{1}{2}I$ and $A + \frac{1}{2}I$ are orthogonal, prove that $A^2 = -\frac{3}{4}I$. Deduce that A is of even order.

(13) Show that $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is nilpotent and find its index.

(14) Show that $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

(15) Show that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ is involutary.

(16) Show that $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

(17) If A is a unitary matrix, show that A' and A^{-1} are unitary.

(18) If A is Hermitian and B is any matrix show that $B^* AB$ is Hermitian, when defined.

(19) If A is Symmetric and B is any matrix, show that $B'AB$ is Symmetric when defined.

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Answers

(1) $\begin{bmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{bmatrix}$ (4) $\begin{bmatrix} 11 & 9 & 11 & 5 \\ 14 & 4 & 10 & 8 \end{bmatrix}$, No.

(6) 0

27.7 Answers to SAQ's

SAQ 1 : $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}; A' = \begin{bmatrix} 6 & 4 & 9 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}; A + A' = \begin{bmatrix} 12 & 12 & 14 \\ 12 & 4 & 10 \\ 14 & 10 & 2 \end{bmatrix}$

$$\frac{A + A'}{2} = \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix} \text{ and } \frac{A - A'}{2} = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}$$

$A = \frac{A + A'}{2} + \frac{A - A'}{2}$, where $\frac{A + A'}{2}$ is symmetric and $\frac{A - A'}{2}$ is skew symmetric.

SAQ 2 : You have to show that $A^*A = A$.

$$\overline{A} = \frac{1}{2}\begin{bmatrix} 1-i & -(1+i) \\ 1-i & 1+i \end{bmatrix}; A^* = \overline{A'} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -(1+i) & 1+i \end{bmatrix}$$

$$\therefore A^*A = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$\therefore A$ is unitary.

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Unit – 28 : Adjoint and Inverse of a Square Matrix

28.0 Contents

28.1 Aims and Objectives

After going through this unit you will able to find the inverse of a given matrix using

(i) Adjoint of the matrix and

(ii) Elementary operations

28.2 Introduction

We can represent a given system of linear equations by a matrix equation of the form $AX = B$. The matrix inverse can be useful in solving the given system of equations. Given a system of equations of the form $AX = B$ where A is a square matrix, both sides of the matrix equation may be multiplied by A^{-1} (if it exists) yielding $X = A^{-1}B$. If A^{-1} does not exist there is either no solution or an infinite number of solutions to the system of equations.

In this unit you are going to learn two methods of evaluating inverse of a given matrix, one using the adjoint of a matrix, the other using elementary transformations.

28.3 Adjoint of a Matrix

Def.: If $A = [a_{ij}]$ is a square matrix of any order and A_{ij}, denotes the co-factor of a_{ij} in $|A|$, then the transpose of the matrix $[A_{ij}]$ is called the adjoint of A.

i.e., adj A = Transpose of the co-factor matrix of A.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

Matrix of co-factors of the elements of A is

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

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Transpose of this matrix,

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Theorem 1: If A is a square matrix then $A\,(\text{adj } A) = (\text{adj } A)A = |A|\ \mathrm{I}$, where I is the unit matrix of the same order as A.

Proof. : $\quad A = [a_{ij}]_{n\times n}, \qquad \text{adj } A = [B_{ij}]_{n\times n}$ where $B_{ij} = A_{ji}$

$$\therefore\ A\,(\text{adj } A) = [c_{ij}]_{n\times n} \quad \text{where } c_{ij} = \sum_{k=1}^{n} a_{ik} B_{kj}$$

$$\therefore\ c_{ij} = \sum_{k=1}^{n} a_{ik} A_{jk} \qquad \because\ B_{ij} = A_{ji}$$

$$= a_{i1} A_{j1} + a_{i2} A_{j2} + \dots + a_{in} A_{jn}$$

$$j = i \Rightarrow c_{ii} = \sum_{k=1}^{n} a_{ik} A_{ik} = |A|$$

$$j \neq i \Rightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} A_{jk} = 0$$

c_{ii} are the diagonal elements of $A(\text{adj } A)$ and $i \neq j$, c_{ij} are the non-diagonal elements.

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$$A\,(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}_{n\times n} = |A|\ \mathrm{I}$$

Similarly, we get $(\text{adj } A)\,A = |A|\ \mathrm{I}$.

Note: $|A\,(\text{adj } A)| = |A|\ |\text{adj } A| = |A|^n\ |I|$, since $|A|$ is a factor n times, each once from every row, in the product determinant $|A\,(\text{adj } A)|$.

$|I| = 1$ and we get $|A|\ |\text{adj } A| = |A|^n$

which gives $|\text{adj } A| = |A|^{n-1}$ when $|A| \neq 0$

28.4 Singular and non-singular square matrices

A square matrix A is singular if its detrminant $|A| = 0$ and non-singular if $|A| \neq 0$.

Result 1: If A and B are square matrices of the same order,

$$|AB| = |A|\,|B|$$

Proof: $\quad A = [a_{ij}]_{n\times n}, \qquad B = [b_{ij}]_{n\times n} \Rightarrow$

$$AB = [c_{ij}]_{nxn} \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$|AB| = |c_{ij}|$ and $|A|\,|B|$ multiplied as in matrices give the same $|c_{ij}|$

$$\therefore\ |AB| = |A|\,|B|$$

Properties of adjoint matrix :

(a) If A, B are nxn non-singular matrices, then

$$\text{adj}\,(AB) = \text{adj}\,B\,.\,\text{adj}\,A.$$

Proof: We know $AB\,(\text{adj}\,AB) = |AB|\ I_n$ (1)

consider $AB\,(\text{adj}\,B\,.\,\text{adj}\,A) = A\,(B\,.\,\text{adj}\,B)\,\text{adj}\,A$

$$= A\,(|B|\ I_n)\,\text{adj}\,A$$

$$= (A\ I_n)\,\text{adj}\,A\,.\,|B|$$

$$= (A\,\text{adj}\,A)\,|B| = |A|\ I_n\,|B| \quad (2)$$

from (1) & (2) we get $\text{adj}\ AB = \text{adj}\,B\,.\,\text{adj}\,A.$

(b) $\text{adj}\,(A') = (\text{adj}\,A)'$, for any square matrix A.

Proof: Let $A = [a_{ij}]_{nXn}$, $A' = [a_{ji}]_{nXn}$

co-factor matrix of $A' = [A_{ji}]_{nXn}$ where A_{ji} is the co-factor of a_{ji} in $|A|$,

$$\therefore\ \text{adj}\,A' = [A_{ij}]_{nXn} \quad (1)$$

$$(\text{adj}\,A) = [A_{ji}]'_{nXn} \text{ the transpose of the co-factor matrix.}$$

$$= [A_{ij}]_{nXn}$$

$$\therefore\ (\text{adj}\ A)' = [A_{ij}]_{nXn} \quad (2)$$

$$\therefore\ (1), (2) \Rightarrow \text{adj}\,(A') = (\text{adj}\,A)'$$

(c) In any square matrix A, $\text{adj}\,(A^*) = (\text{adj}\,A)^*$

Proof: Let $A = [a_{ij}]_{nXn}$

$$\therefore\ \overline{A} = [\overline{a_{ij}}]_{nXn}$$

$$\therefore\ A^* = [\overline{A}]' = [\overline{a_{ji}}]_{nXn},\ \text{adj}\,[A^*] = [\overline{A_{ij}}]_{nXn} \text{ where}$$

$[A_{ij}]_{nXn}$ is co-factor of a_{ij} in $|A|$ (1)

But $\text{adj}\ A = [A_{ji}]_{nXn}$, $\overline{\text{adj}\ A} = [\overline{A_{ji}}]_{nXn}$

$$\therefore\ (\text{adj}\,A)^* = \overline{(\text{adj}\ A)}' = [A_{ij}]_{nXn} \quad (2)$$

$(1), (2) \Rightarrow \text{adj}\,(A^*) = (\text{adj}\,A)^*$

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(d) If A is an $n \times n$ square matrix and $|A| \neq 0$

$$\text{adj}\,(\text{adj}\,A) = |A|^{n-2} A$$

Proof: Put $B = \text{adj}\,A$

$$\therefore \text{adj}\,(\text{adj}\,A) = \text{adj}\,B$$

we know $B\,(\text{adj}\,B) = |B|\,I_n$

$$|B| = |\text{adj}\,A| = |A|^{n-1}$$

$\therefore B\,(\text{adj}\,B) = |A|^{n-1}\,I_n$ and pre-multiplying by A

$$AB\,(\text{adj}\,B) = |A|^{n-1} A\,I_n$$

But $AB = A\,(\text{adj}\,A) = |A|\,I_n$

$\therefore$ we have got $|A|\,I_n\,(\text{adj}\,B) = |A|^{n-1} A\,I_n$ and $|A| \neq 0$

$$\therefore I_n\,(\text{adj}\,B) = |A|^{n-2} A\,I_n$$

i.e., $\text{adj}\,B = |A|^{n-2} A$

$$\therefore \text{adj}\,(\text{adj}\,A) = |A|^{n-2} A.$$

(e) If A is a Symmetric matrix, adj A is Symmetric.

If A is Hermitian, adj A is Hermitian.

Proof: Case (1): Let A be Symmetric $\Rightarrow A' = A$

$\therefore (\text{adj}\,A)' = \text{adj}\,(A') = \text{adj}\,A \Rightarrow$ adj A is Symmetric.

Case (2): Let A be Hermitian, $A^* = A$.

$\therefore (\text{adj}\,A)^* = \text{adj}\,A^* = \text{adj}\,(A) \Rightarrow$ adj A is Hermitian.

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Result 2: If A is a square matrix of $n \times n$ order and $B = \text{adj}\,A$, then $|AB + k\,I_n| = (|A| + k)^n$

Proof : $AB + k\,I_n = A\,(\text{adj}\,A) + k\,I_n$

$$= |A|\,I_n + k\,I_n$$

$$= (|A| + k)\,I_n = \lambda\,I_n$$

$$\therefore |AB + k\,I_n| = |\lambda\,I_n| = \begin{vmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda \end{vmatrix}$$

$$= \lambda^n \cdot 1 = (|A| + k)^n$$

Result 3: Adj $k\,I_n = k^{n-1}\,I_n$ where k is a scalar and n is a positive integer.

Proof:

$$k\,I_n = \begin{bmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k \end{bmatrix}_{n \times n}$$

The co-factors of all diagonal elements bear the same sign. Each co-factor $= k^{n-1}$

$\therefore$ other elements are zeros.

$$\therefore \text{ Co-factor matrix of } k\,\mathrm{I}_n = \begin{bmatrix} k^{n-1} & 0 & \dots & 0 \\ 0 & k^{n-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k^{n-1} \end{bmatrix}_{n \times n}$$

$\therefore$ adj $k\,\mathrm{I}_n$ = transpose of the above $= k^{n-1}\,\mathrm{I}_n$

28.5 Inverse of a square matrix (Reciprocal matrix)

(a) *Definition :* If A is a square matrix and another matrix B exists so that $AB = BA = \mathrm{I}$ then A and B are called mutullay inverse.

$$B = A^{-1} \text{ and } A = B^{-1}$$

(b) *Existence of inverse :* The necessary and sufficient condition for the existence of the inverse of a square matrix is that the matrix is non-singular.

i.e., A^{-1} exists $\Leftrightarrow |A| \neq 0$.

Proof: Case (1) Let $|A| \neq 0$, A be an $n \times n$ matrix

$$\therefore A\,(\text{adj } A) = |A|\,\mathrm{I}_n = (\text{adj } A)\,A$$

$$|A| \neq 0 \Rightarrow \frac{A\,(\text{adj } A)}{|A|} = \mathrm{I}_n = \frac{(\text{adj } A)A}{|A|}$$

$\therefore$ There exists B such that $AB = \mathrm{I}_n = BA$

$\therefore$ $B = \dfrac{\text{adj } A}{|A|}$ is inverse of A i.e., A^{-1}

Case (2): Let there be B such that $AB = \mathrm{I} = BA$

$$\therefore |AB| = |I| = 1$$

i.e., $|A|\,|B| = 1 \Rightarrow |A| \neq 0, |B| \neq 0$

$\therefore$ A is non-singular.

(c) *Uniqueness :* The inverse of A when exists, is unique.

Proof: Let $A^{-1} = B$ and $A^{-1} = C$ if possible.

$$\therefore AB = I = BA, \quad AC = I = CA$$

$$B = BI = B(AC) = (BA)C = IC = C$$

$\therefore$ A^{-1} is unique.

(d) *Revrsal rule :* $(AB)^{-1} = B^{-1}A^{-1}$ when A, B are invertible square matrices of the same order.

Proof: $P^{-1} = Q \Leftrightarrow P\,.\,Q = \mathrm{I} = Q\,.\,P$

But $P = AB, Q = B^{-1}A^{-1}$.

$$\therefore PQ = AB\,(B^{-1}A^{-1}) = A\,(BB^{-1})\,A^{-1}$$

$$= AIA^{-1} = AA^{-1} = I$$

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$\therefore\ A^{-1} = Q$, i.e., $(AB)^{-1} = B^{-1}A^{-1}$

Note: The above rule can easily be generalised as

$$(ABC.....H)^{-1} = H^{-1} .... C^{-1}B^{-1}A^{-1}$$

(e) $(A^{-1})' = (A')^{-1}$ when A is a non-singular square matrix.

Proof: $AA^{-1} = I = A^{-1}A$

$\therefore (AA^{-1})' = I' = (A^{-1}A)' \Rightarrow (A^{-1})'A' = I = A'(A^{-1})'$

i.e., like $PQ = I = QP$

$\therefore Q^{-1} = P \quad \therefore (A')^{-1} = (A^{-1})'$

(f) A is a non-singular orthogonal matrix $\Rightarrow A^{-1}$ is orthogonal.

Proof: A is orthogonal $\Leftrightarrow AA' = I = A'A$

$$\therefore (AA')^{-1} = I^{-1} = (A'A)^{-1}$$

$$A'^{-1}(A)^{-1} = I = (A')^{-1}A^{-1} \qquad \because I^{-1} = I$$

$$A^{-1}(A^{-1})' = I = (A^{-1})'A^{-1}$$

i.e., $PP' = I = P'P \Leftrightarrow P$ is orthogonal.

(g) If A is a unitary matrix, then A^{-1} is unitary.

Proof: We will first prove $(A^*)^{-1} = (A^{-1})^*$

$$A^*(A^{-1})^* = (A^{-1}A)^* = I^* = I$$

$$\text{i.e., } PQ = I \qquad \therefore P^{-1} = Q$$

$$\therefore (A^*)^{-1} = (A^{-1})^*$$

Now A is unitary $\Leftrightarrow A^*A = I = AA^*$

$$\therefore (A^*A)^{-1} = I^{-1} = (AA^*)^{-1}$$

$$\text{i.e., } (A^*)^{-1}A^{-1} = I = A^{-1}(A^*)^{-1}$$

$$\therefore (A^{-1})^*A^{-1} = I = A^{-1}(A^{-1})^*$$

$$\text{i.e., } P^*Q = I = QP^*$$

$\therefore P$, i.e., A^{-1} is unitary.

(h) A is a Hermitian matrix $\Rightarrow A^{-1}$ is Hermitian.

Proof: Given $A^* = A$, we know $(A^*)^{-1} = (A^{-1})^*$

To show $(A^{-1})^* = A^{-1}$

$$AA^{-1} = I = A^{-1}A$$

$$\therefore (AA^{-1})^* = I^* = (A^{-1}A)^*$$

$\therefore (A^{-1})^*A^* = I = A^*(A^{-1})^*$ and $A^* = A$ gives

$(A^{-1})^*A = I = A(A^{-1})^*$ i.e., like

$BA = I = AB$ where $A^{-1} = B$

$\therefore A^{-1} = (A^{-1})^*$ and A^{-1} is Hermitian.

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(i) If A is a non-singular square matrix.

$$(A^k)^{-1} = (A^{-1})^k \quad (k \text{ is a positive integer})$$

Proof: $(A^k)^{-1} = (A \,.\, A.... A)^{-1}$

$= A^{-1}......A^{-1}\, A^{-1}\, A^{-1}$ k factors

$= (A^{-1})^k$

(j) If A is a symmetric matrix, then A^{-1} is symmetric.

Proof: A symmetric $\Rightarrow A' = A$,

we know $(A')^{-1} = (A^{-1})'$, but $A' = A \Rightarrow (A')^{-1} = A^{-1}$

$\therefore$ we have $(A^{-1})' = A^{-1}$ and A^{-1} is symmetric.

(k) If A and B are non-singular symmetric square matrices of the same order which commute then $A^{-1}B$, AB^{-1} and $A^{-1}B^{-1}$ are symmetric.

Proof: $\left.\begin{array}{l} A^{-1}, B^{-1} \text{ exist}, \; AB = BA \\ A' = A \quad B' = B \end{array}\right\}$ data

We will first show that $A^{-1}B = BA^{-1}$

$AB = BA \Rightarrow A^{-1}(AB) = A^{-1}(BA)$

$\Rightarrow (A^{-1}A)B = (A^{-1}B)A$

$\Rightarrow IB = (A^{-1}B)A$

$\therefore B = (A^{-1}B)A$

$\therefore BA^{-1} = (A^{-1}B)AA^{-1} = (A^{-1}B)\, I = A^{-1}B$

Now $(A^{-1}B)' = (BA^{-1})' = B'(A^{-1})' = B(A')^{-1} = BA^{-1} = A^{-1}B$ $\quad [\because A' = A, B' = B]$

$\therefore (A^{-1}B)' = A^{-1}B$, $\therefore A^{-1}B$ is symmetric.

But $A^{-1}B = BA^{-1} \Rightarrow BA^{-1}$ is symmetric.

$(BA^{-1})^{-1} = AB^{-1}$ is symmetric.

Also $(BA)' = A'B' = AB = BA$ from data.

$\therefore BA$ is symmetric.

$\therefore (BA)^{-1} = A^{-1}B^{-1}$ is symmetric.

(l) If $A = \text{adj } B$ and P, Q are unimodular matrices, then

adj $(Q^{-1}BP^{-1}) = PAQ$.

Proof: adj $(Q^{-1}BP^{-1})$ = adj P^{-1} adj B . adj Q^{-1} (1)

$\therefore$ Let M be a square matrix,

$$|M| \neq 0, \text{ then } M^{-1} = \frac{\text{adj } M}{|M|}$$

$$\therefore \text{ adj } M = |M| \, M^{-1}$$

$$\therefore \text{ adj } P^{-1} = |P^{-1}| \, P \text{ etc., and } |P| = |P^{-1}| = |I| = 1$$

$$\therefore (1) \Rightarrow \text{adj}\,(Q^{-1}BP^{-1}) = |P^{-1}| \, PA \, |Q^{-1}| \, Q$$

$$= \frac{1}{|P|} PA \frac{1}{|Q|} Q$$

$$= PAQ, \quad \because |P| = |Q| = 1$$

(m) If A and B are square matrices so that $AB = \mathbf{0}$, then $A = \mathbf{0}$ or $B = \mathbf{0}$ or A and B are both singular.

Proof: $AB = \mathbf{0}$, Let $|A| \neq 0$

$\therefore$ A^{-1} exists

$$A^{-1}(AB) = A^{-1}(\mathbf{0}) = \mathbf{0}$$

$$\therefore (A^{-1}A)B = \mathbf{0} \text{ i.e., } IB = B = \mathbf{0}$$

Similarly when $|B| \neq 0$, we can get $A = \mathbf{0}$.

When $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$ and $|A|$ and $|B|$ do not both vanish, and let $|A| \neq 0$. We can pre-multiply AB with A^{-1} and get $B = \mathbf{0}$, a contradiction of $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$

$\therefore$ It follows that $|A| = 0$ and $|B| = 0$

28.5.1 Inversion of a matrix, by the method of solving a system of equations

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Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ to find A^{-1}.

consider the equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = k_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = k_2$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = k_n$$

These equations take the form, $AX = B$ where A is as above, and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ \cdot\cdot \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} k_1 \\ k_2 \\ \cdot\cdot \\ \cdot\cdot \\ k_n \end{bmatrix}$$

By actually solving the equations we get, say

$$
\begin{aligned}
x_1 &= p_{11}k_1 + p_{12}k_2 + p_{13}k_3 + \ldots\ldots + p_{1n}k_n \\
x_2 &= p_{21}k_1 + p_{22}k_2 + p_{23}k_3 + \ldots\ldots + p_{2n}k_n \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ldots \quad \ldots \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ldots \quad \ldots \\
x_n &= p_{n1}k_1 + p_{n2}k_2 + p_{n3}k_3 + \ldots\ldots + p_{nn}k_n
\end{aligned}
$$

$$AX = B \Rightarrow \quad X = A^{-1}B$$

$\therefore$ The above results take the form $X = A^{-1}B$, when $|A| \neq 0$

$$X = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} B$$

i.e., $X = A^{-1}B$ and we pick up A^{-1}.

Note: This method is not practically very easy when inverses of higher order matrices are required to be found out.

28.6 Elementary operations

(a) *The operations known as elementary row operations are given below :*

1. The interchange of any two rows;

   r_{ij} denotes the interchange of the i th and the j th rows.

2. All the elements of a row are multiplied by the same constant $k \neq 0$.

   $r_i(k)$ denotes the multiplication of the i th row by $k \neq 0$.

3. To the elements of any row, adding the corresponding elements of any other row, multiplied by $k \neq 0$.

$r_{ij}(k) \Rightarrow$ To the elements of i th row, k times those of the j th row are added.

Note: Similarly we can define the elementary column operations as :

C_{ij} $\Rightarrow$ inter-change of the i th and j th columns.

$C_i(k)$ $\Rightarrow$ i th column is multiplied by $k \neq 0$

$C_{ij}(k)$ $\Rightarrow$ To the elements of the i th column, k times those of the j th column are added, in order.

(b) *Equivalent Matrices :* Matrix A is equivalent to matrix B i.e., $A \sim B$ if B is derived from A by a finite number of elementary row or column operations. A and B are of the same order.

(c) *Elementary Matrices :* Any matrix derived from the identity matrix by means of elementary row (column) operations is called an elementary row (column) matrix.

Notation

(1) E_{ij} is the elementary row matrix derived from I, by the operation r_{ij} .

(2) $E_i(k)$ is the elementary row matrix resulting from the operation $r_i(k)$ in I.

(3) $E_{ij}(k)$ is the elementary row matrix resulting from the operation $r_{ij}(k)$ in I.

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(4) $E_{ij}'(k)$ is transpose of $E_{ij}(k)$.

$E_{ij}'(k)$ is directly derived from I, by the operation $C_{ij}(k)$.

(d) Every elementary row (column) operation in matrix A can be interpreted as the product of A and the corresponding elementary row (column) matrix.

Explanation:

Pre-multiplication by E_{ij} $\Leftrightarrow$ row operations in A..

Post-multiplication by E'_{ij} $\Leftrightarrow$ column operations in A..

(1) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}$

Inter-change of 2nd and 3rd rows in this :

$$\text{Take } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ so that } I_3A \text{ is defined.}$$

$$E_{23}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

$\therefore$ $E_{23}A$ is equivaleInt to r_{23} in A.

Interchange of 2nd and 3rd columns of A :

Take I_4 so that AI_4 may be defined.

$$AE_{23} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{13} & a_{12} & a_{14} \\ a_{21} & a_{23} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{32} & a_{34} \end{bmatrix}$$

Similarly all other operations can be checked up.

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(e) The inverse of an elementary matrix is an elementary matrix.

Proof: (1) $E_{ij} \Rightarrow$ Interchange of i th and j th rows in I_n

$\therefore$ Another interchange of the i th and j th rows will give I_n from E_{ij}. From (d), this is given by $E_{ij} \cdot E_{ij}$.

$$\therefore E_{ij} \cdot E_{ij} = I_n$$

$$\therefore E_{ij}^{-1} = E_{ij} \qquad (1)$$

(2) $E_i(k) \Rightarrow$ multiplication of the i th row, by $k\ (\neq 0)$.

Multiplying i th row again by $\frac{1}{k}$ gives I_n from $E_i(k)$. From (d) this is $E_i\left(\frac{1}{k}\right)E_i(k) = I_n$

$$\therefore \left\{E_i(k)\right\}^{-1} = E_i\left(\frac{1}{k}\right) \qquad (2)$$

(3) $E_{ij}(k) \Rightarrow$ Adding to the i th row, k times the j th row.

If $-k$ times the j th row is again added to the i th row, we get I_n from the above.

$$\therefore E_{ij}(-k) \cdot E_{ij}(k) = I_n$$

$$\therefore \left\{E_{ij}(k)\right\}^{-1} = E_{ij}(-k) \qquad (3)$$

(1), (2), (3) establish the result.

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(f) If matrix $A \sim B$, then there exist non-singular matrices R and C so that $B = RAC$.

Proof: B is derived from A by means of elementary row and column operations. An elementary row operation $\Rightarrow$ pre-multiplication of A by an elementary matrix and a corresponding column operation is effected by post-multiplication of A by an elementary column matrix.

If B is derived from A by m row operations and n column operations, then

$$B = (E_m E_{m-1} \ldots E_2 E_1) A (C_1 C_2 \ldots C_n)$$

$$\therefore B = RAC \qquad R = E_m E_{m-1} \ldots E_1$$

$$C = C_1 C_2 \ldots C_n$$

Note: (1) $R^{-1} B C^{-1} = A$, since $B = RAC$.

Note: (2) Any non-singular square matrix can be expressed as a product of elementary matrices.

Proof: Let A be an $n \times n$ square matrix.

A non-singular.

$\therefore A \sim I_n$, $\therefore$ There exists R and C non-singular matrices so that $A = R\, I_n\, C$

$$= (E_m E_{m-1} \ldots E_1) I_n (C_1 C_2 \ldots C_n)$$

$$= E_m E_{m-1} \ldots E_1 \; C_1 C_2 \ldots C_n$$

$$= \text{product of elementary matrices}$$

(g) To find A^{-1}, we reduce A to the form of I_n, by means of row operations i.e.,

$$(E_m E_{m-1} \ldots E_2 E_1) A = I_n$$

$$\text{i.e.,} \quad = EA = I_n$$

$$\therefore E = A^{-1}$$

Examples

1. Find the inverse of

$$\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix} \quad \text{where } a^2+b^2+c^2+d^2 = 1$$

Sol: The co-factors of respective elements in the given matrix are :

$a-ib$ for $a+ib$, $\quad -(-c+id)$ for $c+id$,

$-(c+id)$ for $(-c+id)$ and $a+ib$ for $a-ib$

$\therefore$ co-factor matrix is given by

$$A = \begin{bmatrix} a-ib & c-id \\ -c-id & a+ib \end{bmatrix}$$

$$\text{and } adj\,A = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}, \quad |A| = a^2+b^2+c^2+d^2 = 1$$

$$\therefore A^{-1} = \frac{adj\,A}{|A|} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$

(2) If ω is a cube root of unity and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ show that $B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ is the inverse of A.

Sol: ω is a cube root of unity,

$\therefore \omega^3 = 1,\ 1+\omega+\omega^2 = 0\ \ \omega^4 = \omega$ etc.

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+1+1 & 1+\omega^2+\omega & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega^3+\omega^3 & 1+\omega^2+\omega \\ 1+\omega^2+\omega & 1+\omega^4+\omega^2 & 1+\omega^3+\omega^3 \end{bmatrix}$$

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$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly we could get $BA = I$.

$\therefore AB = BA = I \Rightarrow B = A^{-1}$

3. Find the inverse of $A = \begin{bmatrix} 9 & 7 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{bmatrix}$

Sol: $|A| = 9(-2-32) - 7(14-24) + 6(28+3)$

$= -306 + 70 + 186 = -50$

The respective co-factors of elements of A are

Ist row

$\begin{vmatrix} -1 & 8 \\ 4 & 2 \end{vmatrix} = -34$ for 9

$-\begin{vmatrix} 7 & 6 \\ 4 & 2 \end{vmatrix} = 10$ for 7 $\begin{vmatrix} 7 & -1 \\ 3 & 4 \end{vmatrix} = 31$ for 6

2nd row

$-\begin{vmatrix} 7 & 6 \\ 4 & 2 \end{vmatrix} = 10$ for 7

$\begin{vmatrix} 9 & 6 \\ 3 & 2 \end{vmatrix} = 0$ for -1, $-\begin{vmatrix} 9 & 7 \\ 3 & 4 \end{vmatrix} = -5$ for 8.

3rd row

$\begin{vmatrix} 7 & 6 \\ -1 & 8 \end{vmatrix} = 62$ for 3

$-\begin{vmatrix} 9 & 6 \\ 7 & 8 \end{vmatrix} = -30$ for -4, $\begin{vmatrix} 9 & 7 \\ 7 & -1 \end{vmatrix} = -58$ for 2.

$$\therefore \text{Matrix of co-factors} = \begin{bmatrix} -34 & 10 & 31 \\ 10 & 0 & -15 \\ 62 & -30 & -58 \end{bmatrix}$$

$$\therefore adj\,A = \begin{bmatrix} -34 & 10 & 62 \\ 10 & 0 & -30 \\ 31 & -15 & -58 \end{bmatrix}$$

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$$\therefore A^{-1} = \frac{adj\,A}{|A|} = \frac{1}{50}\begin{bmatrix} -34 & 10 & 62 \\ 10 & 0 & -30 \\ 31 & -15 & -58 \end{bmatrix}$$

Note : For 4 x 4 and higher order matrices, the above method of finding the inverse will be very lengthy and inconvenient. Other methods learnt earlier may be useful.

4. By solving matrix equation find A^{-1} where

$$A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$$

Sol : Consider the equation $AX = B$ where

$$\underset{(A)}{\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}} \underset{(X)}{\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}} = \underset{(B)}{\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}}$$

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$$\therefore 2x + y - z + 2u = k_1$$

$$x + 3y + 2z - 3u = k_2$$

$$-x + 2y + z - u = k_3$$

$$2x - 3y - z + 4u = k_4$$ By solving these, we get

$$x = \frac{1}{18}[2k_1 + 5k_2 - 7k_3 + k_4]$$

$$y = \frac{1}{18}[5k_1 - k_2 + 5k_3 - 2k_4]$$

$$z = \frac{1}{18}[-7k_1 + 5k_2 + 11k_3 + 10k_4]$$

$$u = \frac{1}{18}[k_1 - 2k_2 + 10k_3 + 5k_4]$$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

$$\therefore X = A^{-1}B \text{ and } A^{-1} = \frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}$$

5. If A is square matrix of nth order,

Prove that $|adj\,(adj\,A)| = |A|^{(n-1)^2}$

Sol : Put $adj\,A = B$

$$|adj\;B| = |B|^{n-1}$$

$$\therefore |adj\,(adj\,A)| = |adj\;A|^{n-1} \text{ and } |adj\;A| = |A|^{n-1}$$

$$\therefore |adj\,(adj\,A)| = \left\{|A|^{n-1}\right\}^{n-1} = |A|^{(n-1)^2}$$

6. If $A = diag\;\left\{a_{11}, a_{22}, \ldots\ldots a_{nn}\right\}$, show that

$A^{-1} = diag\;\left\{a_{11}^{-1}, a_{22}^{-1}, \ldots. a_{nn}^{-1}\right\}$ given that

$a_{11}, a_{22}, \ldots\ldots a_{nn} \neq 0$

Sol :
$$A = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

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$|A| = a_{11}, a_{22}, \ldots\ldots a_{nn} \neq 0$, A^{-1} exists.

co-factor matrix of A, i.e.,

$$B = \begin{bmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & A_{nn} \end{bmatrix}$$

A_{11} = Co-factor of a_{11} in $|A|$

$$= a_{22}\; a_{33} \ldots\ldots a_{nn} = \frac{|A|}{a_{11}} \text{ etc.}$$

$$\therefore adj\,A = B' = \begin{bmatrix} |A|/a_{11} & 0 & \ldots & 0 \\ 0 & |A|/a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & |A|/a_{nn} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = diag\ \left\{a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1}\right\}$$

7. Find A^{-1} given $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ using elementary transformations.

Sol : Write $A = IA$. Using elementary row transformations convert this equation to the form $I = BA$ to find $B = A^{-1}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$r_{21}(-3)$ i.e., $r_2 - 3r_1 \Rightarrow$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$r_{31}(-1) \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$$r_{12}(1/2) \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

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$$r_3(-4) \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 0 & 4 & -4 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ -3 & 1 & 0 \\ 4 & 0 & -4 \end{bmatrix} A$$

$$r_{32}(1) \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} A$$

$$r_2(-1/4), r_3(-1/4) \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

$$r_{13}(-1) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

i.e., $I = BA$

$$\therefore B = A^{-1} = \begin{bmatrix} -1 & 3 & -4 \\ 3 & -1 & 0 \\ -1 & -1 & 4 \end{bmatrix} \frac{1}{4}$$

Note the method of getting zeros first below the diagonal, in the above example.

SAQ 1 : Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \text{ using adjoint.}$$

SAQ 2 : Show that a skew symmetric matrix of odd order is singular.

SAQ 3 : Use elementary operations to find A^{-1} where $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

28.7 Summary

If the given matrix is non singular, that is, its determinant is not zero, we can obtain the inverse of the matrix.

We have calculated the inverse of a matrix by using the adjoint of the matrix i.e., $A^{-1} = \dfrac{adj\ A}{Det\ A}$ and by using the elementary row (column) operations.

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28.8 Sample Examination Questions

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$, verify that $A\,(\text{adj}A) = |A|\,I = (adjA)\,A$ and obtain A^{-1}.

(2) If $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$ find the inverse of A by determining adj A

(3) Find A^{-1} by solving a matrix equation $AX = B$

where $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$

(4) Find A^{-1} using elementary transformations on A

where (i) $A = \begin{bmatrix} 1 & 1 & -2 & -4 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 3 & 6 \\ 2 & 1 & -6 & -10 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

(5) Find the reciprocal (inverse) of the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and If } A = \begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$$

then the transform SAS^{-1} is a diagonal matrix.

(6) If A is non-singular square matrix,

show that $A^{-1} = (A'A)^{-1}A' = A'(AA')^{-1}$.

Answers

(1) $\frac{1}{11}\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$ (2) $\frac{1}{25}\begin{bmatrix} 25 & -10 & -15 \\ -10 & 4 & 11 \\ -15 & 1 & 9 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$ (4) (i) $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(5) 2 diag $\{a, b, c\}$

28.9 Answers to SAQ's

SAQ 1 : The co-factor matrix is $\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$ and

the adjoint matrix is $\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

The det.of the given matrix is -1.

Hence the inverse of the given matrix is obtained by $\frac{Adj}{Det} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

SAQ 2 : Given that $A' = -A \Rightarrow |A'| = -|A|$ (since A is odd order) $\Rightarrow |A'| + |A| = 0$.

But $Det(A) = Det(A') \Rightarrow 2|A| = 0 \Rightarrow |A| = 0 \Rightarrow A$ is singular.

SAQ 3 : Write $A = IA$

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$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Interchange r_1 and r_2 i.e., r_{12} to get 1 in a_{11} place.

$$\therefore r_{12} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$r_{34}(-1) \text{ and} r_{41}(-2) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$$r_{12}(-1)\, r_{42}(-1) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$$r_{32}(1) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$$r_{43}(3) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$$r_{14}(-1)\, r_{24}(-2) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ -3 & 4 & -6 & 4 \\ 1 & 0 & 1 & -1 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

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$$r_{34}(-2) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ -3 & 4 & -6 & 4 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$$r_{23}(-2) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

$\therefore I = BA,\ B = A^{-1}$ and we have $A^{-1} = = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

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Unit – 29 : Rank and Linear Equations

29.0 Contents

29.1 Aims and objectives

After going through this unit you will be able to :

(i) Evaluate the rank of the given matrix using elementary transformations,

(ii) Reduce the given matrix into Echelon and Normal Forms of matrices,

(iii) Test the consistency of the given system of linear equations and solve it.

29.2 Rank of a Matrix

Def : A matrix A is of rank $r \Leftrightarrow$

(i) There is atleast one non-singular square submatrix of order r for A,

(ii) Every square submatrix of A having order $(r + 1)$ and those of higher orders vanish.

If we call the square submatrices of A as "minors" of A, then the order of the highest non-vanishing minor, is the rank of A.

When A is a square matrix of order n, then the difference between n and rank (A) (n – rank (A)) is called the 'nullity' of A.

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Consequences of the definition of rank

The rank of A is denoted as $\rho(A)$. We readily understand the following results from the definition of $\rho(A)$.

(1) If there is one minor of order r which does not vanish then $\rho(A) \geq r$, since the minors of higher order $r + 1$ have to be tested for vanishing according to the definition of $\rho(A)$.

(2) If all minors of order $(r + 1)$ vanish then $\rho(A) \leq r$, since minors of order r and less, have to be examined for non-vanishing, according to the definition of $\rho(A)$.

(3) The rank of a unit matrix of order n is n, from the definition of rank.

(4) The rank $\rho(A)$ of an $m \times n$ matrix, satisfies the condition $\rho(A) \leq m$, $\rho(A) \leq n$ since the minors are of orders $\leq m$ and $\leq n$.

(5) $\rho(A) = \rho(A')$, since the corresponding minors of $|A|$ and $|A'|$ are of the same value.

(6) The rank of a null matrix = 0, since any minor of any order vanishes.

(7) The rank of $|a_{ij}|$, $a_{ij} = 1$, $\forall i$ and $\forall j$ can be taken as unity.

(8) $\rho(A) = \rho(A^*) = \rho(A)$ because A and $\overline{A}$ have conjugate elements. If any minor of A reduces to $P + iQ$ the corresponding minor of $\overline{A}$ reduces to $P - iQ$ and $P + iQ = 0 \Leftrightarrow P - iQ = 0$

$$P + iQ \neq 0 \Leftrightarrow P - iQ \neq 0$$

$\therefore \rho(A) = \rho(\overline{A}), \rho(\overline{A}) = \rho(\overline{A}') = \rho(A^*)$ using result (5) above.

Examples

1. Find the rank of $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

Sol: Every minor of 3rd order is 0, because we get $r_2 \equiv r_3$ in each minor, on taking out common factor 3 from r_2 in each.

$$\begin{vmatrix} 4 & 3 \\ 12 & 3 \end{vmatrix} \neq 0, \text{ a minor of } 2^{nd} \text{ order}$$

$$\therefore \rho(A) = 2$$

2. Find the rank of $A = \begin{bmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{bmatrix}$

Sol: $|A| = -i(0 - i^2) - i(i^2 - 0) = 0$, the minor of 3rd order

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = 1 \neq 0, \text{ a minor of } 2^{nd} \text{ order}$$

$$\therefore \rho(A) = 2$$

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3. Show that the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear $\Leftrightarrow \rho(A) < 3$, where

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

Sol: Given points are collinear $\Leftrightarrow$

$$|A| = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\therefore$ all minors of 3rd oder $= 0$

If all minors of $(r+1)^{th}$ order $= 0, \rho(A) \leq r$.

$\therefore \rho(A) \leq 2,$

$\therefore$ collinearity of three points $\Leftrightarrow \rho(A) < 3$.

29.2.1 Determination of rank using elementary opeations

Theorem 1 : Elementary operations made in a matrix, do not alter the rank of a matrix.

(Two equivalent matrices have the same rank).

Proof:

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Consider any minor M_r of order r.

(1) If two rows of A are interchanged and if these rows do not appear in M_r, then $|M_r|$ is not changed.

If the interchanged rows appear in M_r, then $|M_r|$ becomes $-|M_r|$.

(2) If any rows of A is multiplied by k, then $|M_r|$ remains $|M_r|$ if the row does not appear in M_r. If this row belongs to M_r then $|M_r|$ becomes $k\,|M_r|$.

(3) If two rows of A are added, then $|M_r|$ is not changed if this new row is not in M_r. If the new row so obtained belongs to M_r, then $|M_r|$ splits into a sum of two determinants of the same order.

$\therefore$ In A if all minors of $(r+1)^{th}$ order vanish, then after the above elementary transformations also, they become zeros because

$$|M_r| = 0 \Rightarrow -|M_r| = 0 \text{ in (1) above}$$

$$\Leftrightarrow k\,|M_r| = 0 \text{ in (2) above}$$

$$\text{and } |M_r| = |M_r'| + |M_r''| = 0 + 0 \text{ in (3).}$$

$\therefore$ If $A \sim B$ and $\rho(A) = r$, all $(r+1)^{th}$ order minors of A vanish.

$\therefore$ Those of B also vanish.

$\therefore \rho(B) \leq r$.

$\therefore \rho(B) \leq \rho(A)$ (1)

But $A \sim B \Rightarrow B \sim A$

$\therefore \rho(A) \leq \rho(B)$ (2)

(1), (2) $\Rightarrow \rho(A) = \rho(B)$

Note: We can find the rank of a matrix A *by* reducing it to B, by elementary transformations.

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Examples

1. Find $\rho(A)$ where $A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$

Sol: $r_{21}(-1), r_{32}(-1), r_{43}(-1), r_{54}(-1) \Rightarrow$

$$A \sim \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$r_{32}(-1),\ r_{43}(-5),\ r_{54}(-1) \Rightarrow$

$$A \sim \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ All minors of } 3^{rd} \text{ order } = 0$$

$$\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = -1 \neq 0 \text{ a } 2^{nd} \text{ order minor}$$

$\therefore \rho(A) = 2$

2. Find the rank of the matrix A, using elementary opeations.

$$A = \begin{bmatrix} 75 & 0 & 116 & -39 & 0 \\ 171 & -69 & 402 & 123 & 45 \\ 301 & 0 & 87 & 417 & -169 \\ 114 & -46 & 268 & 82 & 30 \end{bmatrix}$$

Sol: $r_2(1/3), r_4(1/2) \Rightarrow$

$$A = \begin{bmatrix} 75 & 0 & 116 & -39 & 0 \\ 57 & -23 & 134 & 41 & 15 \\ 301 & 0 & 87 & 417 & -169 \\ 57 & -23 & 134 & 41 & 15 \end{bmatrix}$$

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$$r_{42}(-1) \Rightarrow A \sim \begin{bmatrix} 75 & 0 & 116 & -39 & 0 \\ 57 & -23 & 134 & 41 & 15 \\ 301 & 0 & 87 & 417 & -169 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Every 4^{th} order minor in this = 0.

$$\text{A third order minor} = \begin{vmatrix} 75 & 0 & 0 \\ 57 & -23 & 15 \\ 301 & 0 & -165 \end{vmatrix}$$

$$= 75(-23)(-169) \neq 0$$

$$\therefore \rho(A) = 3$$

29.3 Echelon form of a matrix

A row in which all elements are zeros, is called a zero-row, (linearly independent row vector). Other rows in which all elements are not zeros are called non-zero rows. (linearly dependent row vectors).

Similarly we can define zero columns and non-zero columns.

Matrix A is in echelon form $\Leftrightarrow$

(1) The zero rows come only after the non-zero rows.

(2) The number of zeros preceding the first non-zero element in any non-zero row is less than that in the next row.

(3) The first non-zero element in every non-zero row should be unity.

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Example

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in Echelon form}$$

In Echelon form the number of non-zero rows (The number of linearly dependent row vectors) is the rank of the matrix, because, if the number of such rows is r, then every minor of order $(r+1)$ vanishes and atleast one minor of order r does not vanish.

Examples

(1) Find the rank of A by reducing it to echelon form

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

Sol:

$$r_{42}(-3) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

$$r_4(-1/2) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$r_{31}\ (-2), r_{21}\ (-1) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$r_{42}\ (-1) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{, Echelon form}$$

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$\therefore \rho(A)$ = number of non-zero rows = 3

Ex. 2: Reduce the matrix A to echelon form and find its rank where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Sol:

$$r_{12} \Rightarrow A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$r_{41}\ (-1), r_{31}\ (-3) \Rightarrow A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$r_{42}(-1), r_{32}(-1) \Rightarrow A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = number of non-zero rows in this = 2.

29.4 Normal form (Canonical form)

The following forms of matrices are called the normal forms.

(1)
$$\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ — & — & — & — & — \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

where I_r is a unit matrix of order r ($r < m$, $r < n$).

The null matrix below I_r is of order $(m-r) \times r$

The null matrix to the right of I_r is of order $r \times (n-r)$

The null matrix across the diagonal is of order $(m-r) \times (n-r)$

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(2)
$$\begin{bmatrix} I_r \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ — & — & — & — & — \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

where I_r is a unit matrix of order $r = n$,

The null matrix below I_r is of order $(m-r) \times n$.

(3)
$$\begin{bmatrix} I_r & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

Where I_r is a unit matrix of order $r = m$.

The null matrix right of I_r is of order $m \times (n-r)$

(4) $[I_r]_{m \times n}$ = The unit matrix of order $r \times r$, $r = m = n$.

Theorem 2: Every matrix of $m \times n$ order of rank r can be reduced to one or other of the normal forms

Proof

$$\text{Let } A = \begin{bmatrix} x_1 & a_1 & b_1 & c_1 & \dots & d_1 \\ x_2 & a_2 & b_2 & c_2 & \dots & d_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_m & a_m & b_m & c_m & \dots & d_m \end{bmatrix}$$

Make $r_1\left(\frac{1}{x_1}\right)$ operation to get 1 as the first diagonal element. Next make $r_{21}(-x_2)$ operation so that the 2nd element in 1st column i.e., a_{21} element becomes zero. Similarly get zero in the place of a_{31} and so on. We have now got the first column as 1,0,0, ..., 0.

Next make $C_{21}(-a_1)$ operation. This gives zero as the first element of second column i.e., at a_{12}. Next $C_{31}(-b_1)$ gives zero at a_{13} and so on till we get zero at the 1st element of the n^{th} column i.e., at a_{1n}. Now A is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_2 & b_2 & c_2 & \dots & d_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & a_m & b_m & c_m & \dots & d_m \end{bmatrix}$$

In the above, we got a_{11} replaced by 1, used 1 as the 'pivot' element to get zeros in the first column and first row. Now make $r_2\left(\frac{1}{a_2}\right)$ operation to get 1 in place of a_2 element of 2nd row. Use this 1 as pivot and get zeros in second column and second row, as above.

Repeat this process until we get the normal form of A. If $\rho(A) = r$, the normal form of A should be $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ when $r < m, r < n$.

If $r = n$, the normal form of A is $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

If $r = m$, the normal form of A is $\begin{bmatrix} I_r & 0 \end{bmatrix}$

If $r = m = n$, the normal form of A is $\begin{bmatrix} I_r \end{bmatrix}$

Theorem 3: If A is any $m \times n$ matrix of rank $r < m, n$ then there exist non-singular matrices P and Q, so that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Proof: A can be reduced to the normal form by means of rwo and column transformations. Every row opeartion is equivalent to pre-multiplication by an elementary matrix. Therefore the row operations made are equivalent to the product of elementary matrices $P_x, P_{x-1} \dots P_2 P_1 = P$ ...

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pre-multiplying A. Similarly the column operations made are equivalent to $Q = Q_1 Q_2 \dots Q_{y-1}, Q_y$, product of elementary matrices post-multiplying A.

$\therefore$ A is reduced to the normal form by the process

$\therefore$ $P_x . P_{x-1} \dots P_2 \; P_1 \; A \; Q_1 \; Q_2 \dots Q_{y-1} \; Q_y$. i.e.,

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem 4: Every non-singular matrix is the product of elementary matrices.

Proof: $|A| \neq 0$ and let A be of order $n \times n$ so that $\rho(A) = n$.

$\therefore$ A reduces to the normal form $[\, I_r \,]$, $r = m = n$.

Reduction to normal form $\Rightarrow$ row and column operations $\Rightarrow$ pre and post multiplication of A by elementary matrices as

$\therefore$ $P_x . P_{x-1} \dots P_2 \; P_1 \; A \; Q_1 \; Q_2 \dots Q_{y-1} \; Q_y = I_n$ (1)

Each elementary matrix is non-singular.

$\therefore$ The inverse exist.

$\therefore$ Pre-multiplying (1) by

$P_1^{-1} \; P_2^{-1} \dots P_{x-1}^{-1} \; P_x^{-1}$ we get from (1)

$IAQ_1 \; Q_2 \dots Q_{y-1} \; Q_y = P_1^{-1} \; P_2^{-1} \dots P_{x-1}^{-1} \; P_x^{-1} \; I_n$

Post multiplying by $Q_y^{-1} \; Q_{y-1}^{-1} \dots Q_2^{-1} \; Q_1^{-1}$ we get,

$IAI = P_1^{-1} \; P_2^{-1} \dots P_x^{-1} \; I_n \; Q_y^{-1} \; Q_{y-1}^{-1} \dots Q_1^{-1}$, a product of elementary matrices.

Theorem 5: The rank of a matrix does not change if it is multiplied by a non-singular matrix.

i.e., $\rho(AB) = \rho(A)$ where B is non-singular.

Proof : B is non- singular.

$\therefore$ B equals to a product of elementary matrices.

$\therefore$ By taking AB or BA we are making elementary column or row operations. The rank of A is not changed by elementary transformations. Hence the result.

Theorem 6: If A be any $m \times n$ matrix of rank r then there exists a non-singular matrix P such that

$$PA = \begin{bmatrix} G \\ 0 \end{bmatrix} \sim \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

[P denotes row operations. The Theorem means that mere row operations can get A to the above form]

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Proof: If $\rho(A) = r < m, n$ we know matrices P, Q exist so that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

$$Q = Q_1 \ Q_2 \dots Q_{y-1} \ Q_y \text{ column operations on } A.$$

$$Q^{-1} = Q_y^{-1} \ Q_{y-1}^{-1} \dots Q_2^{-1} \ Q_1^{-1}$$

$\because$ elementary matrices are non-singular.

Post-multiplying (1) by Q^{-1}, we get

$$PAI = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_y^{-1} \ Q_{y-1}^{-1} \ \dots \ Q_2^{-1} \ Q_1^{-1}$$

In the R.H.S., each column operation does not affect the number of zeros in the last rows, but the zeros in the last columns can become non-zero elements. $\therefore$ we get

$$PAI = PA = \begin{bmatrix} G \\ 0 \end{bmatrix} \sim \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

Note: rank of A = rank of PA = rank of G = rank I_r.

Theorem 7: If A is any $m \times n$ matrix of rank r then there exists a non-singular matrix Q such that $AQ = [\ H \quad 0\ ] \sim [\ I_r \quad 0\ ]$

[Q denotes column operations. The Theorem means that mere column operations can get A to the above form]

Proof: If A is any $m \times n$ matrix of rank $r < m, n$ there exist non-singular matrices P and Q so that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

$P = P_x \ P_{x-1} \dots P_2 \ P_1$ is the product of elementary row operations. Elementary matrices are non-singular.

$\therefore P^{-1} = P_1^{-1} \ P_2^{-1} \dots P_x^{-1}$ exists. Pre-multiplying (1) by

$$P^{-1} \text{ we get } IAQ = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

In the right hand side, row transformations do not affect the zeros of the last columns but the zeros of the last rows can become non-zero elements.

$$\therefore \text{ We get } AQ = [\ H \quad 0\ ] \sim [\ I_r \quad 0\ ]$$

Note: rank of A = rank of AQ. $\because Q$ is non-singular

= rank of H = rank of I_r

Theorem 8: The rank of a product of two matrices can not exceed the rank of either matrix. $\rho(AB) \le \rho(A)$ or $\rho(B)$.

Proof:

A is an $m \times n$ matrix, $\rho(A) = r_1$

B is an $n \times p$ matrix, $\rho(B) = r_2$

AB is an $m \times p$ matrix, $\rho(AB) = r$

To show : $r \leq r_1, r_2$:

A is any $m \times n$ matrix.

$\therefore \exists$ a non-singular matrix P so that

$$PA = \begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} \text{ and } \rho(A) = \rho(PA) = \rho(G) = r_1$$

$$\therefore PAB = \begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} B, \quad \rho(AB) = r \Rightarrow \rho(PAB) = r,$$

$\because P$ is non-singular.

$$\therefore \text{ rank of } \begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} B = r \qquad (1)$$

But $\rho(A) = \rho(G) = r_1 \Rightarrow$ G contains r_1 non-zero rows.

In (1) these r_1 non-zero rows multiply the columns of B, along with the zero rows in $\begin{bmatrix} G \\ \mathbf{0} \end{bmatrix}$

Multiplication of the columns of B, by the zero rows cannot increase the number of non-zero rows in $\begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} B$.

$\therefore$ The number of non-zero rows of $\begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} B \leq r_1$.

$$\therefore \quad \rho(AB) = \rho(PAB) = \rho\left(\begin{bmatrix} G \\ \mathbf{0} \end{bmatrix} B\right) \leq r_1.$$

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$\therefore \quad \rho(AB) \leq \rho(A)$ using this

$\rho(AB) = \rho(AB)' = \rho(B'A') \leq \rho(B') = \rho(B)$

$\therefore \rho(AB) \leq \rho(B)$

Examples

1. Find the rank of A, by reducing it to the normal form.

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 & -1 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -3 & 1 \\ 0 & 1 & 2 & 2 & -6 \end{bmatrix}$$

Sol: $r_3(3)$, then $r_{31}(-1) \Rightarrow A \sim \begin{bmatrix} 3 & -2 & 0 & -1 & -1 \\ 0 & 2 & 2 & 1 & -5 \\ 0 & -4 & -9 & -8 & 4 \\ 0 & 1 & 2 & 2 & -6 \end{bmatrix}$

$$r_{12}(1) \Rightarrow \qquad A \sim \begin{bmatrix} 3 & 0 & 2 & 0 & -6 \\ 0 & 2 & 2 & 1 & -5 \\ 0 & -4 & -9 & -8 & 4 \\ 0 & 1 & 2 & 2 & -6 \end{bmatrix}$$

$$C_3(3), \text{ then } C_{35}(1), \text{ and } C_{51}(2) \Rightarrow A \sim \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & -5 \\ 0 & -4 & -23 & -8 & 4 \\ 0 & 1 & 0 & 2 & -6 \end{bmatrix}$$

$$r_1\left(\frac{1}{3}\right) \Rightarrow \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & -5 \\ 0 & -4 & -23 & -8 & 4 \\ 0 & 1 & 0 & 2 & -6 \end{bmatrix}$$

$$r_{32}(2), r_4(2) \text{ then } r_{42}(-1) \Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & -5 \\ 0 & 0 & -21 & -6 & -6 \\ 0 & 0 & -1 & 3 & -7 \end{bmatrix}$$

$$C_{34}(-1) \Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & -5 \\ 0 & 0 & -27 & -6 & -6 \\ 0 & 0 & -4 & 3 & -7 \end{bmatrix} \quad C_4(2), \text{ then } C_{42}(-1), \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -5 \\ 0 & 0 & -15 & -12 & -6 \\ 0 & 0 & -4 & 6 & -7 \end{bmatrix} \quad C_5(2), \text{ then } C_{52}(5), \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -15 & -12 & -12 \\ 0 & 0 & -4 & 6 & -14 \end{bmatrix} \quad r_2\left(\frac{1}{2}\right), \; r_3\left(\frac{1}{3}\right), \; r_4\left(\frac{1}{2}\right) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & -4 & -4 \\ 0 & 0 & -2 & 3 & -7 \end{bmatrix} \quad r_4(5), \; r_3(2) \text{ then } r_{43}(-1) \Rightarrow$$

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$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -10 & -8 & -8 \\ 0 & 0 & 0 & 23 & -27 \end{bmatrix}, C_{45}(-1) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -10 & 0 & -8 \\ 0 & 0 & 0 & -50 & -27 \end{bmatrix} C_3\left(\frac{-1}{10}\right), \; C_4\left(\frac{-1}{50}\right) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & -27 \end{bmatrix} C_{53}(8) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -27 \end{bmatrix} C_5\left(\frac{-1}{27}\right) \text{ then } C_{54}(-1) \Rightarrow$$

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$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = [\, I_4 \;\; \mathbf{0} \,]$$

2. Determine the rank of A by reducing it to the normal form

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Sol: $r_{21}(-2), r_{31}(-4), r_{41}(-4) \Rightarrow$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix} C_1\left(\frac{1}{2}\right), \text{ then } C_{21}(-1) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

C_{31} (-3), then C_{51} $(-1) \Rightarrow$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix} C_3\left(\frac{-1}{5}\right), \text{ then } C_5\left(\frac{-1}{7}\right) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix} C_{24} \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} r_{32}(-1), r_{42}(-3) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_{32}(-1) \Rightarrow$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$\therefore$ rank of $A = 2$

3. Find the matrices P and Q so that PAQ is the normal form of $A = \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}$

Sol : Let

$$\begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}_{2\times3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $A = I_2\ A\ I_3$ so that the products are defined.

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$$r_1\left(\frac{1}{2}\right), \text{ then } r_{21}(1) \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_{21}(-1), C_{31}(3) \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_1\left(\frac{1}{3}\right) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & 3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_{32}(1) \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & 8/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \; [I_2 \;\; 0] = PAQ$; as above.

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Assignment – 1

(1) Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

Ans : 3

(2) Find the rank of the matrix

$$\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

Ans : 3

(3) Find the rank of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{bmatrix}$$

when (1) $a = b = c$
(2) $a = b \neq c$
(3) $a \neq b = c$

Ans : 3, 2, 1

Hint : $|A| = (a-b)(b-c)(c-a)$

(4) Find the rank of A by reducing it to echelon form

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Ans : 2

(5) Find the rank of matrix A, by reducing it to the echelon form

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 2 & 3 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Ans : 3

(6) Find the rank of matrix A, by reducing it, using elementary operations.

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Ans : 2

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(7) Find the ranks of the following matrices, by reducing them to the normal form

(1) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

(4) $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Ans : (1) 3, (2) . 3, (4) 2

(8) Show that $A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$ is a matrix of rank 2.

(9) Find non-singular matrices P and Q so that PAQ is in the normal form of A where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$$

Ans : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -2/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29.5 Systems of linear equations

(i) By the inversion method :

We have solved a system of non-homogeneous equations, using Cramere's rule in unit –27.

The given system of equations $A\,X = B$, can be solved when

$|A| \neq 0, \because A^{-1}$ exists.

We have $A^{-1}(AX) = A^{-1}B$, i.e., $X = A^{-1}B$,

Ex. 1: (1) Solve the equations by finding the inverse of the coefficient matrix.

$$\begin{aligned} 2x + y + 6z &= 46 \\ 7x + 4y - 3z &= 19 \\ 5x - 6y + 4z &= 15 \end{aligned}$$

Sol: coefficient matrix $A = \begin{bmatrix} 2 & 1 & 6 \\ 7 & 4 & -3 \\ 5 & -6 & 4 \end{bmatrix}$

The given equations are $= \begin{bmatrix} 2 & 1 & 6 \\ 7 & 4 & -3 \\ 5 & -6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 46 \\ 19 \\ 15 \end{bmatrix}$

i.e., $AX = B$; $\quad \therefore X = A^{-1}B$

$$A^{-1} = \frac{adj\,A}{det\,A};$$

$$\begin{aligned} |A| &= 2(16-18) - 1(28+15) + 6(-42-20) \\ &= -4 - 43 - 372 = -419 \end{aligned}$$

co-factors

$A_{11} = \begin{vmatrix} 4 & -3 \\ -6 & 4 \end{vmatrix} = -2 \qquad A_{12} = -\begin{vmatrix} 7 & -3 \\ 5 & 4 \end{vmatrix} = -43$

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$$A_{13} = \begin{vmatrix} 7 & 4 \\ 5 & -6 \end{vmatrix} = -62 \qquad A_{21} = -\begin{vmatrix} 1 & 6 \\ -6 & 4 \end{vmatrix} = -40$$

$$A_{22} = \begin{vmatrix} 2 & 6 \\ 5 & 4 \end{vmatrix} = -22 \qquad A_{23} = -\begin{vmatrix} 2 & 1 \\ 5 & -6 \end{vmatrix} = -17$$

$$A_{31} = -27, \quad A_{32} = 48, \quad A_{33} = 1$$

$$adj\,A = \begin{bmatrix} -2 & -40 & -27 \\ -43 & -22 & 48 \\ -62 & 17 & 1 \end{bmatrix}, \qquad A^{-1} = \frac{adj\,A}{-419};$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} B$$

$$= \frac{1}{419} \begin{bmatrix} 2 & 40 & 27 \\ 43 & 22 & -48 \\ 62 & -17 & -1 \end{bmatrix} \begin{bmatrix} 46 \\ 19 \\ 15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419} \begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore x = 3, \quad y = 4, \quad z = 6$

(ii) Non-homogeneous linear equations, consistency, rank test

(1) Introductory results :

Consider the m equations, in n unknowns, $m \neq n$.

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots\dots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots\dots + a_{2n} x_n = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \dots\dots + a_{3n} x_n = b_3$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \dots\dots + a_{mn} x_n = b_m$$

The coefficient matrix is A is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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The augmented matrix AB;

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & \vdots & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & \vdots & b_n \end{bmatrix}$$

C_1 the column vector is $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$. Similarly we can have the column vectors, $C_2, C_3 \ldots$;

$$\therefore A = [C_1 \; C_2 \; C_3 \ldots \ldots C_n]$$

The vectors are linearly dependent $\Leftrightarrow$

$\exists \; x_1, \; x_2, \; x_3 \ldots \ldots, x_n$, scalars which are not all zeros so that

$$C_1 \, x_1 + C_2 \, x_2 + \ldots + C_n \, x_n = 0$$

When the rank of $A = r$, $\exists$ atleast one non-zero minor of order r i.e.,

$\Delta_r \neq 0$. We can denote the column vectors of this minor as $C_1, C_2, C_3, \ldots \ldots C_r$ without loss of generality.

$$\therefore \; \Delta_r = |C_1 \; C_2 \; C_3 \ldots \ldots C_r| \neq 0$$

This means that the vectors $C_1, C_2, C_3, \ldots \ldots C_r$ are "linearly independent" because for linear relation

$$C_r = x \, C_1 + x_2 C_2 + \ldots \ldots \Rightarrow \Delta_r = 0;$$

Explanation of this result :

rank $A = r \Rightarrow \exists \, \Delta_r \neq 0$ and $\Delta_{r+1} = 0$

$\Delta_{r+1} = |C_1 \; C_2 \ldots \ldots C_r \; C_{r+1}| = 0 \Rightarrow C_{r+1}$ is a linear combination of $C_1 \; C_2 \; \ldots \ldots C_r$ which are themselves linearly independent.

$\therefore \exists \, k_1, k_2 \ldots \ldots k_r$ scalars not all zeros so that

$$C_{r+1} = C_1 k_1 + C_2 k_2 + \ldots . + C_r k_r \neq 0$$

(iii) Rouche's Theorem

Theorem 9 : The system of non-homogeneous equations given by $AX = B$ is consistent $\Leftrightarrow \rho(A) = \rho(AB)$, where A is coefficient matrix and AB the augmented matrix.

Proof: $A : [C_1 \; C_2 \; C_3 \ldots \ldots C_n]$ in terms of the column vectors.

$AB = [C_1 \; C_2 \; C_3 \ldots \ldots C_n B]$ and $AX = B \Rightarrow$

$$C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots . + C_n x_n = B \qquad (1)$$

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Case (1) Let $AX = B$ be consistent. $\therefore$ (1) has solutions

$x_1 = k_1, x_2 = k_2, \ldots\ldots x_n = k_n.$

$$\therefore \quad k_1 C_1 + k_2 C_2 + \ldots\ldots + k_n C_n = B \qquad (2)$$

Let $\rho(A) = r_1$

$\therefore$ minor $\Delta_r \neq 0, \Delta_{r+1} = 0, \Delta_{r+2} = 0, \ldots\ldots$

$\therefore$ $\Delta_r = |C_1 \; C_2 \ldots \ldots C_r| \neq 0 \Rightarrow C_1, C_2, \ldots \ldots, C_r$ are linearly independent.

$\Delta_{r+1} = |C_1 \; C_2 \ldots \ldots C_r \, C_{r+1}| = 0 \Rightarrow C_{r+1}$ is dependent on $C_1, C_2, \ldots \ldots C_r$ which are already linearly independent.

$\Delta_{r+2} = 0, \Delta_{r+3} = 0, \ldots\ldots \Rightarrow C_{r+2}, C_{r+3}, \ldots\ldots, C_n$ are all dependent on $C_1, C_2, \ldots \ldots C_r$.

Now $B = C_1 k_1 + C_2 k_2 + \ldots + C_r k_r + C_{r+1} k_{r+1} + \ldots\ldots + C_n k_n$

Since $C_{r+1}, C_{r+2}, \ldots\ldots, C_n$ depend on $C_1, C_2, \ldots\ldots, C_r$,

we have B dependent only on $C_1, C_2, \ldots\ldots, C_r$.

$AB = [C_1, C_2, \ldots\ldots, C_r \, C_{r+1} \ldots\ldots C_n \, B]$

$\therefore$ AB contains r linearly independent column vectors.

$\therefore$ $\Delta_r \neq 0, \Delta_{r+1} = 0, \ldots\ldots$

$\therefore$ $\rho(AB) = r$ and we have $\rho(AB) = \rho(A)$

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Case (2) Let $\rho(A) = \rho(AB) = r$,

$\therefore$ AB contains r linearly independent column vectors.

$\therefore$ $AB = [C_1, C_2, \ldots\ldots, C_r \, C_{r+1} \ldots\ldots C_n \, B]$

$\Delta_{r+1} = 0$ etc. $\Rightarrow C_{r+1}, C_{r+2}, \ldots\ldots C_n, B$ are linear combinations of $C_1, C_2, \ldots\ldots, C_r$.

$\therefore$ $\exists \; k_1, k_2 \ldots\ldots k_r$ scalars, not all zeros so that

$B = k_1 C_1 + k_2 C_2 + \ldots\ldots + k_r C_r$

$\therefore$ $B = k_1 C_1 + k_2 C_2 + \ldots\ldots + k_r C_r + 0.C_{r+1} + \ldots\ldots + 0.C_n$

$\therefore$ The system of equations $AX = B$, shown in (1).

have the solutions $x_1 = k_1, x_2 = x_2 \ldots\ldots, x_r = k_r, x_{r+1} = 0, x_{r+2} = 0, \ldots\ldots, x_n = 0$

$\therefore$ $AX = B$ are consistent.

Note (1) $\rho(A) = \rho(AB) = r < n \Rightarrow [C_1 \; C_2 \ldots \ldots C_r]$ is not singular.

$C_1 \; C_2 \ldots C_r$ are independent and we get from (1) r independent equations in r unknowns

$C_1 x_1 + C_2 x_2 + \ldots + \ldots C_r x_r + \ldots\ldots + C_n x_n = B$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} x_2 + \ldots + \begin{bmatrix} a_{1r} \\ a_{2r} \\ \cdot \\ \cdot \\ a_{mr} \end{bmatrix} x_r$$

$$+ \ldots\ldots + C_{r+1}\, x_{r+1} + \ldots\ldots + C_n\, x_n = B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

$\therefore \quad a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots + a_{1r}\, x_r = b_1 - a_{r+1}\, x_{r+1} + \ldots\ldots\ldots - a_{1n}\, x_n$

$\therefore$ r unknowns can be solved for, in terms of the other $(n - r)$ unknowns which are arbitrary.

Hence an infinity of solutions exist in this case.

When $r = n$, the n unknowns can be uniquely solved.

Ex.1: Test the consistency of the equations using the rank test :

$$\begin{aligned} 5x + 3y + 14z &= 4 \\ y + 2z &= 1 \\ x - y + 2z &= 0 \\ 2x + y + 6z &= 2 \end{aligned}$$

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Sol.: The equations are

$$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{i.e., } AX = B$$

$$\therefore AB = \left[\begin{array}{ccc:c} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

We reduce A to the echelon form to find $\rho(A)$:

$$r_{13} \Rightarrow AB \sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

$$r_{31}(-5),\ r_{41}(-2) \Rightarrow AB \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \\ 0 & 3 & 2 & 2 \end{array}\right]$$

$$r_{32}(-8),\ r_{42}(-3) \Rightarrow AB \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \\ 0 & 0 & -4 & -1 \end{array}\right]$$

$$r_4(-3),\ \text{then } r_{43}(1) \Rightarrow AB \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$\rho(A)$ = number of non-zero rows in $A = 3$

$\rho(AB)$ = number of non-zero rows in $AB = 4$

$\therefore \rho(A) \neq \rho(AB)$, Equations are not consistent.

2. Find the values of λ and μ for which the equations

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu, \text{ have} \end{aligned}$$

(1) No solutions (2) a unique solution (3) an infinity of solutions

Sol :

$$AB = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$r_{21}(-1),\ r_{32}(-1) \Rightarrow AB \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Case (1) If $\lambda = 3$, $\rho(A) = 2$. If $\lambda = 3$, $\mu \neq 10$, Then $\rho(AB) = 3$

$\rho(A) \neq \rho(AB)$ $\therefore$ Equations are not consistent.

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$\therefore$ They have no solution.

Case (2) $\lambda = 3, \mu = 10 \Rightarrow \rho(A) = \rho(AB) = 2 = r$

$2 = r < n = 3$, the number of unknowns.

$r < n \Rightarrow$ Infinity of solutions.

Case (3) $\lambda \neq 3, \mu \neq 10 \Rightarrow$

$\rho(A) = \rho(AB) = 3 = r = n \Rightarrow$ unique solutions exist.

3. Determine λ so that the equations

$x + y + z = 1, \quad x + 2y + 4z = \lambda, \quad x + 4y + 10z = \lambda^2$ may be consistent and solve them.

Sol :

$$AB = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

$$r_{21}(-1), r_{31}(-1) \Rightarrow AB \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

$$r_{32}(-3) \Rightarrow AB \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

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$\rho(A) = 2$. For consistency $\rho(AB) = \rho(A) = 2 \Leftrightarrow \lambda^2 - 3\lambda + 2 = 0$ i.e., $(\lambda - 1)(\lambda - 2) = 0$, $\lambda = 1$ or 2.

Case (1) $\lambda = 1 \Rightarrow$

$$AB \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (1)$$

$\therefore$ Equations are from (1)

$x + y + z = 1, \quad y + 3z = 0, \quad \rho(A) = 2 = r < n = 3$

$\therefore$ An infinity of solutions exist. Let $z = k$,

$\therefore y = -3k, \quad x = 1 - y - z = 1 + 2k$

$\therefore$ Solutions are $(x, y, z) = (1 + 2k, -3k, k)$ where k is arbitrary.

Case (2) : $\lambda = 2$; Again an infinity of solutions exist.

$$AB \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 3 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$ Equation are $x + y + z = 1, y + 3z = 1$

Let $z = k,\ y = 1 - 3k,\ x = 2k$ are the solutions.

4. Solve

$$x + y + z + \omega = 1$$
$$2x - y + z - 2\omega = 2$$
$$3x + 2y - z - \omega = 3$$

Explain the nature of the solutions.

Sol :

$$AB = \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 1 \\ 2 & -1 & 1 & -2 & \vdots & 2 \\ 3 & 2 & -1 & -1 & \vdots & 3 \end{bmatrix}$$

$$r_{21}(-2), r_{31}(-3) \Rightarrow AB \sim \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 1 \\ 0 & -3 & -1 & -4 & \vdots & 0 \\ 0 & -1 & -4 & -4 & \vdots & 0 \end{bmatrix}$$

$$r_{23}(-3)\, r_{32} \Rightarrow AB \sim \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 1 \\ 0 & -3 & -1 & -4 & \vdots & 0 \\ 0 & 0 & -11 & -8 & \vdots & 0 \end{bmatrix} \quad (1)$$

$\rho(AB) = \rho(A) = 3 = r < n = 4$

$\therefore$ Equations are consistent. An infinity of solutions arise. The equations are, from (1) :

$$x + y + z + \omega = 1$$
$$-3y - z - 4\omega = 0$$
$$-11z - 8\omega = 0$$

Let $\omega = k$,

$\therefore\ z = \frac{-8k}{11},\ y = \frac{-k}{11},\ x = 1 - \frac{9k}{11}$

where k is arbitrary.

(iv) Systems of Homogeneous Equations

(1) n equations in n unknowns :

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots a_{1n}x_n = 0$ (1)

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots a_{2n}x_n = 0$ (2)

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$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots a_{nn}x_n = 0 \quad (n)$

i.e., $AX = 0$, is a system of Homogeneous linear equations.

When $|A| \neq 0$, such a system will have trivial solutions, because

$|A| \neq 0 \Rightarrow A^{-1}$ exists, $\quad \therefore\ A^{-1}(AX) = A^{-1}\,\mathbf{0} = \mathbf{0}$

$\Rightarrow IX = \mathbf{0},\ X = \mathbf{0}$

$\therefore\ x_1 = 0,\ x_2 = 0, x_3 = 0 \ldots\ldots, x_n = 0$ (trivial solutions).

When $|A| = 0$, A^{-1} does not exist. Then the system of equations will have an infinity of solutions which are not trivial.

Let $x_1 = kA_{11},\ x_2 = kA_{12}, \ldots x_n = kA_{1n}$

where k is arbitrary and A_{11} is co-factor of a_{11} in $|A|$ etc.

$\therefore$ (1) $\Rightarrow k\,[a_{11}A_{11} + a_{12}A_{12} + \ldots\ldots + a_{1n}A_{1n}] = 0$. i.e., $k\,|A| = 0$, $\therefore\ |A| = 0$ is necessary for consistency. Also when $|A| = 0$, (1) is satisfied, as also the others equations.

$$\because \sum a_{rs}A_{ij} = 0$$

As k is arbitrary, the number of solutions are infinite.

Note : The rank-test : When $\rho(A) = r < n$ as we are going to prove $(n-r)$ unknowns will have arbitrary values. Hence the necessary and sufficient condition for non-trivial solutions is $r < n$. If there are m equations in n unknowns and $m < n$, then $r \leq m < n$ and non-trivial solutions always exist. Hence we prove the theorem for m equations in n unknowns. $(r \leq m < n)$:

Theorem 10 : The equation $AX = \mathbf{0}$ has $(n-r)$ independent solutions where $\rho(A) = r$ and n is the number of unknowns and A is an $m \times n$ matrix.

Proof: $\rho(A) = r$

$\therefore$ The number of linearly independent column vectors of A is r.

$A = [C_1\ C_2\ C_3 \ldots C_r\ C_{r+1} \ldots C_n]$

$C_1, C_2, \ldots, C_r$ are linearly independent.

$C_{r+1}, C_{r+2}, \ldots, C_n$ depend on $C_1, C_2, \ldots C_r$.

$\therefore\ C_{r+1} = k_1C_1 + k_2C_2 + \ldots + k_rC_r$

i.e., $C_1k_1 + C_2k_2 + \ldots + C_rK_r - C_{r+1} + C_{r+2} \cdot 0 + \ldots + C_n \cdot 0 = 0 \ldots (1)$

$AX = \mathbf{0} \Rightarrow C_1x_1 + C_2x_2 + \ldots + C_rx_r + C_{r+1}x_{r+1} + \ldots + C_nx_n = \mathbf{0}$

comparing this with (1), we get one set of solutions :

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ k_r \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

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As above $C_{r+2} = p_1 C_1 + p_2 C_2 + ... + p_r C_r$ and we get a second solution set

$$X = \begin{bmatrix} k_1 \\ k_2 \\ . \\ . \\ k_r \\ -1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$ and so on. We can proceed this way upto C_n.

$\therefore$ The number of independent solutions in considering C_{r+1} to C_n is $(n-r)$

$\therefore$ $r < n \Leftrightarrow$ there exits non-trivial solutions.

Examples :

1. Solve

$$2\omega + 3x - y - z = 0$$
$$4\omega - 6x - 2y + 2z = 0$$
$$-6\omega + 12x + 3y - 4z = 0$$

Sol : $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix}$

$r_{31}(3), r_{21}(-2) \Rightarrow A \sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -12 & 0 & 4 \\ 0 & 21 & 0 & -7 \end{bmatrix}$

$r_2(1/4), r_3(1/7) \Rightarrow A \sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 3 & 0 & -1 \end{bmatrix}$

$r_{32}(1) \Rightarrow A \sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (1)

Number of non-zero rows of $A = 2$ $\therefore$ $\rho(A) = 2 = r$

Number of unknowns $= 4 = n$

$\therefore$ Number of independent solutions $= n - r = 2$

From (1), the given equations are,

$-3x + z = 0,\ 2\omega + 3x - y - z = 0$

Let $x = k_1, z = 3k_1$ and $y = k_2$,

$\therefore$ $\omega = \frac{k_2}{2}$

$x = k_1,\ y = k_2$ are the independent solutions.

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2. Solve, using the rank - test :

$$x + 2y + 3z + 4\omega = 0$$

$$8x + 5y + z + 4\omega = 0$$

$$5x + 6y + 8z + \omega = 0$$

$$8x + 3y + 7z + 2\omega = 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 5 & 1 & 4 \\ 5 & 6 & 8 & 1 \\ 8 & 3 & 7 & 2 \end{bmatrix}$$

$$r_{21}(-8), r_{31}(-5), r_{41}(-8) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -11 & -23 & -28 \\ 0 & -4 & -7 & -19 \\ 0 & -13 & -17 & -30 \end{bmatrix}$$

$$r_{23}(-3), r_{43}(-3) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & -4 & -7 & -19 \\ 0 & -1 & 4 & 27 \end{bmatrix}$$

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$$r_{32}(4), r_{42}(1) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & 0 & -15 & 97 \\ 0 & 0 & 2 & 56 \end{bmatrix}$$

$$r_{21}(1/2), \text{then } r_{43}(1/15) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & 0 & -15 & 97 \\ 0 & 0 & 0 & 28+97/15 \end{bmatrix}$$

$\rho(A) = 4 = r$, number of unknowns $= 4 = n$

$\therefore$ Number of independent solutions $= n - r = 4 - 4 = 0$

$\therefore$ only trivial solutions exist. $x = y = z = \omega = 0$.

3. Solve
$$\left.\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}\right\}$$

Sol : $A \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, $|A| = 9 \neq 0$

When $|A| = 9 \neq 0$, $r = n = 3$ only trivial solutions exist.

$\therefore \quad x_1 = x_2 = x_3 = 0$

Number of independent solutions is $n - r = 3 - 3 = 0$.

29.6 Summary

We have seen that :

(i) The elementary operations on a matrix do not alter the rank of a matrix.

(ii) In echelon form the number of non-zero rows is the rank of the matrix.

(iii) A system of non-homogenous equations given by $AX = B$ is consistent if $\rho(A) = \rho(AB)$.

29.7 Sample Examination Questions

(1) Establish consistency using the rank test and solve

$$\begin{aligned} x + 2y + 3z &= 14 \\ 3x + y + 2z &= 11 \\ 2x + 3y + z &= 11 \end{aligned}$$

Ans : $(x, y, z) = (1, 2, 3)$

(2) Solve the above question by using the inverse of the coefficient matrix.

(3) Solve
$$\begin{aligned} 2x - 3y + z &= 9 \\ x + y + z &= 6 \\ x - y + z &= 2, \end{aligned}$$
using the rank test.

[Ans : $x = 11$, $y = 2$, $z = -7$]

(4) Show that the equations

$$\begin{aligned} -2x + y + z &= a \\ x - 2y + z &= b \\ x + y - 2z &= c \end{aligned}$$

have no solutions unless $a + b + c = 0$ and solve them when

$a = 1$, $b = 1$, $c = -2$

[Ans : $z = k$, $y = k-1$, $x = k-1$]

(5) Examine consistency using the rank test and solve

$$\begin{aligned} 3x - 2y - \omega &= 2 \\ 2y + 2z + \omega &= 1 \\ x - 2y - 3z + 2\omega &= 3 \\ y + 2z + \omega &= 1 \end{aligned}$$

[Ans : $x = 1$, $y = 0$, $z = 0$, $\omega = 1$]

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(6) Solve :

$$\begin{aligned} 3x + 7y + 5z &= 4 \\ 26x + 2y + 3z &= 9 \\ 2x + 10y + 7z &= 5 \end{aligned}$$

Ans : $z = k$, $y = \dfrac{7-11k}{16}$, $x = \dfrac{5-k}{16}$

(7) Obtain the non-trivial solutions of :

$$\begin{aligned} 2x - 2y + 5z + 3\omega &= 0 \\ 4x - y + z + \omega &= 0 \\ 3x - 2y + 3z + 4\omega &= 0 \\ x - 3y + 7z + 6\omega &= 0 \end{aligned}$$

Ans : $\omega = k$, $z = \dfrac{7k}{9}$, $y = 4k$, $x = \dfrac{5k}{9}$

(8) Obtain the non-trivial solutions of :

$$\begin{aligned} x + y + z + \omega &= 0 \\ x + 3y + 2z + 4\omega &= 0 \\ 2x + z - \omega &= 0 \end{aligned}$$

Ans : $z = k_1$, $\omega = k_2$, $y = -\left(\dfrac{k_1 + 3k_2}{2}\right)$, $x = \dfrac{k_3 - k_1}{2}$

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(9) Solve

$$\begin{aligned} 4x + y + z + 3u &= 0 \\ 6x + 3y + 4z + 7u &= 0 \\ 2x + y + u &= 0 \end{aligned}$$

Ans : $y = k_1$, $z = k_2$, $u = -k_2$, $x = \dfrac{k_2 - k_1}{2}$

Unit - 30 : Eigen Values and Eigen Vectors of a Matrix

30.0 Contents

30.1 Aims and Objectives

After going through this unit you must be able to evaluate,

(i) the eigen values and eigen vectors of a given matrix,

(ii) diagonalise the given matrix by making use of the eigen values and eigen vectors of the matrix.

30.2 Characteristic Equation

Characteristic matrix : If A is a square matrix of order $n \times n$ and I the unit matrix of the same order, then $A - \lambda I$, where λ is a scalar, is called the characteristic matrix of A.

If $A = [a_{ij}]_{nxn}$, then

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

is called the characteristic matrix of A.

(2) *Characteristic polynomial :* The determinants of the characteristic matrix gives the characteristic polynomial, on expansion.

$|A - \lambda I| = \phi(\lambda)$ on expansion and ϕ is the characteristic polynomial of the matrix A.

(3) *Characteristic equation :* The equation $|A - \lambda I| = 0$ is the characteristic equation of the matrix A

(4) *Eigen values (characteristic roots) :* The roots of the characteristic equation in λ i.e., $|A - \lambda I| = 0$ are called the eigen values of the matrix A. They are also known as the characteristic roots or the latent roots of the matrix A.

The set of eigen values of the matrix A is called the "spectrum" of A.

(5) *Eigen vectors :* A non-trivial solution of the equation $AX = \lambda X$, where λ is an eigen value of A, is called an eigen vector X or a characteristic vector X.

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$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{is an eigen vector.}$$

30.2.1 Results following the definitions

(1) X is a characteristic vector so that $AX = \lambda X$

$\Leftrightarrow \lambda$ is a characteristic root of matrix A.

Proof : X is eigen vector $\Leftrightarrow \lambda$ is a non-trivial solution of $AX = \lambda X$.

$\Leftrightarrow$ $(A - \lambda I) X = 0$ has non-trivial solutions.

$\Leftrightarrow$ $\rho (A - \lambda I) = r < n$, the number of unknowns.

$\Leftrightarrow$ $|A - \lambda I| = 0, \lambda$ is a root of this equation,

i.e., a characteristic root.

(2) If X is a characteristic vector arising from $AX = \lambda X$, then kX, where $k \neq 0$ is also a characteristic vector of the same equation.

Proof : Non - zero vector X satisfies the equation $AX = \lambda X$ (1)

$\therefore$ $k \neq 0 \Rightarrow kX \neq 0$. Writing kX for X in (1)

$L.H.S. = A(kX) = k(AX) = k(\lambda X) = \lambda (kX)$

$\therefore$ $kX \neq 0$, kX satisfies $AX = \lambda X$.

$\therefore$ kX is also an eigen vector of the same matrix.

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Note : $k \neq 0$ is an arbitrary scalar.

$\therefore$ Corresponding to the same characteristic root λ, the number of characteristic vectors are many.

(3) A characteristic vector X of the equation $AX = \lambda X$, cannot correspond to more than one characteristic root λ of the matrix A.

Proof : Let $AX = \lambda_1 X$ and $AX = \lambda_2 X$ if possible.

This $\Rightarrow$ $(\lambda_1 - \lambda_2) X = 0$, where X is a non-zero vector.

$\therefore$ $\lambda_1 = \lambda_2$ and X corresponds to only one λ.

(4) Characteristic roots of a diagonal matrix are the diagonal elements

Proof : Let
$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

The characteristic equation is

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \dots & 0 \\ 0 & a_{22}-\lambda & 0 & \dots & 0 \\ 0 & 0 & a_{33}-\lambda & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

i.e., $(\bar{a}_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda) = 0$

$\therefore \quad \lambda = a_{11}, a_{22}, \dots, a_{nn}$

Hence the eigen values of A are the diagonal elements of A.

(5) A and A' have the same eigen values.

Proof : The determinants $|A-\lambda I|$ and $|A'-\lambda I|$ are same in value since the rows of the one are the columns of the other.

$\therefore \quad |A-\lambda I| = 0,\ |A'-\lambda I| = 0$ are same.

$\therefore$ Eigen values of A and A′ are the same.

(6) The square matrix A and $P^{-1}AP$ have the same characteristic roots, where P is a non-singular square matrix of the same order as A.

Proof : Let $B = P^{-1}AP$, B and A are of the same order.

$$B-\lambda I = P^{-1}AP-\lambda P^{-1}P = P^{-1}AP - P^{-1}\lambda IP$$

$$= P^{-1}(AP-\lambda IP) = P^{-1}(A-\lambda I)P$$

$$|B-\lambda I| = |P^{-1}|\ |A-\lambda I|\ |P| = |A-\lambda I|\ |P^{-1}P| = |A-\lambda I|\ |I|$$

$\therefore \quad |B-\lambda I| = |A-\lambda I|,$

$\therefore$ A and B have the same characteristic roots.

(7) The eigen values of a unitary matrix are of unit modulus.

Proof : Let A be unitary $\quad \therefore \quad A^* A = I.$

If λ is a characteristic root of A and X an eigen vector of $AX = \lambda X$, then

$(AX)^* = (\lambda X)^* = \bar{\lambda} X^*$

$X^* A^* = \bar{\lambda} X^* \quad \therefore X^* A^* (AX) = \bar{\lambda} X^* \lambda X = \lambda \bar{\lambda} X^* X$

$\Rightarrow X^*(A^*A)X = \lambda \bar{\lambda}\ X^* X$

$\Rightarrow X^* I X = \lambda \bar{\lambda}\ X^* X \quad (\because A^*A = I)$

$\Rightarrow |X^* X|\ |I| = |\lambda \bar{\lambda}|\ |X^* X|$

$\lambda \neq 0,\ X^* \neq 0\ |X^* X| \neq 0,$

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$\therefore\ |\lambda\,\overline{\lambda}\,| = \ |I| = 1$

$|\lambda\,\overline{\lambda}\,| = \ |\lambda|^2 = 1, \quad |\lambda| = 1.$

$\therefore$ The eigen values of a unitary matrix are of unit modulus.

(8) The characteristic roots of an orthogonal matrix are of unit modulus.

Proof : A is orthogonal $\Rightarrow A'A = I$

If λ is a characteristic root of A and X an eigen vector of the equation $AX = \lambda X$ then

$$(AX)' = (\lambda X)' = \lambda X'$$

$$\therefore\ X'A' = \lambda X' \text{ and}$$

$$(X'A')(AX) = \lambda X'\ \lambda X \Rightarrow X'\ (A'A)\,X = \lambda^2 X'X$$

$$X'X\,(I - \lambda^2) = 0 \qquad \because A'A = I$$

$$X \neq 0,\ X' \neq 0$$

$$\therefore\ |X'X| \neq 0 \qquad \therefore\ |X'X|\ |I - \lambda^2| = 0$$

$$\Rightarrow \lambda^2 = 1 \quad |\lambda| = 1,$$

$\therefore$ Eigen values of orthogonal matrix are of unit modulus.

(9) All eigen values of a Hermitian matrix are real.

Proof : A is Hermitian $\Rightarrow A^* = A$

If λ is a characteristic root of A and X an eigen vector of equation $AX = \lambda X$, then

$$X^*(AX) = X^*(\lambda X) = \lambda X^*X \qquad (1)$$

$$(X^*AX)^* = (\lambda X^*X)^* \Rightarrow X^*A^*X = \overline{\lambda}\,X^*X$$

$$\text{But } A^* = A, \qquad \therefore\ X^*AX = \overline{\lambda}\,X^*X \qquad (2)$$

(1), (2) $\Rightarrow \overline{\lambda} = \lambda$, since $X^*X \neq 0$

$\Rightarrow \lambda$ is real.

(10) All eigen values of a real symmetric matrix are real.

Proof : A is real $\Rightarrow \overline{A} = A$, A is symmetric $\quad \therefore A' = A$.

$\therefore (\overline{A})' = A' = A$ i.e., $A^* = A$.

A is Hermitian and result (9) holds good.

(11) The eigen values of a skew Hermitian matrix are purely imaginary or zeros.

Proof : We know from Unit 27 that,

A is Skew Hermitian $\Rightarrow iA$ is Hermitian

$\Rightarrow$ all eigen values of iA are real.

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If λ is an eigen value of A, then $i\lambda$ is that of iA.

$\therefore$ $i\lambda$ is real, which happens when $\lambda = 0$ or purely imaginary.

(12) The characteristic roots of a real skew symmetric matrix are all purely imaginary or zeros.

Proof : A is real $\Rightarrow \overline{A} = A$, $(\overline{A})' = A' = -A$, since A is skew symmetric.

$\therefore$ $A^* = -A$ and A is Skew Hermitian and result (11) applies.

30.3 Cayley - Hamilton Theorem

Every square matrix A satisfies its characteristic equation.

i.e. $|A - \lambda I| = 0$ when expanded, gives

$p_0 \lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \ldots\ldots + p_n = 0$, the characteristic equation. A stisfies this equation so that $p_0 A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots\ldots + p_n I = 0$

(Proof of this result is excluded from the syllabus).

Use of the above result :

$$p_0 A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots\ldots + p_{n-1} A + p_n I = 0.$$

post-multiplying by A^{-1}, we get

$$p_0 A^{n-1} + p_1 A^{n-2} + p_2 A^{n-3} + \ldots\ldots + p_{n-1} I + p_n A^{-1} = 0 A^{-1} = 0$$

$$\therefore A^{-1} = -\frac{1}{p_n}\left[p_0 A^{n-1} + p_1 A^{n-2} + \ldots\ldots + p_{n-1} I\right]$$

Thus the above theorem can be used to find the inverse of a matrix.

Examples

1. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$$

Sol :

$$|A - \lambda I| = \left|\begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$

$$= \begin{vmatrix} -2-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2) + 5, = \lambda^2 - 2\lambda - 3$$

$\therefore$ The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$

$$(\lambda - 3)(\lambda + 1) = 0$$

i.e., $\lambda = 3$ or -1

eigen values of A are -1 and 3.

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The eigen vectors are the non-trival solutions of X from $AX = \lambda X$, i.e., $(A - \lambda I) = 0$

$$\begin{bmatrix} -2-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad (1)$$

Case (1) $\lambda = 3 \Rightarrow \begin{bmatrix} -5 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, from (1)

i.e., $-5x_1 - x_2 = 0,\ 5x_1 + x_2 = 0$ or $\frac{x_1}{1} = \frac{x_2}{-5}$

$$\therefore X = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \text{ is an eigen vector.}$$

Case (2) $\lambda = -1 \Rightarrow \begin{bmatrix} -1 & -1 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, from (1)

i.e., $x_1 + x_2 = 0$, or $\frac{x_1}{1} = \frac{x_2}{-1}$

$$\therefore X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ is the second eigen vector.}$$

2. Determine the characteristic roots and the characteristic vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol : The characteristic equation is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$ $\quad r_2 + r_3 \Rightarrow$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \quad C_3 - C_2 \Rightarrow$$

$$\begin{vmatrix} 6-\lambda & -2 & 4 \\ 0 & 2-\lambda & 0 \\ 2 & -1 & 4-\lambda \end{vmatrix} = 0$$

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$\Rightarrow \quad (2-\lambda)\left[(6-\lambda)(4-\lambda)-8\right]=0$

$(\lambda-2)(\lambda^2-10\lambda+16)=0$

$(\lambda-2)(\lambda-2)(\lambda-8)=0$

$\lambda = 2, 2, 8$ are the eigen values.

The eigen vectors are the non-trivial solutions of $AX=\lambda X$ ie., $(A-\lambda I)\ X=\mathbf{0}$

$$\text{i.e.,} \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \qquad (1)$$

$$\text{Case (1) } \lambda = 2 \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \text{ from (1)}$$

$$r_{31}(-1/2) \text{ and } r_{23}(1) \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$\rho(A) = 1 = r < n = 3$

$\therefore$ Number of independent solutions $= n - r = 2$

$\therefore 4x_1 - 2x_2 + 2x_3 = 0, x_2 = k, x_3 = l$

$\therefore 4x_1 = 2(k-l), x_1 = \left(\frac{k-l}{2}\right)$, where k and l are arbitrary

$$\therefore X = \begin{bmatrix} \frac{k-l}{2} \\ k \\ l \end{bmatrix}, \text{ an infinity of vectors arise. } k=0, l=2 \Rightarrow$$

$$X = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \text{ and } l=0, k=2 \Rightarrow$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Case (2) When $\lambda = 8$ we get from $(A-\lambda I)X = \mathbf{0}$ i.e., from (1),

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

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$$r_{23}(1) \text{ and } r_{31}(1) \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -6 & -6 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$r_1(-1/2), r_2(-1/6), r_3(-1/3) \Rightarrow$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$r_{32}(-1) \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\rho(A) = 2$, $\therefore$ Number of independent solutions = 3 – 2 = 1

The equations are, $x_1 + x_2 - x_3 = 0, x_2 + x_3 = 0, x_3 = k$

$\therefore \quad x_2 = -k, \quad x_1 = 2k$

$$\therefore X = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}, \ k \text{ being arbitrary is the eigen vector.}$$

When $k = 1$ $X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the vector

Verify Cayley Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}. \text{ Hence find } A^{-1}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$C_{32}(-1) \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -3+\lambda \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$r_{32}(1) \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -3+\lambda \\ 0 & 1-\lambda & 0 \end{vmatrix} = 0$$

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i.e., $(\lambda-3)\begin{vmatrix} 2-\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix} = 0, (\lambda-3)(\lambda-2)(\lambda-1) = 0$

i.e., $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

A should satisfy this equation according to Cayley–Hamilton theorem

Verification

$$A^2 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & -3 \\ -5 & 6 & -3 \\ 5 & -5 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & -3 & -3 \\ -5 & 6 & -3 \\ 5 & -5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -7 & -7 \\ -19 & 20 & -7 \\ 19 & -19 & 8 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 11A - 6I =$

$$\begin{bmatrix} 8 & -7 & -7 \\ -19 & 20 & -7 \\ 19 & -19 & 8 \end{bmatrix} - \begin{bmatrix} 24 & -18 & -18 \\ -30 & 36 & -18 \\ 30 & -30 & 24 \end{bmatrix} + \begin{bmatrix} 22 & -11 & 11 \\ -11 & 22 & -11 \\ 11 & -11 & 22 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

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To find A^{-1},

$$A^3 - 6A^2 + 11A - 6I = \mathbf{0},\ 6I = A^3 - 6A^2 + 11A$$

$$6I\ A^{-1} = A^2 - 6A + 11I$$

$$\therefore A^{-1} = \frac{1}{6}\left[A^2 - 6A + 11I\right]$$

$$\therefore A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -3 & -3 \\ -5 & 6 & -3 \\ 5 & -5 & 4 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \frac{11}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 3 & 3 \\ 1 & 5 & 3 \\ -1 & 1 & 3 \end{bmatrix}$$

4. Show that the matrix

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$ satisfies the characteristic equation. Hence find A^{-1}.

Sol.: $|A - \lambda I| = 0$, the characteristic equation

$$\Rightarrow \begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

i.e., $$-\lambda(\lambda^2+a^2) - c\,(c\,\lambda - ab) - b\,(ac + b\,\lambda) = 0$$

$$-\lambda(\lambda^2 + a^2 + b^2 + c^2) + abc - abc = 0$$

$$\lambda^3 + \lambda\,(a^2 + b^2 + c^2) = 0$$ is the equation

A should satisfy $A^3 + A\,(a^2 + b^2 + c^2) = 0.$

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Verification

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ca & cb & -a^2-b^2 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ca & cb & -a^2-b^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c(a^2+b^2+c^2) & -b(a^2+b^2+c^2) \\ c(a^2+b^2+c^2) & 0 & -a(a^2+b^2+c^2) \\ -b(a^2+b^2+c^2) & a(a^2+b^2+c^2) & 0 \end{bmatrix}$$

$$= (a^2 + b^2 + c^2)\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} = -\,(a^2 + b^2 + c^2)\,A$$

$$\therefore\ A^3 + (a^2 + b^2 + c^2)\,A = 0;$$ To find A^{-1},

$$A^2 + (a^2+b^2+c^2)I = \mathbf{0}A^{-1}, \text{ post-multiplying by } A^{-1}.$$

Again post-multiplying by A^{-1} we get,

$$A + (a^2+b^2+c^2)A^{-1} = \mathbf{0}$$

Taking $(a^2+b^2+c^2) \neq 0$

$$\therefore A^{-1} = \frac{-1}{(a^2+b^2+c^2)}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ use Cayley–Hamilton theorem to express

$2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Sol.:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore (3-\lambda)(2-\lambda) + 1 = 0$, i.e., $\lambda^2 - 5\lambda + 7 = 0$

$\therefore$ By Cayley–Hamilton theorem, A satisfies the equation

$$A^2 - 5A + 7I = \mathbf{0} \qquad (1)$$

pre-multiply by A, A^2, A^3 to get

$$A^3 - 5A^2 + 7A = \mathbf{0} \qquad (2)$$

$$A^4 - 5A^3 + 7A^2 = \mathbf{0} \qquad (3)$$

$$A^5 - 5A^4 + 7A^3 = \mathbf{0} \qquad (4)$$

$$\begin{aligned} 2A^5 - 3A^4 + A^2 - 4I &= 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I, \text{ using (4)} \\ &= 7A^4 - 14A^3 + A^2 - 4I \\ &= 7(5A^3 - 7A^2) - 14A^3 + A^2 - 4I, \text{ using (3)} \\ &= 21A^3 - 48A^2 - 4I, \text{ using (2)} \\ &= 57A^2 - 147A - 4I \\ &= 57(5A - 7I) - 147A - 4I \text{ using (1)} \\ &= 138A - 403I, \text{ the linear polynomial in } A. \end{aligned}$$

Assignments–I

(1) Find the latent roots of

$$A = \begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

(2) Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

(3) Determine the characteristic roots and characteristic vectors of the matı

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ Find the characteristic roots and the eigen vectors of A.

(5) Verify Cayley–Hamilton theorem for the matrix

$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ Hence find A^{-1}

(6) If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ use Cayley–Hamilton theorem to find A^{-2}.

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(7) If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find the characteristic roots of A.

Show that A satisfies Cayley–Hamilton theorem and find A^{-1}.

(8) If $\lambda = \lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are the characteristic roots of matrix A.

Prove that $|\lambda_1, \lambda_2, \ldots\ldots \lambda_n| = |A|$.

(Hint: $|A - \lambda I| = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots (\lambda - \lambda_n)$ (put $\lambda = 0$)

Answers

(1) $\lambda = a, b, c$

(2) $\lambda = -1, i, -i;\ X = [0 \ \ 1 \ \ -1]', [1+i \ \ 1 \ \ 1]'$ and $[1-i \ \ 1 \ \ 1]'$

(3) $\lambda = 0, 3, 15;\ X = [k \ \ 2k \ \ 2k]', [2l \ \ l \ \ -2l]', [2m \ \ -2m \ \ m]'$.

(4) $\lambda = 1, 1, 5;\ X = [k \ \ k \ \ k]', [2 \ \ -1 \ \ 0]', [1 \ \ 0 \ \ -1]'$.

(5) $A^{-1} = \dfrac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$;

or $\quad A^{-2} = \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(7) $\quad \lambda = 1, 1, 4; \; A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

30.4 Diagonalisation of a matrix

If a square matrix A of order n has n linearly independent eigen vectors, then matrix B can be found so that $B^{-1}AB$ is a diagonal matrix i.e., $A \sim B^{-1}AB = D$.

Proof:

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$$

The eigen equation

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

Let $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigen values.

$$X_1 = [x_{11} \;\; x_{21} \;\; x_{31} \;\ldots\; x_{n1}]',$$

$$X_2 = [x_{12} \;\; x_{22} \;\; x_{32} \;\ldots\; x_{n2}]'$$

$$\ldots \quad \ldots \;\; \ldots \;\; \ldots \;\; \ldots \;\; \ldots$$

$$\ldots \quad \ldots \;\; \ldots \;\; \ldots \;\; \ldots \;\; \ldots$$

$$X_n = [x_{1n} \;\; x_{2n} \;\; x_{3n} \;\ldots\; x_{nn}]'$$

be the corresponding n linearly independent eigen vectors.

$$\therefore \quad (A - \lambda_1 I) X_1 = \mathbf{0}$$

$$(A - \lambda_2 I) X_2 = \mathbf{0} \;\ldots\ldots\ldots\ldots, \quad (A - \lambda_n I) X_n = 0$$

$$(A - \lambda_1 I) X_1 = \mathbf{0} \Rightarrow$$

$$(a_{11}-\lambda_1) x_{11} + a_{12} x_{21} + a_{13} x_{31} + \ldots + a_{1n} x_{n1} = 0$$

$$a_{21} x_{11} + (a_{22}-\lambda_1) x_{21} + a_{23} x_{31} + \ldots + a_{2n} x_{n1} = 0$$

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$$\begin{matrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix}$$

$$a_{n1}x_{11} + a_{n2}x_{21} + a_{n3}x_{31} + \ldots + (a_{nn}-\lambda_1)x_{n1} = 0$$

These equations give,

$$\begin{cases} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + \ldots + a_{1n}x_{n1} = \lambda_1 x_{11} \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + \ldots + a_{2n}x_{n1} = \lambda_2 x_{21} \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ a_{n1}x_{11} + a_{n2}x_{21} + a_{n3}x_{31} + \ldots + a_{nn}x_{n1} = \lambda_n x_{n1} \end{cases}$$

$\therefore (A-\lambda_2 I)X_2 = 0, (A-\lambda_3 I)X_3 = 0$ etc. give similar systems of equations.

Let $B = [X_1 \; X_2 \; X_3 \; \ldots \; X_n]_{n\times n}$

That is the columns of B are the eigen vectors of A; and $X_1, X_2, X_3, \ldots X_n$ are linearly independent.

$\therefore \quad |B| \neq 0$

$\therefore \quad B$ is non-singular so that B^{-1} exists.

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} + \ldots + a_{1n}x_{n1} & \ldots \\ a_{21}x_{11} + a_{22}x_{21} + \ldots + a_{2n}x_{n1} & \ldots \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots & \\ a_{n1}x_{11} + a_{n2}x_{21} + \ldots + a_{nn}x_{n1} & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \cdots & \lambda_n x_{1n} \\ \lambda_1 x_{21} & \lambda_2 x_{22} & \cdots & \lambda_n x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 x_{n1} & \lambda_2 x_{n2} & \cdots & \lambda_n x_{nn} \end{bmatrix}$$

in virtue of equqtions (1) and similar results.

$$= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} = BD$$

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where $D = \text{diag}\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$

$\therefore AB = BD$ where $D = \text{diag}\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$

$\therefore B^{-1}AB = B^{-1}(BD) = ID = D$

$\therefore A \sim D$

To find A^m, m being a positive integer:

If A can be diagonalised $A \sim D = B^{-1}(AB)$

$$D = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n\}$$

$D^m = \text{diag}\{\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots, \lambda_n^m\}$ by actual multiplication.

$$D^2 = (B^{-1}AB)(B^{-1}AB) = B^{-1}(ABB^{-1}A)B = (B^{-1}A^2B)$$

$D^3 = (B^{-1}A^2B)(B^{-1}AB) = (B^{-1}A^3B)$ and so on.

$D^m = (B^{-1}A^mB)$, by induction.

$\therefore BD^m = A^mB$ and $BD^mB^{-1} = A^m$

$\therefore$ A^m can be found if the eigen values which give D and the eigen vectors which give B are determined.

Examples

1. Diagonalise the matrix, without performing the row a d co umn transformations.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol.: From Ex.(2), of 30.3, the eigen values of A are 2, 2, 8. The eigen vectors are

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

X_1, X_2, X_3 are linearly independent since a and b cannot be found so that $X_3 = aX_1 + bX_2$.

$[(2, -1, 1) = a(-1, 0, 2) + b(1, 2, 0) \Rightarrow b - a = 2,\ 2a = 1, 2b = -1$, absurd]

$\therefore$ By the diagonalization theorem,

$$A \sim D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

2. Determine A^5, using the idea of diagonalization of A, where

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

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Sol.: The spectrum of A is the set of eigen values is

$$\{2,\ 2,\ 8\} = \{\lambda_1,\ \lambda_2,\ \lambda_3\}$$

The eigen vectors are $X_1 = [-1\ \ 0\ \ 2]'$

$$X_2 = [1\ \ 2\ \ 0]',\ X_3 = [2\ \ -1\ \ 1]'$$

X_1, X_2, X_3 are linearly independent.

$$\text{Hence } A \sim D = \text{diag}\,\{2,\ 2,\ 8\}$$

$$D^5 = \text{diag}\,\{2^5,\ 2^5,\ 8^5\}$$

$$A^5 = BD^5B^{-1},\ B = [X_1\ \ X_2\ \ X_3]$$

To find B^{-1}

$$B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B$$

$$r_{31}(2) \Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} B$$

$r_1(2),\ r_{32}(-1) \Rightarrow$

$$\begin{bmatrix} -2 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} B$$

$r_3(1/6),\ r_{12}(-1) \Rightarrow$

$$\begin{bmatrix} -2 & 0 & 6 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 2/6 & -1/6 & 1/6 \end{bmatrix} B$$

$r_{13}(-5),\ r_{23}(1) \Rightarrow$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/6 & -1/6 & -5/6 \\ 2/6 & 5/6 & 1/6 \\ 2/6 & -1/6 & 1/6 \end{bmatrix} B$$

$r_1(-1/2),\ r_2(1/2) \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/6 & 1/12 & 5/12 \\ 1/6 & 5/12 & 1/12 \\ 2/6 & -1/6 & 1/6 \end{bmatrix} B$$

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i.e., $I = B^{-1}B$

$$\therefore \quad B^{-1} = \frac{1}{12}\begin{bmatrix} -2 & 1 & 5 \\ 2 & 5 & 1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$A^5 = BD^5B^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 8^5 \end{bmatrix}B^{-1}$$

$$= \begin{bmatrix} -2^5 & 2^5 & 2.8^5 \\ 0 & 2^6 & -8^5 \\ 2^6 & 0 & 8^5 \end{bmatrix}B^{-1}$$

$$= \frac{1}{12}\begin{bmatrix} -2^5 & 2^5 & 2.8^5 \\ 0 & 2^6 & -2^{15} \\ 2^6 & 0 & 2^{15} \end{bmatrix}\begin{bmatrix} -2 & 1 & 5 \\ 2 & 5 & 1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2^3 & 2^3 & 2^{14} \\ 0 & 2^4 & -2^{13} \\ 2^4 & 0 & 2^{13} \end{bmatrix}\begin{bmatrix} -2 & 1 & 5 \\ 2 & 5 & 1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -8 & 8 & 16384 \\ 0 & 16 & -8192 \\ 16 & 0 & 8192 \end{bmatrix}\begin{bmatrix} -2 & 1 & 5 \\ 2 & 5 & 1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 65568 & -32736 & 32736 \\ -32736 & 16464 & -16368 \\ 32736 & -16368 & 16464 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 21856 & -10912 & 10912 \\ -10912 & 5488 & -5456 \\ 10912 & -5456 & 5488 \end{bmatrix}$$

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Assignment–2

(1) Diagonalise the following matrices if possible:

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

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(2) Find A^8 using the idea of diagonalization of

(i) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$$

(ii) Find A^3 if $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Answers

(1) (i) diag {0, 3, 15} (ii) diag {1, 1, 5}

(2) (i) $$\begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 25 & 78 & 19 \\ 13 & 38 & 6 \\ 0 & 0 & 27 \end{bmatrix}$$

30.5 Quadratic form

Linear combinations of variables, where the coefficients belong to a field, are called homogeneous forms. We know the linear form among these,

i.e., $\sum_{i=1}^{n} a_i x_i = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$ where

$a_i \in K$, a field (linear form).

A homogeneous form in which each term contains two variables x_i and x_j is a quadratic form. We define

a quadratic form over a field K in the variables $x_1, x_2, x_3, \ldots, x_n$, is a homogeneous polynomial

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \text{ where } a_{ij} \in K.$$

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

then the above form is given by X′AX.

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clearly $\quad A\,X = \begin{bmatrix} a_{11}\,x_1 + a_{12}\,x_2 + \ldots\ldots + a_{1n}\,x_n \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots\ldots + a_{2n}\,x_n \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \\ a_{n1}\,x_1 + a_{n2}\,x_2 + \ldots\ldots + a_{nn}\,x_n \end{bmatrix}$

$$= \left[\sum_{j=1}^{n} a_{ij}\,x_j\right],\ 1 \le i \le n$$

$$X'A\,X = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} a_{ij}\,x_j\right] x_i$$

The expansion of this form while taking $a_{ij} = a_{ji}$ is given by

$$\begin{aligned} A\,(x, x) = q = {} & a_{11}\,x_1^2 + a_{12}\,x_1\,x_2 + \ldots + a_{1n}\,x_1\,x_n \\ & + a_{21}\,x_2 x_1 + a_{22}\,x_2^2 + \ldots + a_{2n}\,x_2\,x_n \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + a_{n1}\,x_n x_1 + a_{n2}\,x_n x_2 + \ldots + a_{nn}\,x_n^2 \end{aligned}$$

$$A\,(x, x) = \sum_{i=1}^{n} a_{ij}\,x_i^2 + 2\sum a_{ij}\,x_i\,x_j \qquad \left.\begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}\right\} \quad (1)$$

Note (1): The total number of terms in this form

$= n$ diagonal terms + the other $n^2 - n$ terms compounded into pairs.

$= n + \frac{1}{2}(n^2 - n) = \frac{n\,(n+1)}{2}.$

Note (2): As we took $a_{ij} = a_{ji}$, the coefficient matrix A is symmetric and $A = A'$. The 'discriminant' of the quadratic form q or the 'modulus' of q is given by $|A|$, when A is symmetric $\rho\,(A)$ is the rank of the form $q = A\,(x, x)$.

$\therefore$ $\quad A$ is Symmetric matrix $\Leftrightarrow A$ is quadratic form.

Examples

1. Write down the quadratic form corresponding to the symmetric matrix

$$A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

Sol.: We know the quadratic form in three variables x_1, x_2, x_3 is

$$\sum_{i=1}^{3} a_{ii}\,x_i^2 + 2\sum a_{ij}\,x_i\,x_j,\ (1 \le i \ne j \le n)$$

i.e., $A\,(x,x) = a_{11}\,x_1^2 + a_{22}\,x_2^2 + a_{33}\,x_3^2 + 2\,a_{12}\,x_1 x_2 + 2\,a_{13}\,x_1 x_3 + 2\,a_{23}\,x_2 x_3$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } a_{ij} = a_{ji}$$

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$$\text{But } A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

$\therefore\ a_{11} = 0,\ a_{22} = 1,\ a_{33} = 2,$ and

$a_{12} = a_{21} = 5, a_{13} = a_{31} = -1, a_{23} = a_{32} = 6$

$\therefore\quad A(x, x) = q = 0.x_1^2 + 1.x_2^2 + 2.x_3^2 + 10x_1x_2 - 2x_1x_3 + 12x_2x_3$

Method (2) : we could also find $q = A(x, x) = X'AX$

$$= [x, y, z]\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [5y - z \quad 5x + y\,6z \quad -x + 6y + 2z]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x(5y - z) + y(5x + y + 6z) + z(-x + 6y + 2z)$$

$$= y^2 + 2z^2 + 10xy + 12yz - 2zx$$

Note : $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = A(x, x)$

$$\text{where } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

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2. Obtain the symmetric matrix for the form

$$A(x, x) = x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$$

Sol :

$$A(x, x) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

$\therefore\ a_{11} = 1,\quad a_{22} = 2,\quad a_{33} = 3,\quad a_{12} = 2,\quad a_{13} = 3,\quad a_{23} = 5/2$

$\therefore\ a_{21} = 2,\quad a_{31} = 3,\quad a_{32} = 5/2.$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$$

is the symmetric matrix.

Write down the quadratic form corresponding to the diagonal matrix

$$A = diag\left\{a_{11}\ \ a_{22}\ \ \dots\dots\ \ a_{nn}\right\}$$

Sol: $$A(x, x) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum a_{ij} x_i x_j (1 \le i \ne j \le n)$$

A is a diagonal matrix

$\therefore a_{ij} = 0$ for $i \ne j$

$$A(x, x) = \sum_{i=1}^{n} a_{ii} x_i^2 = a_{11} x_1^2 + a_{12} x_2^2 + \dots + a_{nn} x_n^2$$

30.6 Reduction of a symmetric matrix to diagonal form

Theorem : Every symmetric matrix of rank r can be reduced to a diagonal matrix, the number of non-zero elements in the diagonal being 'r'.

Proof :

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ be symmetric.}$$

Take a_{11} as the pivot element. By a sequqnce of row transformations of the type $r_{ij}(x)$ we can get zeros as the first elements of $r_2, r_3, \dots, r_n$. Since $a_{ij} = a_{ji}$, similar transformations $c_{ij}(x)$ on the columns, will give zeros as the first elements of $c_2, c_3, \dots, c_n$

We know row and column transformations are equivalent to pre and post multiplication by elementary matrices. Since $A' = A$ if E_{ij} is the elementary matrix post-multiplying A corresponding to $c_{ij}(x)$ operations, then E_{ij}' is the elementary matrix pre-multiplying A corresponding to $r_{ij}(x)$ operation. Now A has taken the reduced form

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Taking a_{22} as the pivot element and repeating the above type of row and column transformations, we get zeros down the column a_{22} and to the right of a_{22} in that row. We continue this process until we get zero in diagonal element. If this element is $a_{r+1\, r+1}$ and all diagonal elements further down are zeros, $\rho(A) = r$ and we have reduced A to the diagonal form

$$diag \left\{ a_{11} \;\; a_{22} \;\dots \dots, a_{rr}, 0, 0, \dots\dots 0 \right\}$$

however if $a_{r+1\, r+1} = 0$, and further down the diagonal $a_{r+k\, r+1} \ne 0$, we can get this non-zero element to the position of $a_{r+1\, r+1}$, by appropriate row and column transformations, repeated until we get the form

$$diag \left\{ a_{11} \;\; a_{22} \;\dots \dots, a_{ss}, 0, 0, \dots\dots 0 \right\} \text{ for } A$$

$\therefore \rho(A) = s$ and there are s non-zero elements in the diagonal. We now have

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$$diag \left\{ a_{11} \ \ a_{22} \ \ \ldots \ldots, \ a_{kk}, 0, 0, \ldots\ldots 0 \right\} = B' AB$$

where B = product of elementary matrices E_{ij} post-multiplying A

B' = product of elementary matrices E_{ij}' pre-multiplying A

Note: When B is orthogonal, $|B| \neq 0$, we can have

$B' B = I, B^{-1} B = I$, and $B^{-1} = B'$. Then $B' AB = B^{-1} AB$

$$= diag \left\{ \lambda_1 \ \ \lambda_2 \ \ \ldots \ldots, \ \lambda_n \right\}$$

by diagonalisation theorem. But can we have B orthogonal?

In further study of matrices, we will see that, if A is a real symetric matrix, then there exists an orthogonal matrix B, such that $B'AB$ is a diagonal matrix whose diagonal elements are the characteristic roots of A.

Reduction of a given quadratic form into a sum of squares

Let $q = X'AX$ be the given quadratic form

Let A be reduced to the diagonal form.

$$B'AB = diag \left\{ a_{11} \ \ a_{22} \ \ \ldots \ldots, \ a_{rr}, 0, 0, \ldots\ldots 0 \right\}$$

To reduce $X'AX$ to the sum of squares.

Let $X = BY$ be the transformation required; where

$$X = \begin{bmatrix} x_1 \\ \cdot \\ \vdots \\ x_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\therefore X' AX = (BY)' A(BY) = Y' (B'AB)Y$$

$$= Y' \ diag \left\{ a_{11} \ \ a_{22} \ \ \ldots \ldots, \ a_{rr}, 0, 0, \ldots\ldots 0 \right\} Y$$

$$= a_{11} y_1^2 + a_{22} \ y_1^2 + \ldots + \ a_{rr} \ y_r^2$$

Method: To get a given quadratic form as a sum of squares :

(1) Write down the symmetric matrix A for q

(2) Let $A = IAI$

(3) Diagonalise A on the left and the equation becomes

$$diag \left\{ a_{11} \ \ a_{22} \ \ \ldots \ldots, \ a_{rr}, 0, 0, \ldots\ldots 0 \right\} = B'AB$$

(4) $X = BY$ is the transformation to get a as the sum of squares.

Eamples

1. Obtain the linear transformation to reduce $2x^2 + 2y^2 + 3z^2 - 4yz - 4zx + 2xy$ to a sum of multiples of squares.

Sol : Corresponding to the given quadratic form q, the symmetric matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

Write $A = I\,A\,I$, i.e.,

$$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_{21}(-1/2)\, r_{31}(1) \Rightarrow$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3/2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c_{21}(-1/2)\, c_{31}(1) \Rightarrow$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r_{23}(1) \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$c_{23}(1) \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= B'AB$$

$\therefore$ $X = BY$ is the transformation which gives

$$q = X'AX = Y'(B'AB)Y = Y' \operatorname{diag}\{2, \tfrac{1}{2}, 1\}Y$$

$$= 2u^2 + \frac{1}{2}v^2 + w^2, \text{ where } Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

is the linear transformation required.

i.e., $x = u + \frac{1}{2}v + w,\ y = v,\ z = v + w$

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30.7 Summary

The roots of the characteristic equation $|A - \lambda I| = 0$ are called the eigen values of the given matrix A. A non trivial solution of the matrix equation $AX = \lambda X$ is called an eigen vector of A corresponding to the eigen value λ.

Cayley - Hamilton theorem states that every square matrix satisfies its characteristic equation and this theorem has been used for evaluating the inverse of a given square matrix.

Every symmetric matrix of rank 'r' can be reduced to a diagonal matrix (without using elementary transformations), the number of non-zero elements in the diagonal being 'r'.

30.8 Sample Examination Questions

(1) Find the symmetric matrix of the quadratic form

$$x^2 + y^2 + 4z^2 + 3w^2 - xy + 3yz + 3zw - wx$$

(2) Find the symmetric matrix corresponding to the quadratic form

$$2x_1^2 - 7x_3^2 + 4x_1x_2 - 6x_2x_3$$

(3) Write down the quadratic forms corresponding to the symmetric matrices

(i) $\begin{bmatrix} 2 & 2 & 3 \\ -3 & -2 & -4 \\ 1 & -4 & -3. \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

(4) Reuce the quadratic from $q = x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_2 + 16x_1x_3$ to a sum of multiples of squares. Specify transformation required.

(5) Reduce the quadratic from $A(x, x)$ to a sum of multiples of squares, specifying the transformation required of

$$A(x, x) = x^2 + y^2 + 4z^2 + 9t^2 - 2xy - 4yz + 6yt - 6tx - 12tz.$$

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Miscellaneous exercise on all units of Block – 8

(1) Evaluate

$$\begin{vmatrix} 3 & 7 & 9 & 6 \\ 8 & 4 & 5 & 8 \\ 7 & 10 & 3 & 5 \\ 6 & 2 & 9 & 8 \end{vmatrix}$$

(2) Show that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

(3) Show that

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$$

(4) Multiply

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \text{ and } \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

and deduce that the product of two expressions like $a^3 + b^3 + c^3 - 3\,abc$, can take the same form.

(5) Evaluate

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$

(6) Express

$$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

as the product of two determinants and find its value.

(7) Show that

$$\forall\, n \in Z_+,\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \text{ (use induction).}$$

(8) $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix},\ B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$

find AB and BA if possible. Is $AB = BA$?

(9) Find the condition that the matrtix A be non-singular and if this condition is satisfied find A^{-1}.

where $A = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}$ and check your result.

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(10) If P is a non-singular, square matrix of order m and Q is another square matrix of the same order,

show that $(P+Q)\,P^{-1}\,(P-Q) = (P-Q)P^{-1}\;(P+Q)$

(11) Verify that

$$\frac{1}{9}\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} \text{ is orthogonal.}$$

What can you say about the column vectors?

(12) Solve by the matrix inversion method,

$x_1 + 2x_2 + x_3 = 2,\ \ 3x_1 + x_2 - 2\,x_3 = 1,\ \ 4x_1 - 3x_2 - x_3 = 3$

(13) Find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$$

(14) Show that the equations

$x + 2y - z = 3,\ \ 3x - y + 2z = 1,\ \ 2x - 2y + 3z = 2,\ x - y + z = -1$ are consistent and solve them.

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(15) Find the rank of the matrix

$$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$$

(16) Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

(17) Show that the only real value of λ for which the following equations have non-zero solutions is 6,

$x + 2y + 3z = \lambda x,\ \ 3x + y + 2z = \lambda y,\ \ 2x + 3y + z = \lambda z$

(18) Verify Cayley - Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$ and hence find A^{-1} and check $AA^{-1} = I$

(19) Find two non-singular matrices P and Q so that PAQ gives the normal form of A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

(20) Verify Cayley - Hamilton theorem for matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and compute $2A^8 - 3A^5 + A^4 + A^2 - 4I$ as a linear polynominal in A.

Answers

(1) $\begin{bmatrix} 1 & -1/2 & 0 & -1/2 \\ -1/2 & 1 & 3/2 & 0 \\ 0 & 3/2 & 4 & 3/2 \\ 1/2 & 0 & 3/2 & 3 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & -3 & -7 \end{bmatrix}$

(3) (i) $2x^2 - 2y^2 - 3z^2 - y + 4zx - 8yz$

(ii) $x_1^2 + x_3^2 + 4x_1 x_2 + 6x_1 x_3 + 6x_2 x_3$

(4) $y_1^2 + 8y_2^2 - 2y_3^2;\ x_1 = y_1 - 4y_2,\ x_2 = y_2 - \frac{1}{2}y_3,\ x_3 = y_2 + \frac{1}{2}y_3$

(5) $u^2 + 4v^2 - w^2;\ x = u + w,\ y = w - 3s,\ z = a + \frac{1}{2}w,\ t = s.$

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Miscellaneous exercise

(1) -500 (5) 0

(6) $\begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ (row by column rule)

The value $= 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$

(8) No (11) orthonormal vectors (12) $(x_1, x_2, x_3) = (1,0,1)$

(13) 3. (14) $(x, y, z) = (-1,4,4)$ (15) 3;

(16) $5, -3, -3;\ [1\ 2\ -1]',\ [2\ -1\ 0]'\ [3\ 0\ 1]'$

Author of Units 26-30 : Sri S.V. Ramanaiah.

REFERENCES

1. **C.G. Cullen**

   Matrices and linear tranformations

2. **Ross A. Beamount & Richard W. Ball**

   Introduction to Modern Algebra and Matrices

3. **Franz E. Honn**

   Elementary Matrix Algebra

4. **E.T. Browne**

   Introduction to the theory of determinants and matrices

5. **D.C. Murdoch**

   Linear Algebra for undergraduates

6. **H.W. Turnbull and A.C. Aitken**

   An introduction to the theory of Canonical Matrices

7. **Frank Iyeres**

   Matrices (Schaum Series)

8. **S.S. Seth**

   Matrices and Determinants
   Ram Prasad & Sons, Agra-3.

9. **M.L. Khanna**

   Matrices

   Jaiprakash Nath and Company, Meerut.

10. Kapur and Singal

    Matrices

    R. Chand & Co., New Delhi.

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Mathematics - Course - I

Assignment - I

Section A

(1) Define the Limit of a function at a point. Evaluate the Limits :

(i) $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$

(ii) $\lim\limits_{x \to 0} (\tan x)^{\tan 3x}$

(iii) $\lim\limits_{x \to 1} \frac{\sqrt{x} - 1 + \sqrt{x - 1}}{\sqrt{x^2 - 1}}$

(2) (i) If $u = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = z$

(ii) If $x^m y^n = (x + y)^{m+n}$, prove that $x \frac{dy}{dx} = y$

(3) (i) Evaluate the integrals,

(i) $\int \frac{x \sin x^2}{\cos x^2 + \sin x^2} \, dx$

(ii) $\int_0^{\pi/2} \log \sin x \, dx$

(ii) Find the area enclosed by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$ in the first quadrant.

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Section B

(1) Define the continuity of f at $x = 0$. Examine the continuity of

$$f(x) = x^2 \sin \frac{1}{x} \quad \text{for } x \neq 0$$
$$= 0 \quad \text{for } x = 0.$$

(2) Verify Rolle's theorem for

$x(x + 3)e^{-x/2}$ in $(-3, 0)$

(3) Show that the radius of curvature for the curve $x = c \log \tan \frac{1}{2}\theta$, $y = c \operatorname{cosec} \theta$ is $\frac{y^2}{c}$.

Mathematics - Course-I

Assignment - II

Section A

(1) Solve the following first order differential equations :

(i) $\frac{dy}{dx} + x \tan(y - x) = 1$

(ii) $x\,dy - y\,dx = \sqrt{x^2 + y^2}\;dx$

(iii) $\frac{dy}{dx} + \frac{x^2 + 3y^2}{3x^2 + y^2} = 0$

(2) Explain the method of variation of parameters for solving a second order linear ordinary differential equation and use this method to solve the equation $\frac{d^2y}{dx^2} + 4y = \tan 2x$.

(3) Form the partial differential equations of lowest order for $\frac{z-y^2}{2} = F\left(\frac{1}{x} + \log y\right)$.

Find the solutions of $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{xz}\right)q = \frac{x-y}{xy}$.

Section B

(1) Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda y + c = 0$, λ being the parameter.

(2) Solve the equations :

$$\frac{dx}{dt} = 8x - y$$

$$\frac{dy}{dt} = 4y + \text{[illegible]}$$

(3) Solve $(x^2 + zy + z)\,dx + (y^2 + xz + z)\,dy + (z^2 + xy + x + y)\,dz = 0$.

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Mathematics – Course – I

Assignment – III

Section A

1. Show that $\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix} = x^2\,(x+6)$

2. Solve the following system of equations by Matrix method :

$$\begin{aligned} x - 3y - 8z + 10 &= 0 \\ 3x + y - 4z &= 0 \\ 2x + 5y + 6z - 13 &= 0 \end{aligned}$$

3. Find the eigen values and eigen vectors of the matrix : $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Section B

1. Solve the following system of equations by Cramer's rule :

$$\begin{aligned} 3x + 5y - 7z &= 13 \\ 4x + y - 12z &= 6 \\ 2x + 9y - 3z &= 20. \end{aligned}$$

2. Show that every matrix can be uniquely represented as the sum of a Hermitian and Skew-Hermitian matrices.

3. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, find A^{-1} using elementary transformations.

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Faculty of Science

B.Sc. II year (3 year Degree Course) Examination

Mathematics – Course – I

(Calculus, Differential Equations and Matrices)

Time : 3 Hours] [Max. Marks : 100 / Min. Marks : 35

Section A

Answer any **Four** Questions $4 \times 15 = 60$

1. (i) (a) Define the continuity of a function f in an interval $[a, b]$.

   (b) If $f(x) = \dfrac{\cos 3x - \cos 4x}{x \sin 2x}$ is continuous at $x = 0$, then find $f(0)$.

   (ii) Evaluate :

   (a) $\lim\limits_{x \to 0} \dfrac{x \cos x - \log(1 + x)}{x^2}$

   (b) $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$

2. (i) If $y^{\sin x} = x^{\sinh y}$, find $\dfrac{dy}{dx}$

   (ii) If $y = e^{-m\tan^{-1}x}$ show that, with the usual notation $(1 + x^2)\, y_2 + (2x + m)\, y_1 = 0$

3. (i) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$

   (ii) If $u = (x^2 - y^2)\, e^x \cos y - 2xy\, e^x \sin y$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

4. Obtain the following:

   (i) $\displaystyle\int \frac{x}{x^4 - x^2(a^2 - b^2) - a^2 b^2}\, dx$

   (ii) $\displaystyle\int_0^{\pi/4} \log(1 + \tan x)\, dx.$

   (iii) $\displaystyle\int \frac{dx}{3 + 4\cos x}$

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5. Solve the following,

(i) $x(3 + 2\tan y)\,dx + (3y^2 + x^2 \sec^2 y)\,dy = 0.$

(ii) $(a^2 + x^2)\frac{dy}{dx} + 2xy = x + 1$

(iii) $\frac{dy}{dx} = \frac{2x + 3y + 4}{4x + 6y + 5}.$

6. Explain the method of solution by variation of parameters for solving a linear differential equation of 2nd order. Using this method solve $\frac{d^2y}{dx^2} + 9y = \tan 3x.$

7. (i) Form the partial differential equation of lowest order

(a) $zx = A\cos(x + y) + B\sin(x + y).$

(b) $xyz = f(x + y + z)$

(ii) Solve the following equations

(a) $x(y^2 - x^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

(b) $x\sqrt{a^2 - y^2 - z^2}\,dx + \log x\,(y\,dy + z\,dx) = 0.$

8. Find the eigen values and eigen vetors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

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Section B

Answer any **Five** of the following. $5 \times 8 = 40$

9. Find the following limits :

(i) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1 + x)}$

(ii) $\lim_{x \to 0} (\sin x)^{\tan x}.$

10. If $y = e^{-m \sin h^{-1} x}$, prove that

$(1 + x^2)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2 - m^2)\,y_n = 0.$

11. Show that the radius of curvature for the curve $y = a \log \sec \frac{x}{a}$ at $P(x_1, y_1)$ is $a \sec \frac{x_1}{a}$.

12. Find the area included by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

13. State Lagrange's theorem and verify it for $f(x) = x^3 - 5x^2 - 3x$ in [1, 3].

14. **Evaluate :**

$$\iint_S \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx\,dy \text{ where S is given by } \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1,\ x \ge 0,\ y \ge 0.$$

15. Solve the equations

(i) $(px+y)^2 + py^2$ *by* clairauts's method. Here $p = \frac{dy}{dx}$

(ii) $3p - 4q - z + \tan(4x + 3y)$ by Lagrange's method.

16. Solve $(x+y)^2 \frac{d^2y}{dx^2} + (x+y)\frac{dy}{dx} + 4y = x + 4$.

17. If $a + b + c = 2s$

Prove that $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$.

18. Using Matrix method, solve the following equations :

$x + y + z = 6, x - y + 2z = 5, 3x + y + z = 8.$

Ans : (1) 3, (2) 4, (3) 3, (4) 2

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Andhra Pradesh Open University

(Undergraduate Programmes)

SECOND YEAR SYLLABUS – 1990-91

MATHEMATICS – COURSE : 1

(Calculus, Differential Equations and Matrices)

Block – 1 : Differential Calculus – I

Unit 1 : Limits and Continuity
Unit 2 : Differentiation
Unit 3 : Mean Value Theorems
Unit 4 : Indeterminate forms, L – Hospitals rule

Block – 2 : Differential Calculus – II

Unit 5 : Partial Differentiation
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Block – 3 : Integration

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Block – 5 : Differential Equations – I

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Block – 6 : Differential Equations – II

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Unit 21 : Simultaneous differential equations

Unit 22 : Condition of integrability for $P\,dx + Q\,dy + R\,dz = 0$

Unit 23 : Total differential equations

Block – 7 : Partial Differential Equations

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Unit 25 : Solution of the Lagranges and other standard equations

Block – 8 : Determinants and Matrix Theory

Unit 26 : Determinants

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Unit 28 : Adjoint and inverse of a square matrix

Unit 29 : Rank and Linear equations

Unit 30 : Eigen values and Eigen vectors of a Matrix

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