PHYSICS

COURSE - I
MECHANICS
WAVES AND OPTICS

"We may forgo material benefits of civilization, but we cannot forgo our right and opportunity to reap the benefits of the highest education to the fullest extent..."

-Dr. B.R.Ambedkar

Dr. B.R.AMBEDKAR OPEN UNIVERSITY

HYDERABAD

2004

COURSE TEAM

COURSE DEVELOPMENT TEAM

Editor

Prof. S.V.Subrahmanyam

Associate Editors

Dr. K.Gnana Prasanna

D.V.N.Sharma

Writers

Prof. Anwar Ur Rahaman

Dr. N.Manohara Murthy

Dr. M.Mutha Reddy

Dr. K.S.N.Murthy

Dr.S.Raghavan

Cover design

Chandra

Graphics

Ramesh

COURSE REVISION TEAM

Editor

Prof. K.Gnana Prasanna

Writer

Dr. V.V.Subrahmanya Sarma

Coverdesign and Graphics

Ramesh

Dr. B.R.Ambedkar Open University, Hyderabad.
First Edition 1984
Second Revised Edition 1990
Third Revised Edition 2002 2003, 2004

This text forms part of Dr. B.R.Ambedkar Open University Course. The complete syllabus for the course appears at the end of the text.

Further information on Dr. B.R.Ambedkar Open University Courses may be obtained from The Director (Academic), Dr. B.R.Ambedkar Open University, Road No.46, Jubilee Hills, Hyderabad - 500 033.

Lr.No. 726/Dr. BRAOU/DMP/PTG/FO-15/JO.227/Copies : 1300, 2004-05 /Dated 14-7-2004
Printed at M/s. Shri Gopala Kwik Graphics, Uppal, Hyderabad - 39.

PREFACE

This book deals with the topics in Mechanics, Waves and Optics in the syllabus for the second year of the B.Sc., Course offered by Dr.B.R.Ambedkar Open University. These topics cover the core area of the subject to be studied in the second year of the three year Degree Course in Science. The syllabus is for the sake of convinience divided into Blocks, each of which comprises a number of units. Each unit generally covers a specific area of the subject. The units are prepared by specialists in accordance with a format so designed as to enable the student read and understand them without much difficulty. Each unit begins with a statement of its Aims and followed by the objectives to be achieved after going through the unit. Generally technical terms with which the student may not be familiar are given at the end of each block under the head Glossary.

In blocks 1 to 4 of the book dealing with Mechanics. It is attempted to explain the linear and rotational motion of the bodies and the forces operating on them. Gravitation and collisions form a part of these Blocks. Blocks 5 and 6 deals with waves and oscillations. Blocks 7, 8, 9 exclusively deals wth that branch of physics called optics. Block 10 deals with relativity. The univeristy hopes that this course material will help the students to get acquainted with the concepts and principles of physics ingeneral and Mechanics, Waves & Optics in particular.

CONTENTS

BLOCK -I

CONSERVATION OF LINEAR MOMENTUM

UNIT-1 : CENTRE OF MASS, MOTION OF CENTRE OF MASS, REDUCED MASS

Contents

1.1 AIMS AND OBJECTIVES

This unit explains the concept of mass and weight.

After going through this unit you will be able to

(1) distinguish between center of mass and center of gravity;

(2) distinguish between the inertial mass / and gravitational mass.

(3) describes the motion of system of particles using the concept of the centre of mass;

(4) recognise the importance of the concept of reduced mass of a system.

1.2 INTRODUCTION

The motion of a body can be described if we know the forces acting on it. In this unit we shall discuss the concept centre of mass and its motion in order to explain the conservation of linear momentum.

1.3 CONCEPT OF MASS

Sir Isaac Newton stated the first law of motion as: Every body persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it. The fact that bodies do not change their state (of rest or of uniform motion in a straight line) in the absence of applied forces is often described by assigning a property called inertia to matter. Newton's first law is also called as the law of inertia. The law of inertia applies to all objects. However, there is a difference in behaviour between 'heavy' and 'light' bodies. Experience tells us that heavy bodies require more force to change their state (of rest or of uniform motion in a straight line) while light bodies require less force. This property of the body is given the name mass or more precisely the inertial mass.

1.4 MASS AND WEIGHT

We are also familiar with the word 'weight' of a body. It is the force with which the body is attracted by the earth due to gravity. The weight of a body is related to its mass, both are not one and the same.

The gravitational force acting on a body defined as weight W is given by

$$W = mg \qquad (1.1)$$

where m is the mass and g is the acceleration due to gravity. Eqn. 1.1 may be written as

$$\frac{W}{m} = g \qquad (1.2)$$

The value of 'g' is the same for all bodies at any one place irrespective of their masses. Eqn. (1.2) hence indicates that the weight of a body is directly proportional to its mass at any one place. It was Galilo who at first established the direct proportionality between weight and mass.

Check your progress 1

The weight of a body changes from place to place why?

The mass we have discussed so far is called the inertial mass.

One way to determine inertial mass is to apply a known amount of force to the standard body (whose mass 'm" we take as equal to unity) and also to the body whose inertial mass 'm' is to be determined. Then the ratio of the acceleration produced in the standard body to the acceleration produced in the body under consideration gives the inertial mass of the body in question.

This is because, according to Newton's Law

$$f \propto m, a, \text{ or } f = kma$$

Similarly

$$f \propto m_1 a_1 \text{ or } f = k m_1 a_1$$

Where k is proportionality constant in both the equations.

Deviding one by the other,

$$\frac{m_1 a_1}{ma} = 1 \text{ or } m_1 = \frac{a}{a_1} \text{ since m is taken as unity}$$

Let us suppose the inertial masses of two bodies are m_A and m_B. According to Newton's law of universal gravitation every body attracts every other body in the universe with a force directly proportional to the product of "gravitational masses" of the bodies and inversely proportional to the square of distance between them.

We have to note that the gravitational masses of the bodies are something different from the inertial masses of bodies which we have determined by applying a constant force and measuring accelerations. Instead of the word "gravitational mass" if we use something like "gravitational charge" for that instrinsic property of bodies producing gravitational attraction the distinction would have been much more clear.

Let the gravitational masses of two bodies A and B be μ_A and μ_B situated at a distance of R from Then if the gravitational mass of the earth is μ_E then according to Newton's universal law of gravitation, the gravitational attractive force between A and Earth at the surface of earth is

$$F_A = W_A = G\frac{\mu_A \mu_E}{R^2}$$

This is what is called the weight W_A of the body. A Similarly the weight of the body B is

$$F_B = W_B = G\frac{\mu_B \mu_E}{R^2}$$

Then the acceleration with which these two bodies fall towards the earth are

$$g_B = \frac{W_A}{m_A} = \frac{G\mu_E}{R^2}\frac{\mu_A}{m_A}$$

$$g_B = \frac{W_B}{m_B} = \frac{G\mu_E}{R^2}\frac{\mu_B}{m_B}$$

As we have already discussed in 1.2 and experimentally determined peculiar fact is

$$g_A = g_B$$

This can be the case only if

$$\frac{\mu_A}{m_A} = \frac{\mu_B}{m_B}$$

or

$$\frac{\mu_A}{\mu_B} = \frac{m_A}{m_B}$$

i.e., the gravitational and inertial masses are proportional to each other.

This experimental proportionality of inertial mass to the gravitational mass is the basis for our determination of inertial mass by a balance.

This experimental proportionality is also the basis for Einstein to regard gravitation as a manifestation of inertia and enunciate principle of equivalence.

Let us suppose that we have two particles of masses. m_1 and m_2 on the x-axis at a distance of x_1 & x_2 from the origin. Then the center of mass of the two particles is defined as

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad (1.3)$$

If the two particles are in the XY-plane with coordinates (x_1, y_1) and (x_2, y_2) the coordinates of center of mass is defined by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

If there are more than two particles distributed in space the coordinates of center of mass of these particles are defined by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$z_{cm} = \frac{m_1z_1 + m_2z_2 + ...}{m_1 + m_2 + ...} = \frac{\sum_i m_i z_i}{\sum_i m_i} \qquad (1.5)$$

These three equations can be combined to give the position vector of the center of mass

$$\overrightarrow{R_{cm}} = \left(\vec{i}\, x_{cm} + \vec{j}\, y_{cm} + \vec{k}\, z_{cm}\right) = \frac{\sum_i m_i\left(\vec{i}\, x_i + \vec{j}\, y_i + \vec{k}\, z_i\right)}{\sum_i m_i}$$

$$= \frac{\sum_i m_i \vec{r_i}}{\sum m_i} \qquad (1.6)$$

Here $\sum_i m_i$ is the total mass of the system M, Hence we can write

$$M\vec{R}_{cm} = \sum_i m_i \vec{r_i} \qquad (1.7)$$

If we differentiate (1.7) and put $\frac{d\vec{c_i}}{dt} = \vec{V_i}$ and $\frac{d\overrightarrow{R_{cm}}}{dt} = \overrightarrow{V_{cm}}$ we get

$$M\overrightarrow{V_{cm}} = \sum_i m_i \vec{v_i} \qquad (1.8)$$

The right hand side of the equation 1.8 is the total lincar momentum of all the particles and is called the total momentum of the syatem. The equation 1.8 indicates that the total momentum of the system is equal to the total mass of the system M multiplied by the velocity of the center of mass.

Differentiating once again and writing $\frac{d\vec{v_i}}{dt} = \vec{a_i}$

and $\frac{d\overrightarrow{v_{cm}}}{dt} = \overrightarrow{a_{cm}}$ we get

$$M\overrightarrow{a_{cm}} = \sum_i m_i \vec{a_i} \qquad (1.9)$$

By Newton's second law $m_i\vec{a_i} = \vec{F_i}$ the force acting on the ith particle. The right hand side of equation (1.9) is therefore the resultant force acting on all the particles or the resultant force acting on the system. Now we can write

$$M\overrightarrow{a_{cm}} = \sum_i \overrightarrow{F_{ie}} + \sum_{i \neq j} \overrightarrow{F_{ij}}$$

Where F_{ie} is the external force acting on the ith particle and F_{ij} is the force of interaction (gravitational, electrical contact etc.) on the ith particle due to jth particle. This F_{ij} will be equal and opposite to the force F_{ji} acting on jth particle due to the ith particle (which occurs in the expression for F_{j}). Thus the forces of interaction between particles in the system occur in equal and opposite pairs and hence cancel each other when the resultant force on the system is calculated. If we denote the resultant force acting on the system by $\vec{F}$, then

$$\vec{F} = \sum_i \vec{F_i} = \sum_i \overrightarrow{F_{ie}} = \overrightarrow{F_e}$$

Where F_e is the resultant external force acting on the system.

$$M\overrightarrow{a_{cm}} = \overrightarrow{F_e} \tag{1.9a}$$

Equation (1.9a) indicates that the center of mass has an acceleration as though the total mass of the system is concentrated at it and the resultant external force is applied to it.

1.5 CENTRE OF MASS

If we have a system consisting of many particles which interact with each other and also subject to external forces then a complete description of the motion of the particles composing the system is not simple. However the problem can be simplified by replacing the system of particles by a single equivalent particle. utilising the concept of the centre of mass. Thus if a system consists of n particles, then there exists one point in it which behaves as though the entire mass of the system is concentrated at that point, and if the external forces acting on the system were applied directly to it. This point is calles the centre of mass of the system. The concept of the centre of mass allows us to treat relatively complicated motion in a simple way.

1.6 CENTRE OF MASS OF SYSTEM OF PARTICLES

Worked Example-I :

Two isolated bodies of masses 4kg. and 2kg are moving with velocities $4ms^{-1}$ and $2ms^{-1}$ respectively and are located as shown in Fig. 1.1 at certain instant of time. (a) Find the position and velocity of the centre of mass at the initial time. (b) Calculate the position of the centre of mass and its velocity after 10 sec.

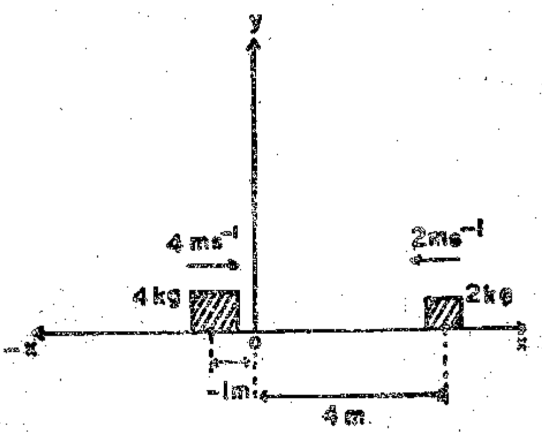


Fig. 1.1 Motion of isolated masses along x-direction.

Solution :

Let the initial position of the centre of mass be at $(x_{cm})_i$ and its velocity be $(v_{cm})_i$, then

$$(x_{cm})_i = \frac{\sum_{j=1}^{n} m_j x_j}{\sum_{j=1}^{n} m_j} = \frac{2(4)+4(-1)}{2+4} = 2/3\text{m}.$$

$$(V_{cm})_i = \frac{\sum_{j=1}^{n} m_j v_j}{\sum_{j=1}^{n} m_j} = \frac{2(-2)+4(4)}{2+4} = 2\text{ms}^{-1}$$

Intially the centre of mass is located at 2/3 m from the origin and it was then moving to the right with a velocity of 2 ms^{-1} . Because the system is isolated the centre of mass moves to the right at this constant speed even after the collision.

Thus after 10s, the centre of mass would have undergone a displacement Δx given by

$$\Delta x = (v_{cm})_i \text{ X } \Delta t = 2\text{ms}^{-1} \text{ x } 10\text{s} = 20 \text{ m}$$

Therefore the position of the centre of mass after 10s is given by

$$(V_{cm})_f = (V_{cm})_i + \Delta x = 2/3\text{m} + 20\text{m} = 20\tfrac{2}{3}\text{m}.$$

Thus we can easily predict where the centre of mass must be at any instant of time after collision, even though it is difficult to say where the isolated particles are located or what their velocities are.

1.7 CENTRE OF MASS OF A RIGID BODY

The centre of mass of a rigid body can be determined by subdividing the body into a large number of small elements of mass Δm located at the points x_i, y_i, z_i The coordinates of the centre of mass are given by

$$x_{cm} = \frac{\Sigma \Delta m_i x_i}{\Sigma \Delta m_i} \qquad (1.10)$$

$$y_{cm} = \frac{\Sigma \Delta m_i y_i}{\Sigma \Delta m_i} \qquad (1.11)$$

$$z_{cm} = \frac{\Sigma \Delta m_i z_i}{\Sigma \Delta m_i} \qquad (1.12)$$

If the number of mass elements tend to infinity, then the smmation in the above expression can be replaced by integrations.

Thus $$x_{cm} = \underset{\Delta m_i \to 0}{\text{Lt}} \frac{\sum \Delta m_i x_i}{\sum \Delta m_i} = \frac{\int x \, dm}{\int dm} = \frac{1}{M}\int x dm \qquad (1.13)$$

$$y_{cm} = \underset{\Delta m_i \to 0}{Lt} \frac{\sum \Delta m_1 y_i}{\sum \Delta m_1} = \frac{\int y\, dm}{\int dm} = \frac{1}{M} \int ydm \quad (1.14)$$

$$z_{cm} = \underset{\Delta m_i \to 0}{Lt} \frac{\sum \Delta m_1 z_1}{\sum \Delta m_i} = \frac{\int zdm}{\int dm} = \frac{1}{M} \int zdm \quad (1.15)$$

In vector notation

$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r} dm \quad (1.16)$$

For homogeneous objects having a point, a line or a plane of symmetry, the centre of mass will lie at the point, on the line and in the plane of symmetry. For example the centre of mass of a homogeneous sphere lies at the centre of the sphere. The centre of a rectangular plane sheet lies on the plane of symmetry and is given by the intersection of its planes of summetry. The poisition of centre of mass of some symmetrical structures is shown in Fig. 1.2

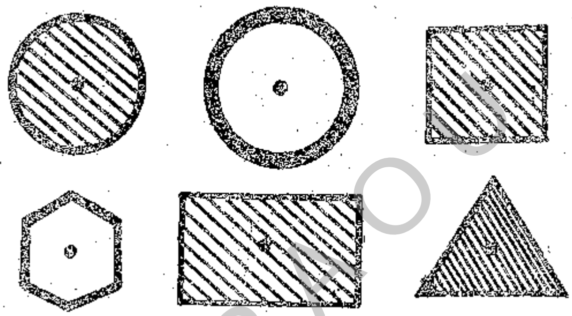


Fig. 1.2 Centre of mass of symmetrical structures

1.8 REDUCED MASS

A small mass m coupled by spring of spring constant k to a rigid support is a simple harmonic oscillator. This is a two body system in which the rigid support serves as a body with infinite mass. When the mass 'm' attached to the spring executes oscillations the rigid support remains at rest in an inertial reference frame and the potential energy U(x) of the oscillating system is a function of the displacement x of the mass alone.

In nature we find two-body oscillating systems in which we can not consider the mass of one of the bodies to be infinite. Molecules such as H_2, CO, HCL etc. are examples of this type. They oscillate along with their axis of symmetry. We have to consider the motions of both bodies in an appropriate inertial reference frame.

Consider two bodies m_1 and m_2 connected by massless spring of force constant k as shown in fig 1.3. The system is free to oscillate on a frictionless horizontal surface. Let $x_1(t)$ and $x_2(t)$ represent the coordinates of the end position of the spring. Centre of gravity of a homogeneous sphere, cube, circular disk, rectangular plate etc., lie at their centres. The centre of gravity of a cylinder lies on the axis of the cylinder.

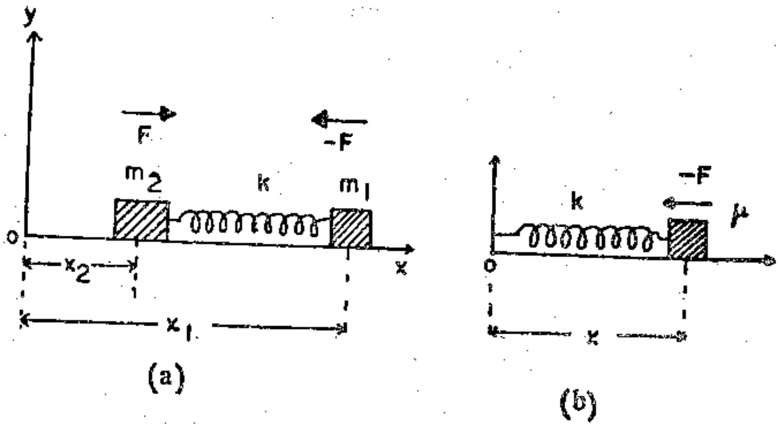


Fig 1.3 a) Masses m_1 and m_2 connected by a spring.
Fig 1-3b) one body system with reduced mass connected by a spring to a fixed point.

The length of the spring at any instant is $(x_1 - x_2)$. If the unstretched length of the spring is I, then the change in the length of the spring is

$$x=(x_1 - x_2) - 1 \tag{1.17}$$

If x is positive the spring is stretched, if x=0 the spring has its normal length and if x is negative it is compressed. Let the spring be stretched i.e., x>0.

Applying Newton's second law to the mass m_1 and m_2 we get

$$m_1 \frac{d^2 x_1}{dt^2} = -kx \tag{1.18}$$

$$m_2 \frac{d^2 x_2}{dt^2} = +kx \tag{1.19}$$

Multiplying Eqns. (1.18) by m_2 and (1.19) by m_1 and subtracting we get

$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_2 kx - m_1 kx \tag{1.20}$$

Rewriting

$$\left[\frac{m_1 m_2}{m_1 + m_2}\right] \frac{d^2}{dt^2}(x_1.x_2) = -kx \tag{1.21}$$

We now designate $\left[\frac{m_1 m_2}{m_1 + m_2}\right]$ as reduced mass of the system μ which had dismensions of mass. μ is less than the smaller of the two masses and hence the name reduced mass.

Since I is constant

$$\frac{d^2}{dt^2}(x_1 - x_2) = \frac{d^2 x}{dt^2} \tag{1.22}$$

Substituting Eqn. (1.21) in Eqn. 1.20 we get

$$\frac{d^2x}{dt^2}+\frac{k}{\mu}x=o \tag{1.23}$$

This is the equation representing the motion of one body system as depicted in Fig. 1.3b If $m_1 = m_2 = m$ then we get μ=m/2 if one of the masses of the system is very small then the reduced mass of the system corresponds to that of the smaller mass. If m_1 is very small

$$\text{i.e.,} \quad \mu=\frac{m_1m_2}{m_1+m_2}=m_1\left[\frac{1}{\left(\frac{m_1}{m_2}+1\right)}\right] \tag{1.24}$$

$$\text{or } \mu=m_1\left(1+\frac{m_1}{m_2}\right)^{-1}\cong\left(1-\frac{m_1}{m_2}\right) \tag{1.25}$$

Since m_1/m_2 is nearly equal to zero

$$\mu=m_1 \tag{1.26}$$

Worked Example-2:

Determine the centre of mass of three particles situated at the corners of a right angled triangle shown in Fig.1.4

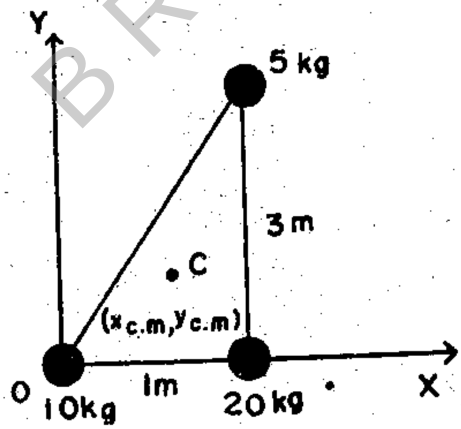


Fig.1.4 Disposition of 3 masses at the corners of a right angled triangle

Solution:

Let us choose the origin of the coordinate system at 10kg mass. [The Coordinates of 10kg, 20kg and 5kg are taken as (0, 0), (1, 0), (1, 3) respectively.] The x-coordinate of the centre of mass C of the system is given by

$$x_{cm}=\frac{10(0)+20(1)+5(1)}{10+20+5}=\frac{25}{35}=5/7$$

The y-coordinate of the centre of mass C of the system is given by

$$y_{cm} = \frac{10(0)+20(0)+5(3)}{10+20+5} = \frac{15}{35} = 3/7$$

The coordinates of the centre of mass C (Xcm, Ycm)=(5/7, 3/7)

Worked Example-3 :

Consider a system of three particles acted upon by external forces as detailed in Fig. 1.5 Find the acceleration of the centre of mass of the system.

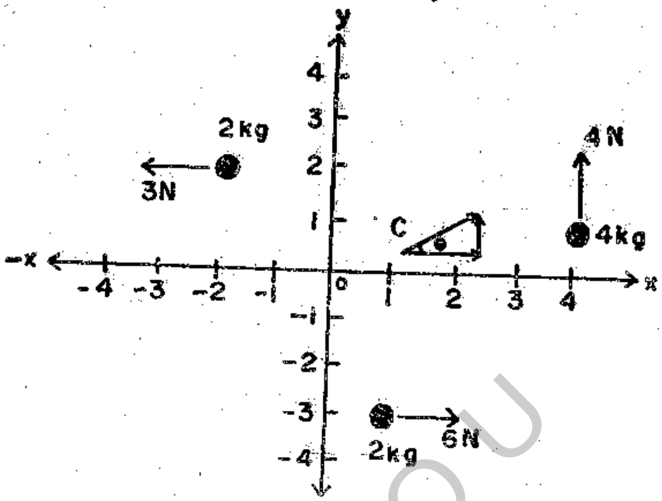


Fig. 1.5 A system of 3 particles acted upon by external forces.

Solution :

The Coordinates of the 4 kg, 2 kg and 2 kg masses are taken as (4, 1), (-2, 2) and (1, 3) respectively.

Then the coordinates of the centre of mass C are

$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{4(4)+2(-2)+2(1)}{4+2+2} = 7/4$$

$$y_{cm} = \frac{\Sigma m_i y_i}{\Sigma m_i} = \frac{4(1)+2(2)+2(-3)}{4+2+2} = 1/4$$

$$\therefore (X_{cm}, Y_{cm}) = 7/4, 1/4)$$

The resultant of the x-component forces acting on the system

$$F_x = 6N - 3N = 3N$$

The resultant of the y-comonent of the external forces acting on the system

$$F_y = 4N$$

The resultant external force has a magnitude

$$F = \left(F_x^2 + F_y^2\right)^{\frac{1}{2}} = \left(3^2 + 4^2\right)^{\frac{1}{2}} = (25)^{\frac{1}{2}} = 5N$$

The resultant force makes an angle θ with the x-axis given by

$$\text{Tan}\,\theta = \frac{F_y}{F_x} = 4/3$$

$$\therefore \theta = 53^\circ$$

The acceleration of the centre of mass is given by

$$a_{cm} = \frac{F}{M} = \frac{5N}{8\,Kg} = 0.65\ ms^{-2}$$

1.9 SUMMARY

Mass or the inertial mass is the intrinsic property of a body which resists any change in its state of rest or of uniform motion in a straight line under the influence of an external force. The weight of a body is the force acting on it due to earths gravity. The mass of a body remains constant where as the weight of a body varies with the acceleration due to gravity. There exists a strict proportionality between inertial mass and gravitational mass. The motion of system of particles can be studied by using the concept of centre of mass of the system. The position r_{cm}, velocity V_{cm} and acceleration a_{cm} of the centre of mass of a system of n particles can be represented by

$$r_{cm} = \sum_{i=1}^{n} m_1 r_i\, i \int \sum_{(i=1)}^{N} n_i$$

$$v_{cm} = \sum_{i=1}^{n} m_1 v_i \int \sum_{(i=1)}^{N} n_i$$

$$a_{cm} = \sum_{i=1}^{n} m_1 v_i\, a_i \int \sum_{(i=1)}^{N} ni$$

The reduced mass of a system of particles is always less than the masses of individual particles constituting the system.

1.10 MODEL ANSWERS

Check your Progress

The weight of a body changes from place to place because it depends upon the value of (g) the acceleration due to gravity which changes from place to place.

1.11 SAMPLE EXAMINATION QUESTIONS

I. Answer the following questions in about 30 lines.

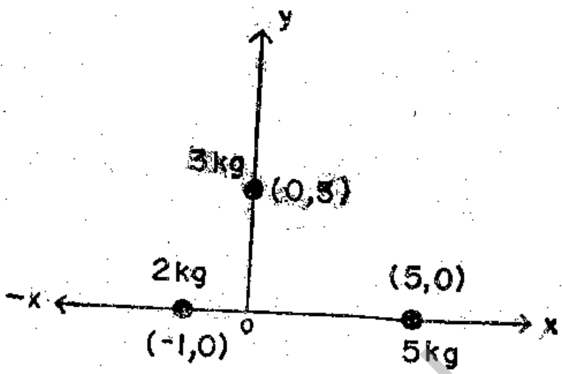
1. Define the term centre of mass of a system of particles. Derive expressions for the position, velocity and acceleration of centre of mass of the system acted upon by external

forces.

II. Solve the Following Problems

1. An object mass 5kg is moving with a velocity of $10ms^{-1}$ towards the right and another object of mass 10kg is moving with a velocity of $5ms^{-1}$ towards the left along a straight line. Determine the velocity of centre of mass. (Ans :X_{cm} =0).

2. Determine the centre of mass of the system of particles as shown below.



System of three particles

[Ans : (x_{cm} y_{cm}) = (2.3, 0.9)].

3. Locate the centre of mass of three particles of mass m_1=2kg, m_2=3kg and m_3=5kg located at the corners of an equilateral triangle of side 1m. Ans:(X_{cm} Y_{cm})=(0.55, $\sqrt{3/4}$)

4. Find the centre of mass of a system of three particles whose masses and position coordinates are given below:

m_1 = 6kg, x_1 = 4m, y_1 = 3m, z_1 = 4m

m_2 = 8kg, x_2 = 2m, y_2 = 1m, z_2 = 2m

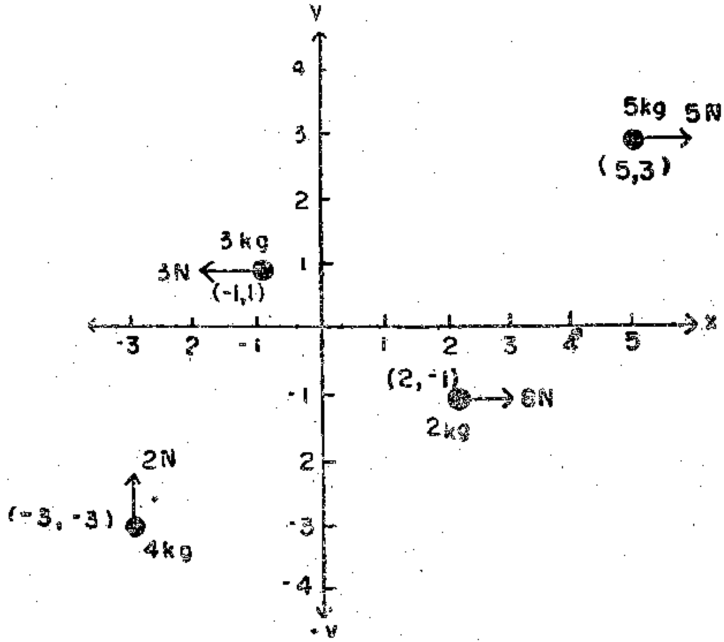
m_3 = 10kg x_3 = -5m, y_3 = 4m, z_3 = 2m

[Ans: x_m, y_m, z_m = 0.42, 2.75, 0.50].

5. Two particles of masses m_1 = 5 kg and m_2 = 10kg have the positrons r_1 = 2i+3j+5k and r_2 = 5i+3j+8k and velocity v_1 = 6i + 4j + 2k and v_2 = 10i-2J + 8k at any instant of time t. Determine the instantaneous position and velocity of the centre of mass of the system.

[Ans: v_{cm} = $(4i + 3j + 7k)_m$, V_{cm} = $(14i + 0j + 14k)_m$ s^{-1}]

6. A system of four particles are acted upon by external forces as detailed below. Determine the position of the centre of mass and its acceleration.



System of four particles acted upon by external forces.

[Ans: (x_{cm}, y_{cm}) = 1,2/7), $a_{cm} = 0.73ms^{-2}$)

7. Determine the centre of mass of HCL molecule separated by a distance 1.8×10^{-10}m. The atomic weights of Hydrogen and Chlorine atoms are 1 amu and 35.5 amu respectively. Determine the reduced mass of the system.

[Ans: 1.75 x 10 m from H atom, $\mu = 0.973$ amu]

8. Consider a system of four particles of masses $m_1 = 2kg$, $m_2 = kg$, $m_3 = 8kg$, and $m_4 = 10kg$ placed at the corners of a square of edge 5m, which is initially at rest. Determine the position of the centre of mass of the system. If an external force of 10 N acts on m_3 towards the right and a force of 5 N acts on m_1 towards the left. Determine the acceleration experienced by the centre of mass of the system.

[Ans: Taking m_1 along y-axis, m_2 in the xy plane m_3 along x axis and m_4 at the origin of the coordinate system, the coordinates of the centre of mass are

(13/5, 7/5); $a_{cm} = 2/5ms^{-2}$.

UNIT-2 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES. CONSERVATION OF LINEAR MOMENTUM.

Contents

2.1 AIMS AND OBJECTIVES

This unit explains the concept of linear momentum and its conservation. In order to make you understand, the concept it is shown that rate of change of linear momentum of a body represents the force acting on a body and that the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of centre of mass.

After going through this unit you will be able to understand the principle of conservation of linear momentum.

2.2 INTRODUCTION

In this unit we shall discuss the conservation of linear momentum. With help of the concept of centre of mass as discussed in Unit I.

2.3 LINEAR MOMENTUM OF A PARTICLE

The linear momentum $\vec{p}$ of a particle of mass 'm' moving with a velocity $\vec{v}$ relative to an observer in an inertial reference frame is defined as

$$\vec{p} = m\vec{v} \qquad (2.1)$$

The linear momentum is a vector quantity. Its direction is the same as the direction of the velocity of the particle.

As per Newton's second law, the force acting on a body of mass 'm' can be expressed as

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \qquad (2.2)$$

If the force acting on a body is zero, then the momentum of the body is constant. That is the linear momentum of an isolated particle remains unchanged with time.

2.4 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Consider a system of n particles having masses $m_1, m_2, m_3 \ldots m_n$. Let us assume that no mass enters or leaves the system. Hence the total mass $M = \sum mi$ remains constant with time. The particles of the system may interact with each other and external forces may also act on the system. Under these conditions each particle may have a velocity and hence a momentum. The system as a whole may have a momentum P which may be simply given by the vector sum of the momentum of individual particles.

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \ldots\ldots \vec{p}_n \quad ..(2.3)$$

or

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots . m_n\vec{v}_n \quad ..(2.4)$$

As per definition the velocity of centre of mass of a system of particles given by

$$\overrightarrow{v_{cm}} = \frac{\sum m_i \overrightarrow{v_i}}{\sum m_i} = \frac{\sum m_i \overrightarrow{v_i}}{M} \quad ..(2.5)$$

We have

$$\vec{P} = Mv_{cm} \quad ...(2.6)$$

The above equation indicates that the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

If F_e represents the resultant extreme force acting on the system.

$$\vec{F}_e = Ma_{cm} \quad ..(2.7)$$

Comparing Eqn. (2.6) with Eqn. (2.7) we have

$$F_{ext} = \frac{dp}{dt} \quad ...(2.8)$$

2.5 CONSERVATION OF LINER MOMENTUM

If the sum of the external forces acating on the system is zero.

$$\frac{d\vec{p}}{dt} = 0 \quad ..(2.9)$$

or

$$\vec{P} = \text{constant} \quad ...(2.10)$$

This result indicates that when the sum of the external forces acting on a system is zero, then the total linear momentum of the system remains unchanged. This result is termed as the conservation of linear momentum.

The principle of conservation of linear momentum has a wide range of applications and is important in the study of many collision processes and interactions. This law is considered to be the corner stone in the study of interactions in the realm of both classical and quantum mechanics.

The law of momentum conservation is so basic that it is worth while to state it in several forms namely (1)The total momentum of an isolated system in constant (2)When a body transfers momentum to a second body in collision, the loss of momentum of one body is balanced by the gain of momentum by the other.

Let us consider an example where the principle of conservation of momentum is applied in order to study the motion of a system. Consider the projectile motion of a bomb shell, shown in Fig. 2.1 which explodes at the point x_1 and the fragments of the shell are blown off in all directions. The only external force acting on the ystem is that of gravity which is constant. The forces responsible for explosion are all internal forces. These forces may change the momenta of all the individual fragments from the values they had when the fragments were confined to the shell, that is, before explosion. However as explained earlier, the internal forces do not change the total momentum of the system. Since the external force acting on the system is the same before and after explosion, the system of particles after explosion moves as a whole in such a way that the centre of mass of the fragments will continue to move in the parabolic trajectory that the unexploded shell would have followed. The trajectory of the centre of mass of the bomb shell after explosion is thus determined since the system obeys the principle of conservation of momentum.

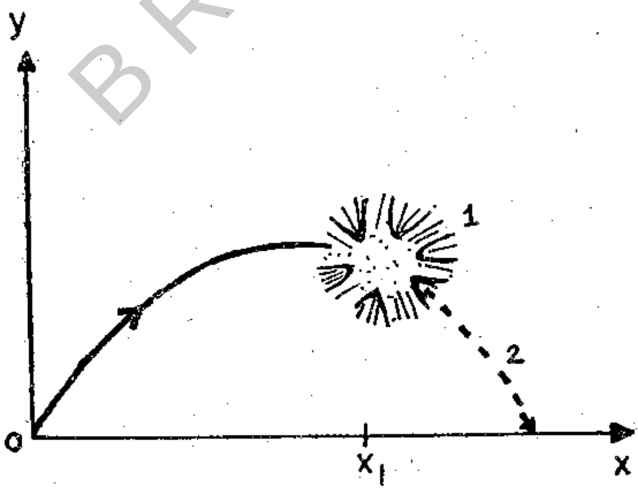
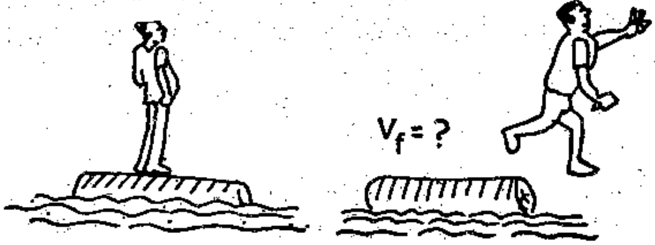


Fig. 2.1 Projectile motion of bomb shell

Worked Example-1:

A man weighing 50kg jumps horizontally from a log of mass 40kg floating on a lake with a speed of $5ms^{-1}$ relative to water. This causes the log to shood backward as show in fig. 2.2 Find the recoil speed of the log just after the man makes a jump.

Solution :



(a) (b)

Fig. 2.2 A man jumping from a stationary log floating on water.
(a) before jumping b) after jumping

Before the jump both the man and the log have zero momentum. After the jump the momentum gained by the man is

$$50 \text{ kg} \times 5 \text{ ms}^{-1} = 250 \text{kg ms}^{-1}$$

If v_f is the velocity of recoil of the log, then its momentum = 40 v_f kg. ms^{-1}

Applying the principle of conservation of momentum we have.

Total momentum before jump = Total momentum after jump.

$$0 = 250 \text{ kg ms}^{-1} + 40 \, v_f \text{ kg ms}^{-1}$$

$$v_f = -\frac{250}{40} = -6.25 \text{ m s}^{-1}$$

The negative sign of v_f indicates that the log suffers a recoil and moves in the direction opposite to the direction of the jump made by the man.

Worked Example-2 :

An alpha particle is emitted from uranium-238 nucleus orginally at rest with a speed of 1.5×10^7 ms^{-1}. Find the recoil speed of the residual nucleus, i.e., thorium-234.

Solution:

Uranium nucleus may be considered to be a bound system of two particles, namely, thorium and ' α – particles. This system which is initially at rest fragements into two separate parts. The momentum of the system before fragmentation must be equal to the momentum of the system after fragmentation. Since radioactive decay is not influenced by any external forces, we can supply the principle of conservation of momentum to the system. Thus

$$M_\alpha V_\infty + M_{th} v_{th} = 0$$

Where $M_\alpha, M_{th}, v_a, v_{th}$ represent the masses and velocities respectively of α and thorium particles.

$$v_{th} = \frac{-M_\alpha}{M_{th}} v_\alpha$$

$$= \frac{-4}{234}(1.5) \times 10^7 \cong -2.6 \times 10^5 \text{ ms}^{-1}$$

The negative sign indicates that the residual nucleus recoils in a direction exactly opposite to the motion of the α – particles.

Worked Example-3 :

A 0.2kg bullet moving with a velocity of 100 ms^{-1} hits a wooden block of mass 5 kg resting on a horizontl table, if the bullet sticks to the block, find the velocity of the wooden block after collision.

Solution :

Ignoring the frictional forces acting on the block by the table, we can apply the principle of conservation of momentum to the system.

The momentum of the system before collision=The momentm of the system after collision.

$$0.2\ kg \times 100\ ms^{-1} + 5kg \times 0\ ms^{-1}$$

$$= 5.2\ v_b$$

$$v_b = \frac{0.2 \times 100}{5.2} \cong 3.87\ m\,s^{-1}$$

The velocity of the wooden block after collision is 3.87 ms^{-1} The block moves along with the direction the bullet initally had before collision.

Worked Example 4:

A system of four particles of masses 5kg, 10kg, 15kg and 20kg x is in motion. Their velocities are 5ms^{-1}, 2ms^{-1}, 8ms^{-1} and 5 ms^{-1} respectively. No external force is acting on the system. After a time t if the velocities of first three masses are changed to 5ms^{-1}, 10ms^{-1}, 2ms^{-1}, find the velocity of the fourth mass and velocity of the centre of mass.

Solution:

Since no external force is acting on the system the principle of conservation of linear, momentum can be taken into consideration.

$$Pi = \sum_i m_i v_i$$

$$= 5\,(5) + 10(2) + 15(8) + 20(5) = 215ms^{-1}$$

After a time t, the momentum of the system of particles is

$$p_f = \sum m_i v_f = 5(5) + 10(10) + 15(2) + 20(V_f)$$

$$= 155kg\ ms^{-1} + 20V_f$$

But $\quad p_i = p_f \therefore 215kg\ ms^{-1} = 155kg,\ ms^{-1} + 20V_f$

$$\therefore v_f = 3ms^{-1}\ \text{Ans}$$

2.6 SUMMARY

The linear momentum P of a body of mass m moving with a velocity V is given by the product mv and it is a vector. The rate of change of momentum of a body represents the force acting on the body. The total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of centre of mass. The total linear momentum of the system remains constant when there are no external forces are acting on the system. This is known as the principle of conservation of linear momentum.

2.7 SAMPLE EXAMINATION QUESTIONS

I Answer the following Question in about 30 lines.

1. Derive an expression for the linear momentum of a system of particles in terms of the centre of mass of the system and the total mass of the system.

II Answer the following questions in about 10 lines.

1. Explain the term momentum of a body Show that the external force acting on a body is given by the time rate of change of momentum of the body.

2. Derive the principle of conservation of momentum for a system of particles. Discuss the importance of this conservation principle.

III Solve the following problems

1. A 5kg cart is moving from left to right with a velocity of $20ms^{-1}$ and strikes a 100kg cart at rest. After collision the 10kg cart is seen moving to the right with a velocity of $10\ ms^{-1}$. Find the final velocity of the cart. [Ans: 0]

2. A standard object of mass 5 kg moving with a velocity of $10ms^{-1}$ makes a headon collision with another object at rest. The two objects stick together after the collision and are observed to move with a velocity of $5\ ms^{-1}$. Find the mass of the second object. [Ans: m=5 kg].

3. A 100 kg hockey player moving at $5\ ms^{-1}$ makes a headon collision with a 75 kg hockey player moving in the opposite direction at $10ms^{-1}$. They stick together after collision. Determine their velocity after collision and their direction of motion.

   [Ans: 1.43 ms-1. Both the players move along the direction in which 75kg player is moving before collision].

4. An ice skater whose mass is 100kg and a 5kg stone are initially at rest on the ice. The ice skater pushes the rock which moves off at $20\ ms^{-1}$. The ice skater recoils backward as he pushes the rock. Determine the velocity of the ice skater [Ans: $1\ ms^{-1}$]

5. A 0.01 kg bullet is fired with a muzzle velocity of $500\ ms^{-1}$ which strikes a wooden block at rest. After collision the bullet sticks to the block and the wooden block is found to move with a velocity of $5\ ms^{-1}$. Determine the mass of the block. [Ans: 0.99kg]

6. A hunter fires 0.02 kg bullet at $500\ ms^{-1}$. If the hunter and the gun together weigh 100kg. Find the recoil velocity of the hunter in the absence of the friction. [Ans: $0.1\ ms^{-1}$]

7. The velocity of centre of mass of a system of 3 particles of masses 5kg, 10kg and 15kg is $50\ ms^{-1}$, If the velocity of 5kg particle is $50\ ms^{-1}$, find the momentum and the velocity of 15 kg particle. [Ans: p=1050kg ms^{-1},=70 ms^{-1}]

2.8 GLOSSARY

Uniform motion	:	A body is uniform motion travels equal distances in equal intervals of time.
Inertia	:	The intrinsic property of body that opposes any change in its state.
Parabolic motion	:	Motion of a particle along a parabola.
Trajectory	:	Path of a particle in motion.
Simple harmonic motion	:	A body is said to move with simple harmonic motion if it vibrates about its mean point such that the force acting on it is always directed towards the centre and is proportional to the displacement at any instant of time.

2.9 RECOMMENDED BOOKS

1.	Resnick, R. and Halliday, D.	Physics Part I	Wiley Eastern Pvt. Ltd. New Delhi.
2.	Sears, F.W. and Zemansky, M.W.	College Physics	Addision Wesley Publishing Co. Inc. London.
3.	Mathur, D.S.	Machanics	Vicks Publishing House Pvt. Ltd.
4.	Bueche, F.	Technical Physics	Harper and Row Publishers, New york.
5.	White, H.E.	Modern College Physics	East-West Press Pvt. Ltd. New Delhi.
6.	Zafiratos, C.D.	Physics	John Wiley and Sons Inc, NewYork.
7.	Taylor, L.W.	Physics The Pioneer Science, Vol.I	Dover Publications Inc., New York

BLOCK -2

ROTATIONAL DYNAMICS

UNIT-3 KINEMATICS

Contents

3.1 AIMS AND OBJECTIVES

This unit explains the rotational motion of rigid bodies confined to a fixed axis of rotation. In order to make you understand the concept a analogy between translation motion and rotational motion was brought out.

After going through this unit you will be able to define linear velocity, linear acceleration, angular velocity and angular acceleration and the relations between them.

3.2 INTRODUCTION

The word 'Kinesis' is a Greek word which means 'motion'. The word 'dynamics' means power. Kinematics is that branch of mechanics that 'describes' motion while dynamics 'explains' the motion. Before we go into details of kinematics of rotating bodies it would be worthwhile to recollect basic principles involved in kinematics of bodies in translatory motion. The following section deals with kinematics of bodies executing translatory motion.

3.3 KINEMATICS IN ONE DIMENSION

In mechanics the motion a body is described by assigning a property to it called particle nature. In nature we do not come across with bodies which have no size. Nevertheless, real objects may at times behave as though they were particles. A body need not be 'small' so as to be treated as a particle. For example the motion of planets round the sun can be described by treating them as particles. Any body which has only translatory motion behaves like a particle. An observer will consider the motion of a body to be translational if the axes of a reference frame which is imagined to be rigidly attached to the body say x^1, y^1 and z^1 remains always parallel to his own reference frame say x, y and z. An object making translatory motion is shown in Fig.3.1. As the body moves, at any position of the body say

A, B or C, the axes of reference frame attached to the body (x^1, y^1) always remains parallel to the axes of reference frame (x, y). The path taken by the body need not be a straight line. Every point of the body undergoes the same displacement as every other. We can treat the body to be a particle since by describing the motion of one point on the body we get the description of the motion of body as a whole.

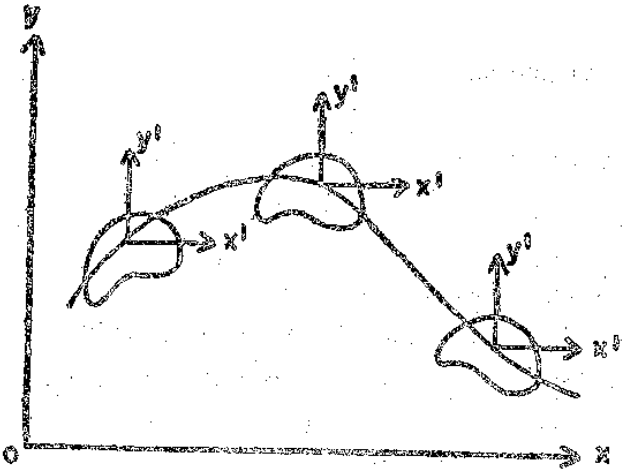


Fig. 3.1 Translatory motion of rigid body.

We shall now briefly describe the parameters characterizing translatory motion. Consider a particle moving in one dimension say along x-direction. A change in position $\Delta\vec{x} = \vec{x}_2 - \vec{x}_1$ is called displacement. The velocity of a particle represents the rate at which the position of the particle changes with time. If a displacement Δx occurs in a time $\Delta t = t_2 - t_1$ then the average velocity of the particle is given by

$$\bar{v} = \frac{\Delta\vec{x}}{\Delta t} \tag{3.1}$$

The instantaneous velocity of the particles can be given by

$$\vec{v} = \underset{\Delta t \to 0}{Lt} \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{dx}}{dt} \text{ or } \vec{dx} = \vec{v}\,dt \tag{3.2}$$

If we know the velocity of a particle, at any instant of time t, its position can be given by

$$x(t) = \int v dt + k_1 \tag{3.3}$$

When K_1 is the constant of integration which can be determined using the condition that

at $t = 0$, let $x(t) = x_o$

$$\therefore x_0 = k_1$$

Hence the above equation can be written as $x(t) = \int v\,dt + x_o$ (3.4)

If the velocity of a particle does not change with time i.e., $\vec{v} = \vec{v}_0 =$ constant, then the position of the particle at any instant of time t can be given by

$$X = \int v_o dt + x_o = v_o t + x_o \tag{3.5}$$

When the particle is executing translatory motion, the velocity of a particle may or may not be constant. If its velocity is changing with time, we say the particle is accelerating. The rate of change of velocity is difined as acceleration and is given by

$$\vec{a} = \frac{\vec{dv}}{dt} \tag{3.6}$$

If we know the acceleration of a particle its velocity at any instant of time t can be written as

$$\vec{v} = \int \vec{a}\, dt + k_2 \tag{3.7}$$

Where k_2 is the constant of integration which can be determined using the the condition that

at t=0, let $v=v_o$

$$\therefore v_o = k_2 \tag{3.8}$$

The average acceleration a is defined, as

$$a = \frac{\Delta v}{\Delta t}$$

An object with constant acceleration is said to be accelerating uniformly. Uniform acceleration of the body occurs when the force acting on the body is constant. A constant acceleration implies that the velocity of the particle is linear function of time. For uniform acceleration the velocity of the particle at any instant of time t can be expressed as

$$v = at + v_o \tag{3.10}$$

The displacement of a particle moving with constant acceleration is given by

$$x = \int v\, dt + C = \int (v_o + at)\, dt + C \text{ where C is the constnat of integration} \tag{3.11}$$

$$x = \int v_o\, dt + \int at\, dt + C = v_o t + \frac{1}{2} a t^2 + C \tag{3.12}$$

C can be determined using the condition that $\vec{v} = \vec{v}_0$ when t=o, let $x=x_o$ then $C=X_o$

Hence

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{3.13}$$

From Eqns. (3.10) and (3.13), we obtain the following relation which is a derived one.

$$v^2 = v_o^2 + 2 a d \tag{3.14}$$

Where d is the distance given by $d=x - x_o$

An object with free-fall undergoes constant accelerating $g=9.8ms^{-2}$. Then the velocity and displacement for a particle failing from rest at t=o are given by

$$x = \frac{1}{2} g t^2 \tag{3.15}$$

$$v^2 = 2 g d \tag{3.16}$$

If the acceleration of an object is not uniform then the velocity and displacement can be given by

$$\Delta v = v_2 - v_1 = \int_{t_2}^{t_2} a dt \tag{3.17}$$

$$\Delta x = x_2 - x_1 = \int_{t_1}^{t_2} v dt \tag{3.18}$$

3.4 ROTATIONAL MOTION

Rotational motion is an important characteristic of most of the machinery used in the present day world. We cannot come across any machine without a gear, wheel or rotating shaft. The simple cart makes use of the rotational motion of the wheel for its translatory motion. The most general type of motion which a rigid body can undergo is a combination of translation and rotation. To describe the motion of a body which rotates as ti moves we have to specify its position and its orientation with respect to the reference frame. Consider the motion of a rigid body shown in Fig 3.2

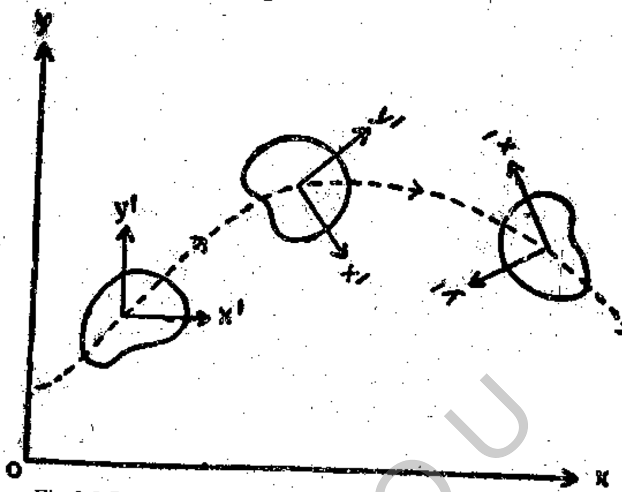


Fig 3.2 Rotational and Translational motion of a rigid body.

The reference frame x^1 y^1 fixed on the body changes its orientation with respect to x, y as the body moves. The body rotates as it moves. To locate the body we have to specify the coordinates of the point O on the body and also specify the orientation of x^1 y^1 reference frame with respect to x, y inertial reference frame.

Plane rotation is associated with bodies which rotate about a fixed axis i.e., without any translation as shown in Fig.3.3

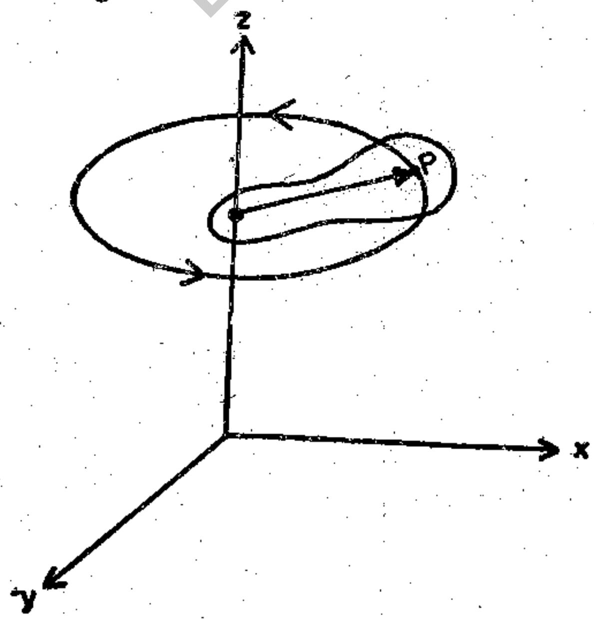


Fig.3.3. Rotation of a rigid body about z-axis.

Here the body rotates about z-axis. Every particle on the rigid body, say P, moves in a circle and the centres of the circular trajectory traced by the particles on the rigid body fall on the straight line oz.oz is called the axis of rotation. We shall now describe the rotation of rigid bodies by defining the variables characterizing the motion. The description of rotation is called rotational kinematics.

3.5. ROTATIONAL KINEMATICS

3.5.1 Angular Velocity

Consider the rotation of a rigid body of some orbitrary shape about a fixed axis is through the point 0 and perpendicular to the plane the figure, as shown in Fig.3.4.

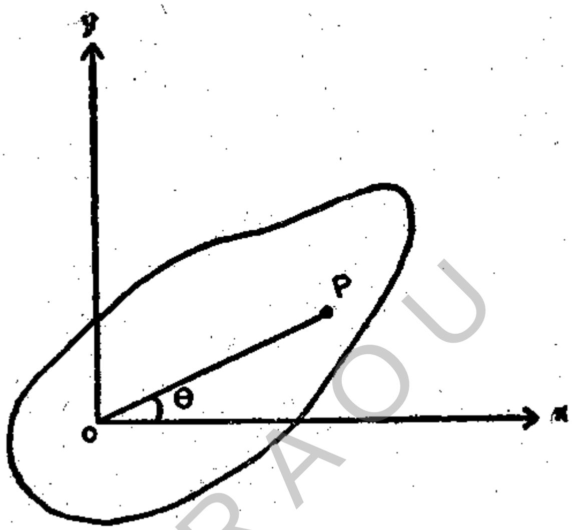


Fig.3.4. Rotation of a rigid body about a fixed axis through the point 0.

The line OP is fixed with respect to the body and rotates with it. The position of the body is given by the angle θ which the line OP makes with some reference line fixed in space such as OX. The equations of motion of the rigid body executing pure rotation can be simplified by defining the angle θ in radians. One radian is the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle as represented in Fig.3.5

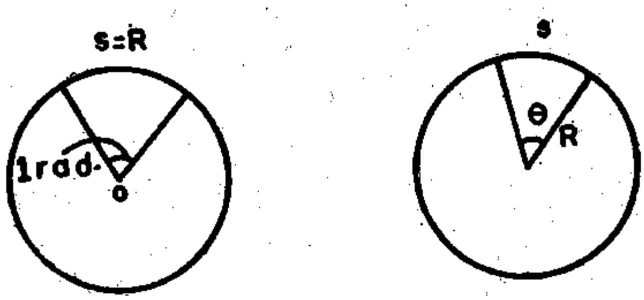


Fig.3.5. Definition of a radian.

If θ represents any orbitary angle subtended by an arc of lengths on the circumference of a circle of radius R as shown in Fig.3.5, then

$$\theta = \frac{s}{R} \text{ radians} \tag{3.19}$$

Let the reference line OP in the rotating body make an angle θ_1 with reference to the line ox at time t_1 and let θ_2 be the angle made by OP after sometime t_2 as shown in Fig.3.6

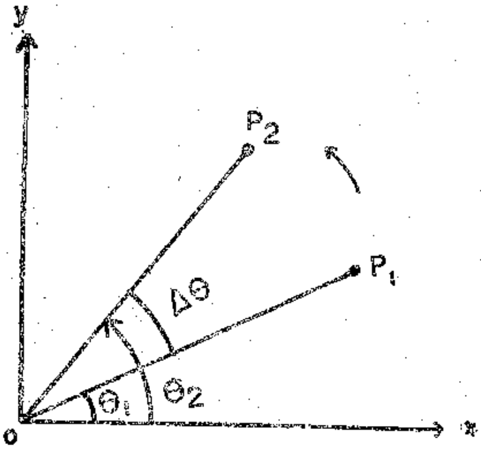


Fig.3.6 Angular displacement of a rigid body under rotation.

The average angular velocity (i) is given by

$$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \qquad (3.20)$$

The instantaneous angular velocity, ω is given by

$$\omega = \underset{\Delta \to 0}{\mathrm{Lt}} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (3.21)$$

Since the body is rigid, any line joining the origin with any point on the body rotates through the same angle in same time. Hence the angular velocity is characteristic of the body as a whole. If θ is in radians (i) is expressed as radians per second.

The following convention is adopted to define the direction of $\vec{\omega}$. If the fingers of the right hand curl round the axis in the direction of rotation of the body, the extended thumb point along the direction of the angular velocity vector.

Check your Pogress-1

One radian is a.............

Worked Example-1:

A wheel makes 5 full revolutions in 1/2s. Find average angular velocity.

Solution:

$$\left.\begin{array}{l}\text{Angular displacement in}\\ 1/2\text{s}\end{array}\right\} \theta = 5(2\pi) = 10\pi \text{ rad.}$$

Average angular velocity

$$\omega = \frac{\theta}{t} = \frac{10\pi\text{rad}}{\frac{1}{2}\text{s}}$$

$$\omega = 20\pi \text{rads}^{-1}$$

3.5.2 Angular Acceleration

When the angular velocity of a body changes with time then the body possess angular acceleration. If ω_1 and ω_2 represent the instantaneous angular velocities of the body at times t_1 and t_2 respectively, then the average acceleration α is given by

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad (3.22)$$

The instantaneous angular acceleration α may be defined as

$$\alpha = \underset{\Delta t \to 0}{Lt} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad (3.23)$$

Angular acceleration is expressed as rad. s^{-2}

3.6 ROTATION WITH CONSTANT ANGULAR ACCELERATION

The analogy between translatory motion and rotational motion can be understood by considering the rotational motion characterized by constant angular acceleration. Let the angular velocity of the body change by equal amounts in equal intervals of time i.e., the body is moving with constant angular acceleration.

Hence

$$\frac{d\omega}{dt} = \alpha = \text{cons tan t} \qquad (3.24)$$

$$d\omega = \alpha\, dt \qquad (3.25)$$

Integrating on either sides

$$\int d\omega = \alpha \int dt \qquad (3.26)$$

$$\omega = \alpha t + C_1 \qquad (3.27)$$

where c_1 is the constant of integration

To evaluate the constant of integration C_1 let $\omega = \omega_\circ$ at t=o (Initial Conditions)

Then $\qquad C_1 = \omega_\circ$

Hence $\qquad \omega = \omega_\circ + \alpha t \qquad (3.28)$

To obtain the angular displacement θ of the body with respect ot OX axis at any instant of time t we rewrite Eqn. 3.28 in a different way as

$$\frac{d\theta}{dt} = \omega_\circ + \alpha t \qquad \text{since } \omega = \frac{d\theta}{dt} \qquad (3.29)$$

$$d\theta = \omega_\circ\, dt + \alpha t\, dt \qquad (3.30)$$

Integrating

$$\int d\theta = \omega_\circ \int dt + \alpha \int t\, dt \quad (3.31)$$

$$\theta = \omega_\circ\, t + \frac{\alpha t^2}{2} + C_2 \quad (3.32)$$

Where C_2 is the constant of integration. The constant of integration C_2 can be evaluated by considering $\theta = 0$ at $t = 0$. Then $C_2 = 0$

Hence

$$\theta = \omega_\circ\, t + \frac{\alpha t^2}{2} \quad (3.33)$$

We can write

$$\alpha = \frac{d\omega}{dt}$$

or

$$\alpha = \frac{d\omega}{d\theta}\left(\frac{d\theta}{dt}\right) = \omega \frac{d\omega}{d\theta} \qquad \text{since} \quad \omega = \frac{d\theta}{dt} \quad (3.34)$$

$$\alpha\, d\theta = \omega\, d\omega \quad (3.35)$$

Integrating on either sides

$$\alpha \int d\theta = \int \omega\, d\omega \quad (3.36)$$

$$\alpha\theta = \frac{\omega^2}{2} + C_3 \quad (3.37)$$

Where C_3 is the constant of integration. It can be determined assuring the initial conditions as when $t = 0$ let $\theta = 0, \omega = \omega_\circ$ then

$$C_3 = -\frac{1}{2}\omega_\circ^2 \quad (3.38)$$

$$\text{Hence } \alpha\theta = \frac{\omega^2}{2} - \frac{\omega_\circ^2}{2} \quad (3.39)$$

$$\omega^2 = \omega_\circ^2 + 2\,\alpha\theta \quad (3.40)$$

3.7 ANALOGIES BETWEEN TRANSLATIONAL AND ROTATIONAL KINEMATICS

There exists close analogy between the parameters characterizing rotational motion with constant angular acceleration and those of translatory motion with constant accelaration of a rigid body. This is illustrated in Table 1.

Table 1: Comparison of Parameters Characterizing Rotational Translational Kinematics of a Rigid Body

Rotational Kinematics	Translational Kinematics
$\omega = \frac{d\theta}{dt}$	$v = \frac{dx}{dt}$
$\alpha = \frac{d\omega}{dt}$	$a = \frac{dv}{dt}$
$\Delta\omega = \int_1^2 \alpha\, dt$	$\Delta a = \int_1^2 a\, dt$
$\Delta\theta = \int_1^2 \omega\, dt$	$\Delta x = \int_1^2 v\, dt$
For the special case of constant acceleration	
$\alpha = \text{const.}$	$a = \text{const.}$
$\omega = \omega_o + \alpha t$	$v = v_o + at$
$\theta = \omega t + \frac{1}{2}\alpha t^2$	$x = v_o t + \frac{1}{2} a t^2$
$\omega^2 = \omega_o^2 + 2\alpha\theta$	$v^2 = v_o^2 + 2ax$
$\theta = \frac{\omega_o + \omega}{2} t$	$x = \frac{v_o + v}{2} t$

3.8 RELATION BETWEEN LINEAR AND ANGULAR KINEMATICS FOR A PARTICLE IN CIRCULAR MOTION

When a rigid body rotates about a fixed axis, every particle in the body moves in a circle. Hence we can describe the motion of such a particle either as linear variable or angular variables. The relation between the linear and angular variables enables us to change from one description to the other as and when required and are useful in the study of dynamics of particles executing rotational motion.

Consider a point P in the rigid body which is at a distance r from the origin of the coordinate system as shown Fig. 3.7 When the body rotates about the fixed axis say OY Perpendicular to the plane of the paper the particle P moves in a circle of radius r.

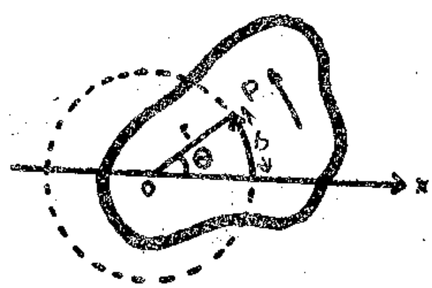


Fig. 3.7 Rotation of a rigid body about y-axis. s=r θ

Let the particle move through an angle θ.

Then

$$s = r\,\theta \tag{3.41}$$

When the angle is very small say $\Delta\theta$ then length of the arc 'Δs' is also very small and the time Δt for the body to rotate through the angle $\Delta\theta$ is also small. Then we can write

$$\Delta s = r\Delta\theta \tag{3.42}$$

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \tag{3.43}$$

When $\Delta t \rightarrow O$ Eqn. (3.43) can be written as

$$\frac{ds}{dt} = r\frac{d\theta}{dt} \tag{3.44}$$

$$v = r\omega \tag{3.45}$$

Eqn. (3.45) relates the linear velocity of the particle in the rigid body with the angular velocity of the particle in the rigid body.

Differentiating Eqn. (3.45) with respect to time to we we get

$$\frac{dv}{dt} = r\frac{d\omega}{dt} \tag{3.46}$$

Here $\frac{dv}{dt}$ represents the magnitude of the tangenial component of acceleration of the particle and $\frac{d\omega}{dt}$ represents the magnitude of the angular acceleration of the rotating body.

We can rewrite Eqn. (3.46) as

$$a_r = r\alpha \tag{3.47}$$

Eqn. (3.47) indicates that the magnitude of the tangential component of linear acceleration of a particle in circular motion is the product of the magnitude of the angular acceleration and the distance of the particle from the axis of rotation.

The radial component of accleration of a particle moving in a circular path is given by

$$a_n = \frac{v^2}{r} \tag{3.48}$$

Since $\quad v = \omega r$

$$a_n = \omega^2 r \tag{3.49}$$

The tangenial and radial components of acceleration of any arbitrary point P in rotating body are shown in Fig 3.8

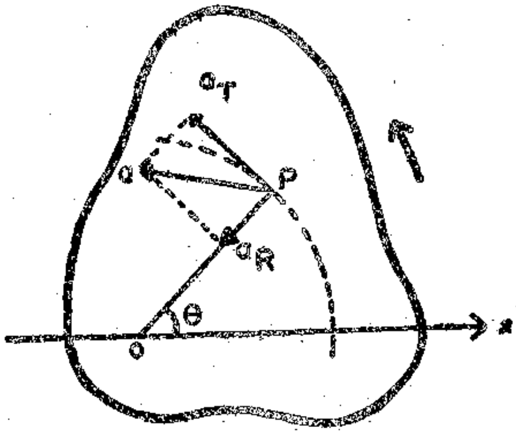


Fig. 3.8 Non-uniform rotation of a rigid body about a fixed axis through the point O.

Worked Example-2:

The angular velocity of a body is 5 rad s^{-1} at time t=0 and its angular acceleration is constant and equals to 4 rad s^{-2}. A line OP in the body is horizontal at time t = 0 which represents the axis of rotation. Find the angle made by OP with the horizontal after a time t = 5s. What is the angular velocity of the body at this time.

Solution:

The angular displacement

$$\theta = \omega_o t + 1/2 \alpha t^2$$

As per problem $\omega_o = 5 \text{ rad. s}^{-1}$

$$\alpha = 4 \text{ rad.s}^{-2}$$

$$t = 5s$$

Hence

$$\theta = 5 \text{ rad.s}^{-1} \text{ x } 5s + \frac{1}{2}.4 \text{ rad s}^{-2}.(5s)^2$$

$$= 75 \text{ rad} \qquad = 11.93 \text{ rotations}$$

The angular velocity of the body

$$\omega = \omega_o + \alpha t$$
$$= 5 \text{ rad.s}^{-1} + 4 \text{ rad.s}^{-2}.5s$$
$$\omega = 25 \text{ rad s}^{-1}.$$

Worked Example-3:

A wheel spins at 6000 rev/min. and moves with constant angular acceleration of 2 rad.s^{-2}. Find the angular displacement after a time t = 3 sec.

Solution:

We have

$$\theta = \omega_o t + \frac{\alpha t^2}{2}$$

$$n = \frac{6000 \text{ rev}}{60} = 100 \text{ rev/sec}$$

$$\omega_o = 2\pi \times n \text{ rad.s}^{-1}$$

$$\omega_o = 200\pi \text{ rad.s}^{-1}.$$

Hence

$$\theta = (200\pi \text{ rad.s}^{-1} \text{ X } 3\text{s}) + \frac{2 \text{ rad.s}^{-2} \text{ X } (3\text{s})^2}{2}$$

$$= 600\,\pi \text{ rad} + 9 \text{ rad}$$

$$\theta = \frac{600\text{X}22}{7} + 9 = 1884 + 9 = 1893 \text{ rad.}$$

Worked Example-4:

A grind stone has constant acceleration of 5 rad. s^{-2} as shown in Fig.3.9. Starting from rest a line such as OP is horizontal.

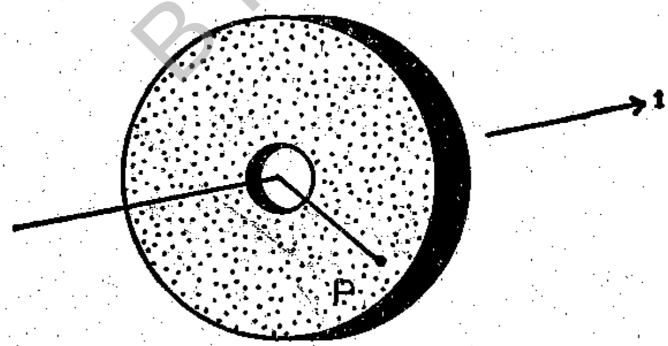


Fig.3.9 Rotation of point P on the surface of a grind stone.

Find the angular displacement of the line OP and the angular speed of the grind stone after 5s. If the radius of the grind stone is 0.4 m calculate the linear speed of the particle at the rim.

Solution:

We have

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2$$

$$\text{at } t = 0, \omega_o = 0$$

Hence

$$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2} X\, 5 \text{ rad. } s^{-2} (5s)^2 = 62.5 \text{ rad.}$$

The angular velocity

$$\omega = \omega_o + \alpha t$$

$$\theta = 0 + 5 \text{ rad.} s^{-2}\ X\ 5\ s = 25 \text{ rad. } s^{-1}$$

The linear velocity of the particle at the rim of the grind stone is V, then

$$V = r\omega = (0.4m)(25 \text{ rad } S^{-1}) = 10mS^{-1}$$

3.9 SUMMARY

The rotational motion of a particle can be described by its angular acceleration. The Angular Velocity (ω) of the rotating body is given by $d\theta / dt$ and its angular acceleration α by $d\omega / dt$. In circular motion the angular parameters are related to the translational parameters as

$$\theta = s/r \qquad \alpha = a/r$$

$$\omega = v/r$$

Rotational kinematic quantities obey relations analogous to those of translational kinematics when α is constant.

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2$$

$$\omega = \omega_o + \alpha t$$

$$\omega^2 = \omega_o^2 + 2\alpha\theta$$

$$\theta = \frac{\omega_o + \omega}{2} t$$

3.10 MODEL ANSWERS

Check your Progress-1

One radian is the angle subtended at the centre of circle by an arc of length equal to the radius of the circle.

3.11 SAMPLE EXAMINATION QUESTIONS

I. Answer the following questions in about 30 lines.

1. Derive an expression for the angular displacement of a rotating particle possessing constant angular acceleration.

2. Derive expressions for angular velocity and angular acceleration in terms of transitional velocity and acceleration for a rotating particle.

3. Obtain an expression relating these two parameters for a rotating system.

II. Answer the following questions in about 10 lines.

1. Bring out the analogy between translational kinematics and rotational kinematics.

2. How is moment of inertia related to the angular momentum of the body?

IV. Solve the following problems.

1. A wheel of 1m diameter starts from rest and accelerates uniformly to an angular velocity of 60 rad in 10 sec. Find the angular acceleration and angle turned through during this time.

(Ans. $\alpha = 6$ rad s^{-2}, $\theta = 300$ rad s^{-1})

2. A point on the rim of a wheel moves 4m when the wheel turns through an angle of 1/5 rad. Find the radius of the wheel.

(Ans. 20m).

3. A wheel is initially rotating with an angular velocity of 5 rad s^{-1}. After 10 sec. its angular velocity is found to be 15 rad s^{-2}. Find the average angular acceleration.

(Ams. 1 rad s^{-2})

4.A motor shaft is turning at 500 rev/sec. When the motor is turned off. It stops after 50s. Find the acceleration.

(Ans. 62.8 rad s^{-2})

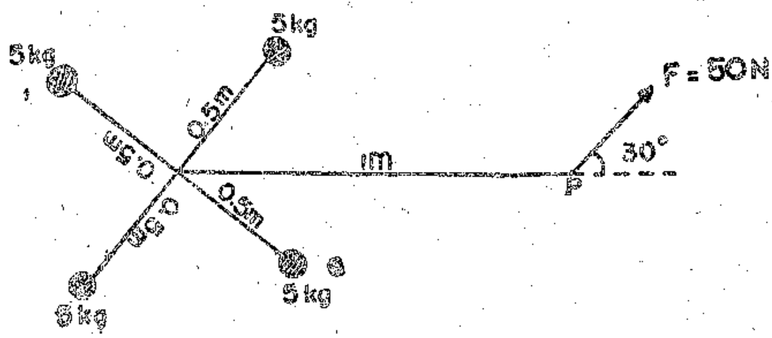
5. A wheels starts from rest and in 10 sec reaches a speed of 500 rev/sec. Find its angular acceleration.

(Ans. 314 rad s^{-2}).

6. A weight is falling with an acceleration of 60 ms^{-2} from a pully of radius 0.5m Find the angular acceleration of the wheel.

(Ans. 12 rad s^{-2}).

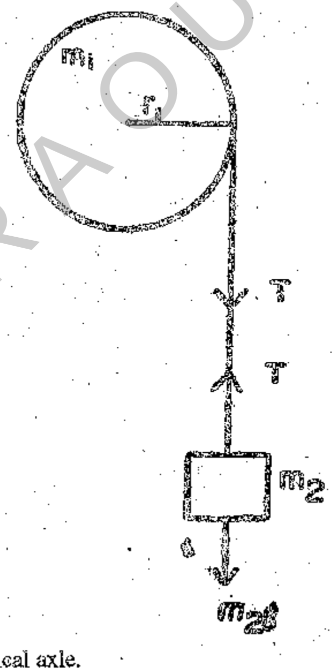
7. Four 5 kg masses are connected 0.5 m spokes to an axle as shown in Fig below. A force of 50N acts on a lever of 1m long to produce an angular acceleration α. Find its magnitude.



Four 5kg. masses connected by spokes to axle.

(Ans. 5 rad s^{-2})

8. A solid cylindrical fly wheel has a cord wrapped around its circumference from which is suspended a weight as shown in Fig. below. Find the angular acceleration of the fly wheel if it has a radius of 0.2m, a mass of 4 kg and the weight has a mass of 10 kg.



weight attached to the cylindrical axle.

(Ans. 40.85 rad s^{-2}).

UNIT 4 TORQUE AND ROTATIONAL MOTION

Contents

4.1 AIMS AND OBJECTIVES

This Unit discusses the effect of the torque on rotating bodies and moment of inertia of rotating bodies. In order to explain the concept (1) it is mathematically proved that the time rate of change of angular momentum of a particle is equal to the torque; (2) the moment of Inertia of rotating bodies about fixed axis is evaluated.

After going though this Unit you will be able (1) to define the torque, and moment of inertia of rotating bodies; (2) evaluate the moment of inertia of a body roatating about any given axis by using parallel axis and perpendicular axis theorems.

4.2 INTRODUCTION

The kinematics of rotating bodies have been discussed in unit 3. The rotational motion of extended objects can be described in terms of their angular acceleration. In this unit we shall study the nature of forces responsible for the rotation of bodies. Such a study is called rotational dynamics.

4.3 TORQUE ACTING ON A PARTICLE

There exists a close analogy between translational kinematics and rotational kinematics. In translational motion linear acceleration is associated with a force. In rotational motion, we can associate with angular acceleration a parameter analogous to force called torque. Torque is not simply the force acting on rotating bodies. This is because if we apply some force on a revolving door, the angular acceleration of the door depends on where the force is applied and how it is directed. A force applied to the hinge line can not produce any angular acceleration where as a force applied perpendicular to the plane of the door as its outer edge produces maximum angular acceleration.

The force as shown in Fig. 4.1a acting on body (nut) causes rotation of the body. The tendency to turn the nut depends upon both the magnitude of the force and length of its lever arm (r)

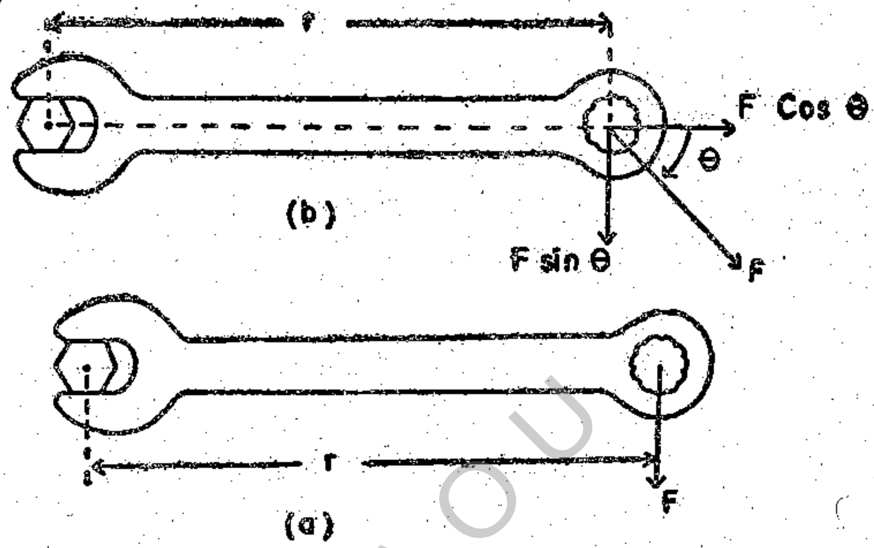


Fig. 4.1 Force acting on a nut causes it to rotate.

The same turning ability can be produced by a smaller force and a larger lever arm or a large force and a smaller lever arm. It is well known that if we want to open the door by applying force close to the hinge we have to apply large force. But small force is sufficient to open the door when applied at a point far from the hinge. The twisting ability of a force is defined as torque. If the force is not acting perpendicular to the lever arm as shown in Fig.4.1b then the perpendicular component of the force acts to produce a twist. The other component simply pulls or pushes sideways on the nut without producing a rotation.

Let us consider a force on a particle at a point P as shown in Fig.4.2.

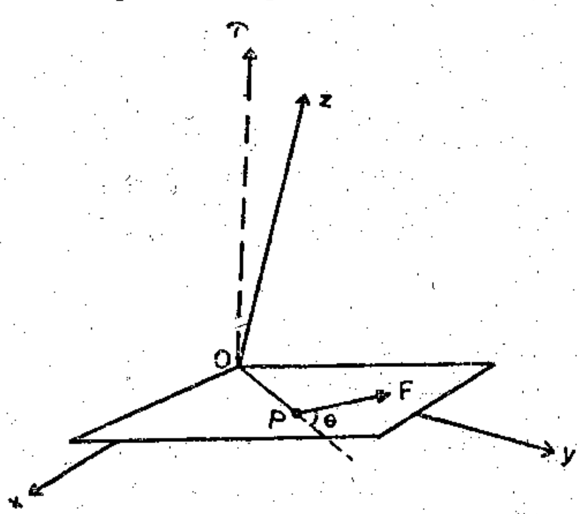


Fig.4.2 Force acting on a particle P causing it to rotate

Let r be the position vector of the point P with respect to the inertia reference frame whose origin is at O. If the particle undergoes rotational motion due to the force then the torque τ acting on the particle with respect to the origin 0 is defined as

$$\vec{\tau} = \vec{r} X \vec{F} \quad (4.1)$$

Torque is a vector quantity. Its magnitude is given by

$$\tau = r F \sin\theta \quad (4.2)$$

Where θ is the angle between r and F. The direction of the torque is normal to the plane containing r and F. The sense is given by the right hand rule namely, if one rotates r into F through the smaller angle between them, with the curled fingers of the right hand, then the extended thumb gives the direction of $\vec{\tau}$

The dimenstional formula of torque is ML^2T^{-2}. The unit of torque is N-m (Newton-meters).

The magnitude of the torque produced by a force F depends on the magnitude and direction of the force but also on the point of application of the force relative to the origin r is, zero and hence the torque is zero. We can rewrite Eqn. (4.2) as

$$\tau = (r_\perp \sin\theta)F = Fr_\perp \quad (4.3)$$

or

$$\tau = (F \sin\theta)r = rF_\perp \quad (4.4)$$

In the above equations $r(= r_\perp \sin\theta)$ represents the component r at right angles to the line of action F and $F_\perp (= F\sin\theta)$ represents component of F at right angles to r.

4.4 ANGULAR MOMENTUM OF A PARTICLE

The analogue of linear momentum in rotational motion is angular momentm. To define angular momentum, consider a particle of mass m and linear momentum at $\vec{p}$ at position $\vec{r}$ with respect to the origin O of an inertial reference frame shown in Fig. 4.3

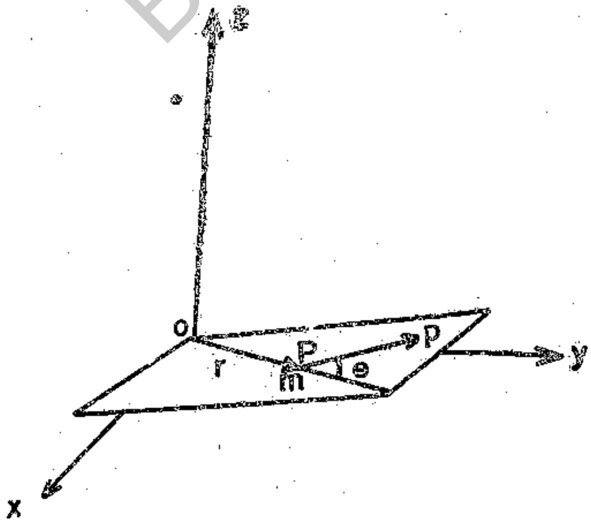


Fig. 4.3 Linear momentum acting on a particle P which is rotating about O

The angular momentum L of the particle with respect to the origin O is defined as

$$\vec{L} = \vec{r} \times \vec{p} \quad (4.5)$$

Angular momentum is a vector. Its magnitude is given by

$$L = r\,p\,\sin\theta \qquad (4.6)$$

Where θ is the angle between r and p. The direction of $\vec{L}$ is normal to the plane containing $\vec{r}$ and $\vec{p}$. The sense is given by the right hand rule namely, if one rotates $\vec{p}$ into $\vec{p}$ through the smaller angle between them, with the uirled figures of the right hand, then the extended right thumb points to the direction of 1.

For the Simple rotating system shown in Fig. 6.4. where the angle between r and p is 90°

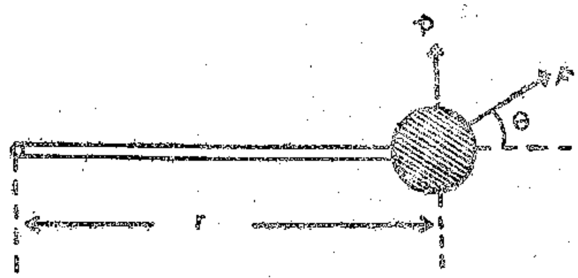


Fig. 4.4 Simple rotating system

the magnitude of angular momentum of the body of mass m attached to a rigid rod of length r is given by

$$L = rp \qquad (4.7)$$

Since p = mv we can rewrite Eqn. (4.7) as

$$L = r\,m\,v \qquad (4.8)$$

we know $v = r\,\omega$ hence (4.9)

$$L = (mr^2)\,\omega \qquad (4.9)$$

Eqn. 4.9 indicates that the magnitude of angular momentum of a body is equal to the product of (mr^2) and the angular velocity ω. In analogy with p=mv, the quantity mr^2 plays the role of inertial mass. The quantiry mr^2 is called the moment of inertia of the revolving mass. The importance of moment of inertia playing the role of inertial mass in rotational motion can be well understood through the following experimental observation.

Consider two wheels of same mass as shown in fig. 4.5

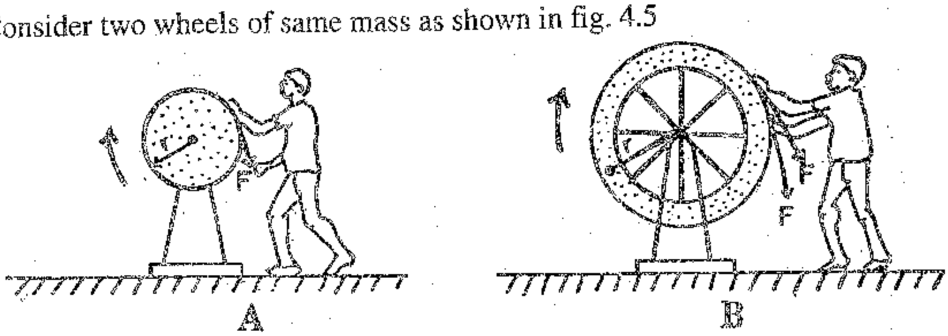


Fig. 4.5 Rotation of two wheels having equal masses but different moments of inertia

The wheel shown in Fig. 4.5b has the mass concentrated in a rim far from the axis of rotation of the wheel. For the wheel shown in Fig 4.5a the mass is closer to the axis of rotation. The general experience is that it is easy to set the wheel shown in Fig. 4.5a into rotation rather than the wheel shown in Fig 4.5b. The wheel shown in Fig 4.5b has more inertia. This indicates that rotational inertia depends not only on mass but also on another parameter i.e. the average distance of the mass from rotation axis. The farther the mass is from its rotational axis, the larger is its rotational inertia.

The relation between torque and angular momentum can be derived as follows.

The force acting on a particle is given by

$$\vec{F} = \frac{\overrightarrow{dp}}{dt} \tag{4.10}$$

Taking vector product of r on both sides of Eqn. (4.10) we get

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{\overrightarrow{dp}}{dt} \tag{4.11}$$

since $\vec{r} \times \vec{F}$ is the torque $\vec{\tau}$

$$\vec{\tau} = \vec{r} \times \frac{\overrightarrow{dp}}{dt} \tag{4.12}$$

differentiating Eqn. 4.5 with respect to time we get

$$\frac{\overrightarrow{dL}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)$$

$$= \vec{r} \times \frac{\overrightarrow{dp}}{dt} + \frac{\overrightarrow{dr}}{dt} \times \vec{p} \tag{4.13}$$

Since $\frac{\overrightarrow{dr}}{dt} = \vec{v}$ and $\vec{p} = m\vec{v}$, we can rewrite Eqn. (4.13)

$$\text{as } \frac{\overrightarrow{dL}}{dt} = \vec{r} \times \frac{\overrightarrow{dp}}{dt} + \vec{v} \times m\vec{v} \tag{4.14}$$

The second term on the right hand side of Eqn. (4.14) is zero because vector product of two parallel vectors is zero. Hence Eqn. 4.14 can be rewritten as

$$\frac{\overrightarrow{dL}}{dt} = \vec{r} \times \frac{\overrightarrow{dp}}{dt} \tag{4.15}$$

Using Eqn. (4.12) in Eqn. (4.15) we get

$$\vec{\tau} = \frac{\overrightarrow{dt}}{dt} \tag{4.16}$$

The above equation indicates that the time rate of change of angular momentum of a particle is equal to the torque acting on it.

4.5 ANGULAR MOMENTUM OF SYSTEM OF PARTICLES

Let us consider a system consisting of many particles. The total angular momentum of the system of particles L about a given point is given by the vectorial sum of angular momenta of individual particles of the system about the same point.

If $L_1, L_2, L_3 \ldots L_n$ are the angular momenta of the individual particles about a point then the total angular momentum L of the system of particles about the same point will be

$$\vec{L} = \vec{L_1} + \vec{L_2} + \vec{L_3} \ldots + \vec{L_n} \quad (4.17)$$

or

$$\vec{L} = \sum_{i=1}^{n} \vec{L_i} \quad (4.18)$$

As time advances, the total angular momentum of the system about a fixed reference point may change.

This change, $\frac{d\vec{L}}{dt}$ arises from two sources. They are (1) torques exerted on the particles of the system by internal forces between the particles and (2) torques exerted on the particles of the system by external forces. According to Neweon's third law, the total internal torque becomes zero, since the torque resulting from each internal action-reaction force pair is zero. (In the strong form of Newton's third law not only action and reaction are supposed to be equal and opposite but they are supposed to act along the line joining the particles and the student can show in this case that the sum of internal torques is zero).

$$\vec{\tau_e} = \frac{d\vec{L}}{dt} \quad (4.19)$$

Where $\vec{\tau_e}$ is the resultant external torque. That is, the time rate of change of the total angular momentum of a system of particles about the origin of an inertial reference frame is equal to the sum of the external torques acting on the system.

$\vec{\tau}$ and $\vec{L}$ are measured with respect to the origin of an inertial reference frame. Eqn. 4.19 does not hold good if $\vec{\tau}$ and $\vec{L}$ are measured with respect to any other reference point within the system except if the reference point is chosen as the centre of mass of the system, even though this point is not fixed as the system moves. This is another important property of the centre of mass. The general motion of a system of particles can be separated into the translational motion of centre of mass $\left(F_e = \frac{d\vec{p}}{dt} \right)$ and rotational motion about its centre of mass.

$$\left(\vec{\tau}_e = \frac{d\vec{L}}{dt} \right)$$

4.6 KINETIC ENERGY OF ROTATION MOMENT OF INERTIA

Consider a rigid body rotating with angular speed ω about an axis fixed in a particular inertial reference frame shown in Fig. 4.6

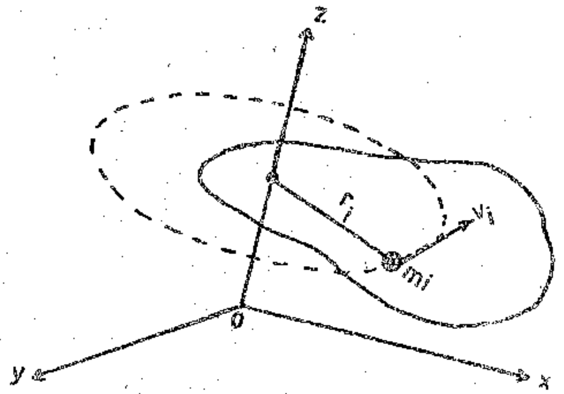


Fig. 4.6 Rigid body rotating about z-axis

The body may be considered to be consisting of a large number of particles of mass m_1, $m_2 \ldots m_n$ at distance r_1, r_2, ... r_n from the axis of rotation. Since the particles are rigidly attached to the body, all these particles have same angular velocity equal to the angular velocity of the body itself. Each particle of mass m_1 moves with a velocity v_1 and hence its kinetic energy is given by

$$k_1 = \frac{1}{2} m_1 v_i^2 \text{ using } v_i = r\omega, \tag{4.20}$$

$$k_1 = \frac{1}{2} m_1 r_i^2 \omega^2 \tag{4.21}$$

The total kinetic energy of the rotating body is given by

$$k = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \ldots + \frac{1}{2} m_n r_n^2 \omega^2 \tag{4.22}$$

$$k = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \tag{4.23}$$

$\sum_i m_i r_i^2$ is known as total rotational inertia of all the particles composing the body. It represents rotational inertia of the body. It is also called moment of inertia of the body. Therefore

$$I = \sum m_i r_i^2 \tag{4.24}$$

The moment of inertia has the dimensional formula as ML^2 and expressed in Kg. m^2 In terms of rotational inertia, the kinetic energy of a rotating body can be expressed as

$$K = \tfrac{1}{2} I\omega^2 \tag{4.25}$$

If we try to arrive at an expression for the angular momentum of rigid body relating about a fixed axis by a similar procedure we would get

$$\vec{L} = I\vec{\omega} \tag{4.25a}$$

Where I is again equal to $I = \sum m_i r_i^2$

The moment of inertia of a body as given by Eqn 4.24 is applicable to bodies which are composed of discrete point masses. For bodies possessing continuous distribution of matter the summation in Eqn. (4.24) must be replaced by integration.

Irrespective of the shape of the body it is always possible to find a radial distance from any given axis at which the mass of the body could be concentrated without altering the moment of inertia of the body about that axis. This distance is called radius of gyration of the body about that axis. If the radius of gyration of a body about the axis of rotation is R, then

$$I = MR^2 \tag{4.26}$$

and

$$R = (I/M)^{1/2} \tag{4.27}$$

Thus the radius of gyration R of a body is the square root of the ratio of moment of inertia of the body to the mass of the body.

Worked Example-1:

Three masses of 5 kg, 10 kg and 15 kg are connected by light rigid rods of equal length of 5 m as shown in Fig.4.7 Determine the moment of inertia of the system about an axis at O perpendicular to the plane containing the masses. Also determine the kinetic energy of the system if its angular velocity is 5 rad.s^{-1}.

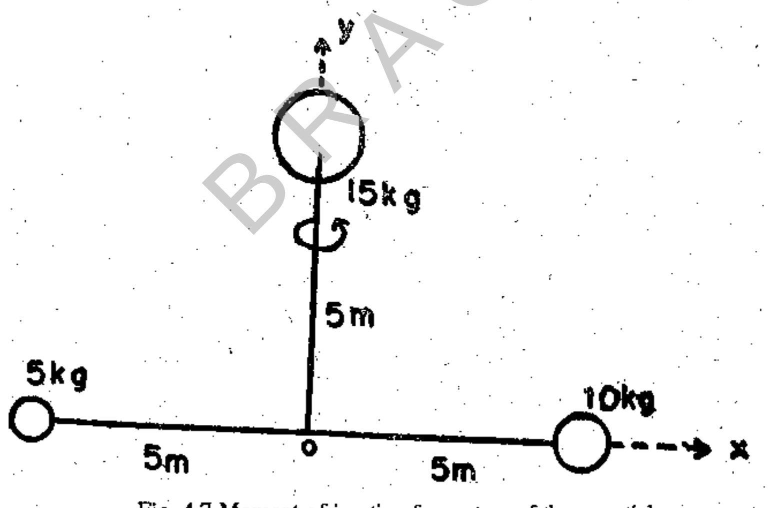


Fig. 4.7 Moment of inertia of a system of three particles.

Solution :

The moment of inertia of the system about

Z-axis I_z $= [5\,(5)^2+10(5)^2+15\,(5)^2 \text{kg.m}^2$

$= [5\,(25)+10\,(25)+15\,(25)]\ \text{kg.m}^2$

$= 125+250+375=750\ \text{kg.m}^2$

The kinetic energy of rotation-K $= (1/2)\, I_z \omega^2$

$= \frac{1}{2} \times 750\ \text{kg. m}^2\ (5\ \text{rad. s}^{-1})^2 = 9375$ Joules

$I_z = 750\ \text{kg.m}^2$

$$K = 9375\ J$$

Worked Example-2 :

A disk of radius 0.1 m and mass 0.5 kg is rolling down an inclined plane of height 0.6m. Assuming that the disk started from rest from the top of the inclined plane. Find its translational velocity and kinetic energy of rotation of the body by the time that disk reaches the ground.

Consider an inclined plane as shown in Fig. 4.8

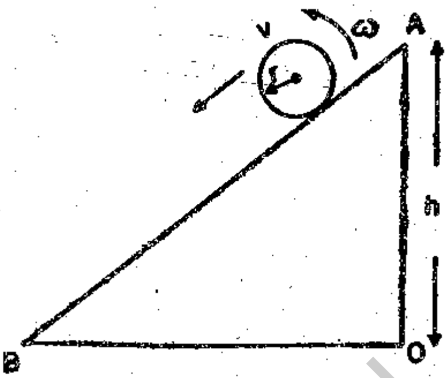


Fig.4.8 Motion of spherical body on an inclined plane

Solution:

According to the principle of conservation of energy the energy of the disk at A=Energy of the disk at B i.e.,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2$$

$$gh = \frac{1}{2}v^2 + \frac{1}{4}v^2$$

$$3/4\, v^2 = gh$$

$$v = \left[\frac{4\,gh}{3}\right]^{1/2}$$

$$v = \left[\frac{(4)\,(9.8)\,0.6}{3}\right]^{1/2} = (7.8)^{1/2}\ ms^{-1}$$

The kinetic energy of rotation when the disk reaches the ground = (½) I ω^2

$$= \frac{1}{2}\left[\frac{(0.5 \times 0.1)^2}{2}\right]\left[\frac{(7.8)}{(0.1)^2}\right]$$

$$= \frac{1}{2}(0.25 \times 0.01\left[\frac{(7.8)}{(0.01)}\right]$$

$$= 0.9375 \text{ J}$$

4.7 MOMENT OF INERTIA OF EXTENDED MASS

The moment of inertia of regularly shaped body can be easily calculated. The moments of inertia of a few simple but important bodies are given in 4.9. The value of I is given against each body in terms of the dimensions of the body. Since $I=MR^2$, the radius of gyration R is easily found from I and M and this value is also shown in Fig.4.9

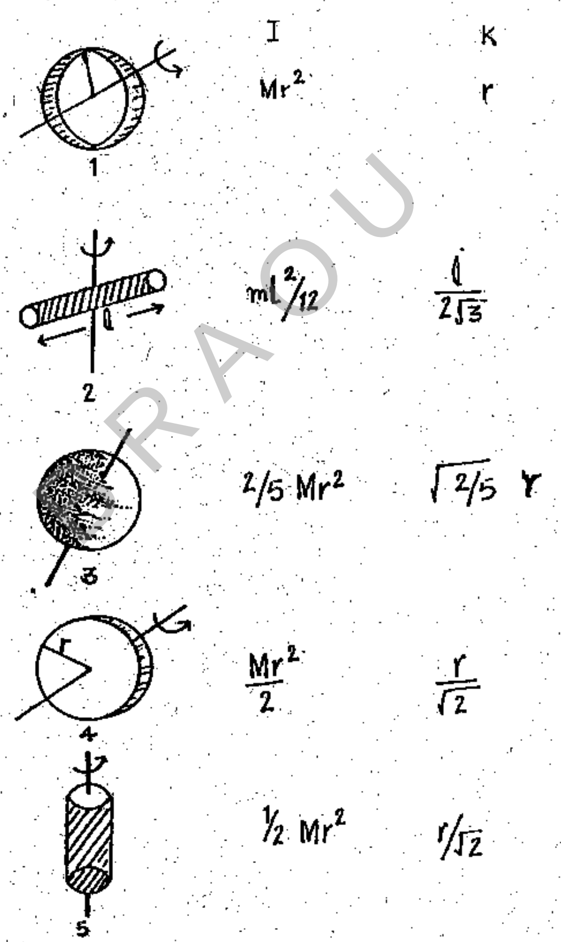


Fig 4.9 Moment of inertia of

1. Hoop about cyclindrical axis
2. Thin rod about an axis perpendicular to length
3. Solid sphere about an axis passing though its centre
4. Disk about an axis passing though its centre
5. Solid cylinder about its vertical axis

To illustrate the method of evaluation of moment of inertia of regularly shaped bodies, a few examples are given below

4.7.1 Moment of Inertia of a Wheel

Let the wheel rotate about the axis passing through its centre as shown in Fig. 4.10

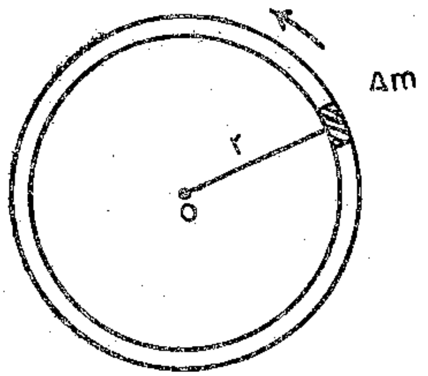


Fig.4.10 Wheel rotating about an axis passing through its centre

The Continuous distribution of mass of the wheel concentrated uniformly at the rim can be divided into n mass elements of mass Δ m. Each mass element is at the same distance from the axis of rotation. The moment of inertia of the body is given by

$$I = \sum_{i=1}^{n} \Delta m_i r_i^2 \quad (4.28)$$

Since $r_i = r$ is the same for each element Δm_i

$$I = r^2 \sum \Delta m_i = r^2 M \quad (4.28a)$$

Where M represents the total mass of the wheel. Therefore the moment of inertia of the wheel about the axis of rotation passing through its centre is given by

$$I = Mr^2$$

4.7.2 Moment of Inertia of a Homogeneous Rigid Rod of Length about an Axis Perpendicular to the Rod Through One End.

Let us consider an elemental length dx of the rod at a distance x from the axis about which the rod executes rotational motion as shown in Fig. 4.11

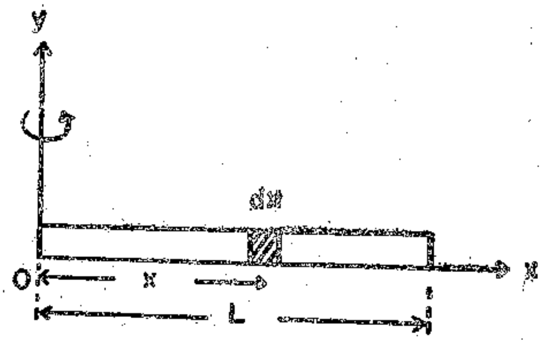


Fig. 4.11 Moment of inertia of a rigid rod about an axis perpendicular to the rod through one end

Let μ represent the mass of unit length of the rod. Then mass of the element length of the rod is μdx. The moment of inertia of the element is given by

$$I = \mu\, dx\, x^2 \qquad (4.29)$$

Moment of inertia of the rod

$$I = \int_0^L \mu x^2 dx = \mu\left[\frac{x^3}{3}\right]_0^L = \frac{\mu L^3}{3}$$

$$I = \frac{ML^3}{3} \qquad (4.30)$$

If M represents the mass of the rod

$$M = \mu L \qquad (4.31)$$

substituting μL from Eqn. (4.31), we get

$$I = \frac{ML^3}{3} \qquad (4.32)$$

If the axis of rotation is perpendicular to the rod but passes through the centre of the rod as shown in Fig. 4.12

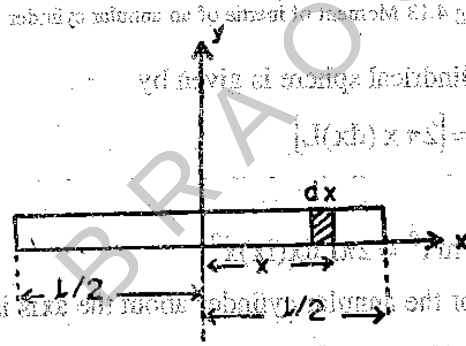


Fig. 4.12 Moment of inertia of a rod perpendicular to its length and passing through its centre

Then the moment of inertia of the rod about the axis passing through its centre is given by

$$= \int_{-1/2}^{1/2} \mu x^2 dx = \mu\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{\mu L^3}{12} \qquad (4.33)$$

since $M = \mu L$

$$I = \frac{ML^2}{12} \qquad (4.34)$$

4.7.3 Moment of Inertia of an Annular Cylinder

Let us consider an annular cylinder of mass M, length L, having R_1 and R_2 as the internal and external radii respectively. Let d represent the density of the cylinder. Consider a mass element as an infinitesimally thin cylinder shell of radius x, thickness dx and length L as shown in Fig.4.13

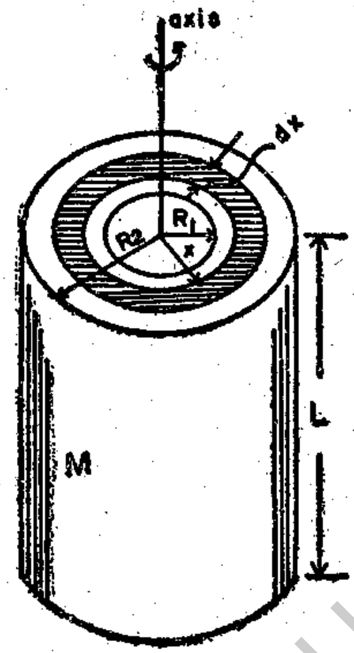


Fig 4.13 Moment of inertia of an annular cylinder

The mass of the thin cylindrical sphere is given by

$$m = [2\pi x\,(dx)L] \qquad (4.35)$$

The moment of inertia of the cylinder shell about the axis is

$$I = mx^2 = 2\pi Ldx(ix)x^2 \qquad (4.36)$$

The moment of inertia of the annular cylinder about the axis is therefore given by

$$I = \int_{R1}^{R2} mx^2 = \int_{R1}^{R2} 2\pi\, Ldx^3(dx) \qquad (4.37)$$

$$I = 2\pi\, Ld\int_{R1}^{R2} x^3(dx) \qquad (4.38)$$

$$I = 2\pi\, Ld\left[\frac{x^4}{4}\right]_{R1}^{R2} \qquad (4.39)$$

$$I = 2\pi\, Ld\left[\frac{R_2^4}{4} - \frac{R_1^4}{4}\right] \qquad (4.40)$$

$$I = \frac{1}{2}\pi\, Ld\left[R_2^2 + R_1^2\right]\left[R_2^2 - R_1^2\right] \qquad (4.41)$$

The mass of the annular cylinder

$$M = \pi\left(R_2^2 - R_1^2\right)Ld \qquad (4.42)$$

Substituting Eqn. 4.42 in Eqn. 4.41 we get, the moment of inertia of the annular cylinder and is given by

$$I = \frac{1}{2}M\left(R_2^2 + R_1^2\right) \tag{4.43}$$

If R = 0, we have a solid cylinder then its moment of inertia about the axis passing through the axis of the cylinder is given by

$$I = \frac{MR^2}{2} \tag{4.44}$$

Check your Progress I

The moment of inertia of an annular cylinder about an axis passing through the axis of the cylinder is

1. $\frac{ML^2}{12}$

2. $\frac{1}{2}M\left(R_2^2 + R_1^2\right)$

3. $M\left(\frac{R^4}{4} + \frac{1^2}{12}\right)$

4.8 PARALLEL AXIS AND PERPENDICULAR AXIS THEOREMS

The moments of inertia for bodies with simple geometry and high symmetry are relatively easy to calculate if the axis of rotation coincides with the axis of symmetry. If axis of rotation does not coincide with the axis of symmetry, then the calculation of moment of inertia of the body about such an axis of rotation is not a simple process even if the body has high degree of symmetry. But making use of parallel axis and perpendicular axis theorem it would be easier to evaluate the moment of inertia of the body about any axis. Let us understand these theorems before we apply them to a specific case.

4.8.1 Parallel Axis Theorem

Fig. 4.14 represents a cross-section of the body which contains the center of mass C of the body. Let this cross-section coincide with the xy-plane. Let us suppose that we want the moment of inertia of the body about the z-axis. Consider a particle m at point P at a distance r_i from the origin of coordinates. Let C be at distance r_c from the origin of coordinates. The moment of inertia of the body with respect to the z-axis is given by

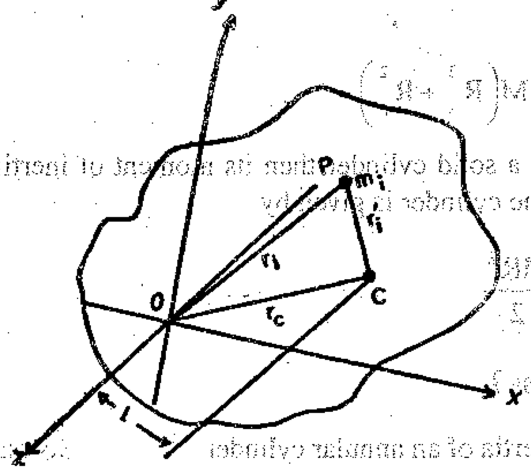


Fig 4.14 Parallel axis theorem

Let the distance of the point P from the centre of mass of the body be given by r_i. Let x_c, y_c represent the coordinates of the centre of mass

$$I = \Sigma_i\, m_i\left(x_i^2 + y_i^2\right) \tag{4.45}$$

The we can write

$$x_i = x_c + x_i \tag{4.46}$$

$$y_i = y_c + y_i \tag{4.47}$$

substituting Eqns. 4.46 and 4.47 in Eqn. (6.45) we get

$$I = \Sigma m_i\left[\left(x_c + x_i^1\right)^2 + \left(y_c + y_i^1\right)^2\right] \tag{4.48}$$

$$I = \sum_i m_i\left[x_c^2 + \left(x_i^1\right)^2 + 2x_c x_i^1 + y_c^2 + \left(y_i^1\right) + 2y_c\left(y_i^1\right)^2\right] \tag{4.49}$$

$$I = \sum_i m_i\left(x_c^2 + y_c^2\right) + \Sigma m_i\left(x_i^{1^2} + y_i^{1^2}\right) + \sum_i m_i 2x_c x_i^1 + \sum_i m_i 2y_c y_i^1 \tag{4.50}$$

$$I = \sum_i m_i\left(x_c^2 + y_c^2\right) + \Sigma m_i\left(x_i^{1^2} + y_i^{1^2}\right) + \sum_i m_i 2x_c x_i^1 + \Sigma_i\, m_i 2y_c y_i^1 \tag{4.51}$$

The second term in Eqn. (4.51) represents the moment of inertia of the body about an axis passing through the centre of mass of the body about and parallel to the z-axis. Let it be represented by I_c. If the distance between the z-axis and the axis parallel to it and passing through the centre of mass of the body is 1 them,

$$r_c^2 = 1^2 = x_c^2 + y_c^2 \tag{4.52}$$

As per the definition of the centre of mass of the body

$$\Sigma_i m_i x_i = \Sigma_i m_i y_i = 0 \tag{4.53}$$

Using Eqns.(4.52) and (4.53) in Eqn.(4.51) we get

$$I = I_c + Ml^2 \tag{4.54}$$

Eqn.(4.54) represents the parallel axis theorem. It states that moment of inertia of a body about any axis is given by the sum of the moment of inertia of the body about an axis parallel to the given axis and passing through the centre of mass of the body and the product of the total mass of the body and the square of the perpendicular distance between the two axes.

4.8.2 Perpendicular Axis Theorem

Consider a cross-section of a rigid body in xy plane as shown in Fig.6.15.

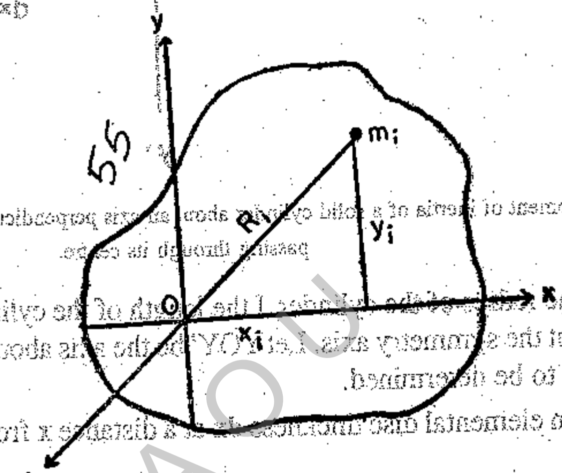


Fig 4.15 Perpendicular axis theorem

The moment of inertia of the body about z-axis is given

$$I_z = \Sigma_i m_i R_i^2 \tag{4.55}$$

$$\text{since } R_i^2 = x_i^2 + y_i^2 \tag{4.56}$$

$$I_z = \Sigma_i m_i \left(x_i^2 + y_i^2\right)$$

$$I_z = \Sigma_i m_i x_i^2 + \Sigma_i m_i y_i^2$$

$\Sigma m_i x_i^2$ represents the moment of inertia of the body about y-axis, say Iy and $\Sigma m_i y_i^2$ represents the moment of inertia of the body about x-axis, say Ix. Hence

$$I_z = I_x + I_y \tag{4.57}$$

Eqn. (4.57) represents the perpendicular axis theorem. It states that the sum of the moments of inertia a body about two mutually perpendicular axes is equal to the moment about an axis which is perpendicular to both of them and passing through their point of intersection.

Let us consider an example to illustrate the applicability of the parallel axis and perpendicular axis theorems to determine the moment of inertia of a body. Let u;s attempt to determine the moment of inertia of a solid cylinder about an axis perpendicular to the axis of the cylinder and passing through its centre as shown in Fig.4.16

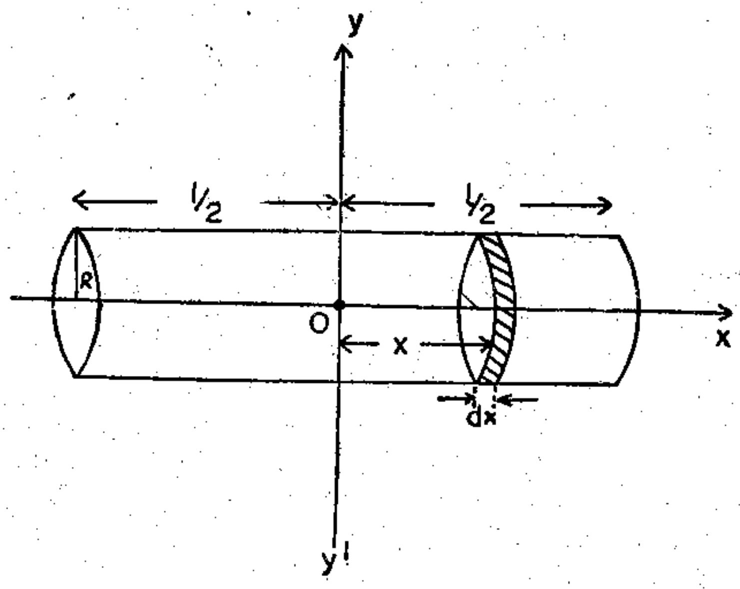


Fig.(4.16) Moment of inertia of a solid cylinder about an axis perpendicular to the axis of the cylinder and passing through its centre.

Let R be the radius of the cylinder, l the length of the cylinder and M its total mass. Let XOX represent the symmetry axis. Let YOY' be the axis about which the moment of inertia of the body is to be determined.

Consider an elemental disc thickness dx at a distance x from the centre of mass O of the body.

The mass of the circular disc is

$$\pi R^2 (dx) d \tag{4.58}$$

Where d represents the density of the cylinder. The moment of inertia of a circular disc about any diameter is given by

$$I_d = \frac{MR^2}{4} = \frac{\pi R^2 d(Ix) R^2}{4} \tag{4.59}$$

(Prove that the moment of inertia of a circular disc is $\frac{MR^2}{4}$ by applying perpendicular axis theorem and the fact that moment of inertia a circular disc about an axis perpendicular to the plane of the disc and passing through its centre, is $\frac{MR^2}{2}$.

Applying parallel axis theorem the moment of inertia of the body (circular disc) about OY is given by

$$I = \frac{\pi R^2 d R^2}{4} dx + \left(\pi R^2 d\, dx\right) x^2 \tag{4.60}$$

Since the cylinder is the sum of such elemental strips the moment of inertia of the solid cylinder about the axis perpendicular to the symmetry axis and passing through its centre of mass is given by

$$I = \int_{+\ell/2}^{-\ell/2} \left[\frac{\pi R^2 dR^2 dx}{4} + \pi R^2 dx^2 dx \right] \tag{4.60}$$

$$I = \pi R^2 d \left[\int_{-\ell/2}^{+\ell/2} \frac{R^2}{4} dx + \int_{-\ell/2}^{+\ell/2} x^2 dx \right] \tag{4.61}$$

$$I = \pi R^2 d \left[\frac{R^2}{4} [x]_{-\ell/2}^{\ell/2} + \left(\frac{x^3}{3} \right)_{-\ell/2}^{\ell/2} \right] \tag{4.63}$$

$$I = \pi R^2 d \left[\frac{R^2}{4} I + \frac{\ell^3}{12} \right] \tag{4.64}$$

$$I = \pi R^2 d\ell \left[\frac{R^2}{4} + \frac{\ell^2}{12} \right] \tag{4.65}$$

The mass of the solid cylinder M = $\pi R^2 \ell . d$.

$$I = M \left[\frac{R^4}{4} + \frac{\ell^2}{12} \right] \tag{4.66}$$

4.9 WORK AND POWER IN ROTATIONAL MOTION

Consider a force 'F' acting at the rim of pivoted wheel of radius R as shown in Fig. 4.17

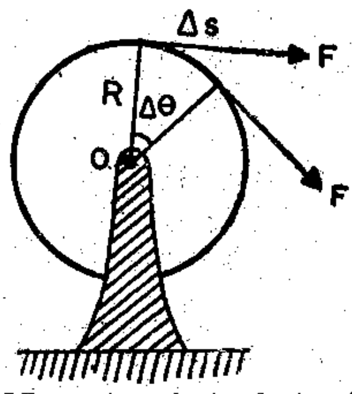


Fig. 4.17 Force acting at the rim of a pivoted wheel

Let the wheel rotate through an angle $\Delta\theta$. If $\Delta\theta$ is small the force acting may be taken to be constant during the correspondingly small interval of time Δt.

The work done by the force F is given by

$$\Delta W = F \Delta s \tag{4.67}$$

since $\Delta s = R\Delta\theta$ we have

$$\Delta W = FR\,\Delta\theta \tag{4.68}$$

Since τ = FR we can rewrite (4.68) as

$$\Delta W = \tau\Delta\theta \tag{4.69}$$

or

$$dW = \tau d\theta \tag{4.70}$$

The work done by the torque acting τ on a body is given by

$$\int dw = \int_{\theta_1}^{\theta_2} \tau d\theta$$

$$W = \tau[\theta]_{\theta_1}^{\theta_2} = \tau(\theta_2 - \theta_1) = \tau\theta \tag{4.71}$$

Where θ is the angular displacement. If we differentiate Eqn. (4.71) with respect to time t we get

$$\frac{dw}{dt} = \tau\frac{d\theta}{dt} \tag{4.72}$$

since

$\frac{dw}{dt} = P$ the power we get

$$P = \tau\omega \tag{4.73}$$

Eqn. 4.73 indicates that the instantaneous power developed by an agent exerting a torque equals the product of the torque and the instantaneous angular velocity.

4.10 RELATION BETWEEN TORQUE AND ANGULAR ACCELERATION OF A ROTATING BODY

We have already shown in 4.25a that the angular momentum of a body about a given axis

$$\vec{L} = I\vec{\omega} \tag{4.74}$$

Differentiating we get

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}I\vec{\omega}$$

Hence

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} \tag{4.75}$$

Worked Example 3 :

A uniform solid disk of mass 5 kg having a radius of 0.5 m is mounted on a frictionless axle. Another small disk of mass 2kg and of radius of 0.2m is fastened to the large disk and a rope is wrapped around the smaller disk. A constant tension of 50 N is applied to the rope. Find the angular acceleration of the system. If the system starts from rest what is its angular velocity after 10 sec? Find also the angular momentum.

Solution:

The torque on the system is Fr. Since

$\tau = I\alpha$ we have

the angular acceleration

$$\alpha = \frac{\tau}{I} = \frac{Fr}{\frac{MR^2}{2} + \frac{mr^2}{2}}$$

Where M = Mass of larger disk

m = Mass of smaller disk

R = Radius of larger disk

r = Radius of smaller disk

F = Tension in the string

Substituting the values of the parameters from the problem in the expression for α we get

$$\alpha = \left[\frac{(50N)(0.2m)}{\frac{5(0.5)^2}{2} + \frac{2(0.2)^2}{2}}\right]$$

$$\alpha = \frac{10}{0.665} = 15 \text{ rad. s}^{-2}.$$

The angular velocity of the system after 10s if it starts from rest position at t=0 is given by

$$\omega = \omega_o + \alpha t$$

$$\omega = o + 15 \text{ rad s}^{-2} \text{ x } 10\text{s} = 150 \text{ rad. s}^{-1}$$

Angular momentum

$L = I\omega = (0.665 \text{ kg m}^2)(150 \text{ rad.s}^{-1}) = 99.75. \text{ kg m}^2 \text{ s}^{-1}$

Worked Example 4: A uniform disk of Radius R=05 m and mass m = 1kg is mounted on an axis

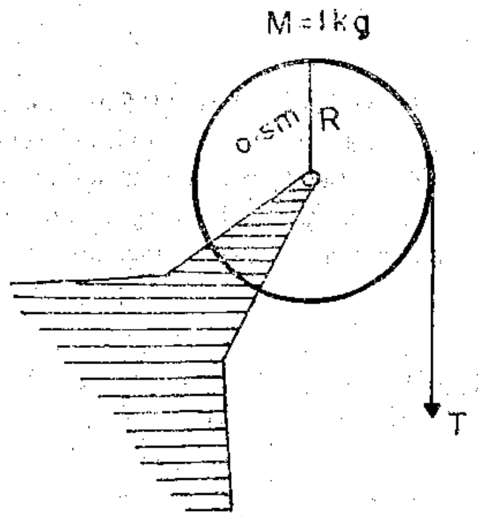


Fig 4.18 Uniform disk mounted on an axle.

supported in fixed frictionless bearings as shown in Fig. 4.18. A light cord is wrapped around the rim of the wheel and a steady down ward pull T = 10 N is exerted on the cord. Find the angular acceleration of the wheel and the tangential acceleration of a point on the rim.

Solution:

The Torque about the central axis

$$\tau = TR = 10\,N \;X\, 0.5$$
$$= 5\ Nm$$

We can also express the torque τ in terms of moment of inertia of the fly wheel and angular acceleration i.e.,

$$\tau = I\,\alpha$$

$$= \frac{MR^2}{2}\alpha$$

$$= \frac{(1\,kg)\,(0.5\,m)^2}{2}\alpha$$

$$= 0.125\,kg\,m.\alpha.$$

$$\alpha = \frac{\tau}{0.125\,kg\,m} = \frac{5\,N\,m}{0.125\,kg\,m^2}\,4 rad.s^{-2}$$

The tangential acceleration at the point on the rim =

$$a = R\,\alpha = 4\ rad.s^{-2}\ x\ 0.5\,m$$
$$a = 2\ m.s^{-2}$$

4.11 SUMMARY

The twisting ability of force acting on a rigid body is called torque τ, The angular momentum L of a rigid body rotating about an axis perpendicular to the plane of the body is given by $I\omega$. Where I is the moment of inertia and ω is the angular velocity. The time rate of change of angular momentum of a particle is equal to the torque acting on it. The kinetic energy of a rotating body is given by $K = (1/2)\, I^2.\omega^2$

Parallel axis and perpendicular axis is theorems are useful in determining the moment of Inertia of a body about any given axis. The parallel axis theorem states that the moment of inertia of a body about an axis parallel to that axis and passing through the centre of mass of the body and the product of the total mass of the body and the square of the distance between the two axis.

The perpendicular axis theorem states that the moment of inertia of a body about an axis perpendicular to a plane of the body is given by the sum of the moments of inertia of the body about two mutually perpendicular axes in the plane of the body which pass through the origin of the coordinate system and through whose inter section point the perpendicular axis passes.

4.12 MODEL ANSWERS

Check your Progress I

The moment of inertia of an annular cylinder about an axis passing through the axis of the cylinder is $^1/_2 M(R_2{}^2 + R^2{}_1)$

4.13 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in about 30 lines.

1. Derive an expression for the moment of inertia of an annular cylinder about an axis passing through the symmetry axis of the cylinder.

2. State and explain the parallel axis and prependicular axis theorems. Explain with an example how these laws are useful to obtain the moment of inertia of rigid body about any axis.

II. Answer the Following Question in about 10 lines.

1. Derive an expression for the kinetic energy of a rotating body.

2. Derive an expression relating work and power of an rotating body with the torque acting on it.

3. Derive an expression relating torque and angular acceleration of a rotating body.

III. Solve the Following Problem.

1. Consider a body consisting of two sperical masses of 10 kg each connected by a light rod of 1 m length. Find the moment of inertia of the body about an axis normal to it and passing through the centre of the system.

2. The speed of a body of mass 20kg moving along a circle of radus 1.5 M. increases at the rate of 0.5M/s find the torque on it.

3. A Sphear of mass 0.1 kg having a diameter of 0.01M rotates without slipping with a velocity of 0.05M/sec calculate the KE of the rotation

[$5x10^{-5}$J].

4. A disc of mass 2kg rolls without slipping on a horizontal plane with a velocity of 4M/S. Find the KE.

[24 J]

5. Calculate the rotational KE of the earth due to spinning of the earth arround its axis given that Mass of the earth is $6x10^{24}$ Kg, radius of the earth 6.4×10^6 Meters.

[$2.6x10^{29}$J]

6. A circular disc of mass 100 kg having a radius of 1M is momented axially and made to rotate. Calculate kinetic energy which executing 120 rot/min

[9349J]

UNIT 5 CONSERVATION OF ANGULAR MOMENTUM

Contents

5.1 AIMS AND OBJECTIVES

This Unit introduces the principle of conservation of angular momentum. In order to make you understand the principle it is shown that the resultant torque acting on a system of particles is zero and that the total angular momentum of the system remains constant irrespective of the charges of angular momenta of individual particles composing the system.

After going through this Unit you will be able to understand the principle underlying the feats of Acrobats and ice skaters.

5.2 INTRODUCTION

In this Unit we introduce the concept of angular momentum, the conservation of angular momentum and the concept of torque. These are particularly important in the treatment of rigid bodies.

5.3 CONSERVATION OF ANGULAR MOMENTUM

We know that the time rate of change of the total angular momentm of a system of particles about a fixed point in an inertial reference frame is equal to the sum of the external torques acting on the system

Hence

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \qquad (5.1)$$

if $\vec{\tau}_{ext} = 0$ then

$$\frac{d\vec{L}}{dt} = 0 \qquad (5.2)$$

Hence

$$\vec{L} = \text{constant} \tag{5.3}$$

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is known as the priciple of the conservation of angular momentum.

This applies to both a system of particles and rigid bodies. For a system of a particles the total angular momentum L about some point is given by

$$\vec{L} = \vec{L_1} + \vec{L_2} + \vec{L_3} + ... \tag{5.4}$$

When the resultant external torque is zero we have as per the principle of conservation of angular momentum

$$\overrightarrow{\sum L_i} = \text{constant} \tag{5.5}$$

Eqn. 5.5 indicates that the angular momenta of the individual particles of the system may change but their vector sum remains constant in the obsence of a net external torque.

In the case of rigid body rotating about an asis, say z-axis, that is fixed in an inertial reference frame we may write

$$L_2 = I\omega \tag{5.6}$$

Hence L_z presents the angular momentum of the body along the rotation axis and I represent the moment of inertia of the body about z-axis. If there is any redistribution of the mass of the body then there occurs a change in the moment of inertia of the body. If the net external torque acting on the body is zero then as a result of change in I there must occur a compensating change in the angular velocity ω of the body so as to keep the angular momentum of the system constant as per the principle of conservation of angular momentum. That is I_o and ω_o represents the moment of inertia and angular velocity of the rigid body initially and after some time due to redistribution of the parts of the body if I and ω represents the moment of inertia and angular velocity of the rigid body, then

$$I_o\omega_o = I\omega = \text{constant} \tag{5.7}$$

Eqn. 5.7 represents the principle of conservation of angular momentum applicable to rigid bodies in rotational motion.

Worked Example-1 :

Consider a star as a uniform sphere of mass M. It has an angular velocity ω_o. As time goes on its radius decreases from r_o to r_r. Find the angular velocity of the star when its radius is r_r.

Solution :

As per the principle of conservation of angular momentum, if the angular velocity of the star when its radius is r_r is ω_f then

$$I_o\omega_o = I_f\omega_f$$

For an uniform sphere of radius r the moment of inertia

$$I = \frac{2Mr^2}{5}$$

$$\therefore I_o = \frac{2Mr_o^2}{5}, I = \frac{2Mr_f^2}{5}$$

Substituting for I_o and I in the formula

$$\frac{2Mr_\theta^2 \omega_\theta}{5} = \frac{2Mr_f^2 \omega_f}{5}$$

$$\therefore \omega_f = \omega_\theta \left[\frac{r_\theta}{r_f}\right]^2$$

The above result indicates the star rotates faster as it collapses.

Worked Example-2:

A horizontal platform in the shape of a cylinderical disk rotates in a horizontal plane about a frictionless axis as shown in fig.5.1. The platform has a mass of 200 kg and a radius of 4m. A man whose mass is 50kg walks slowly from the rim of the platform towards the center. If the angular velocity of the platform and the man is 5 rad/sec when the man is at the rim. Calculate the angular velocity when the man reaches a point which is 1 m from the centre.

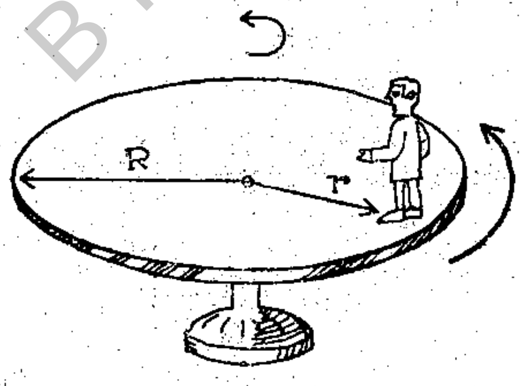


Fig. 5.1 Rotation of horizontal platform

Solution:

The moment of inertia of the platform about the axis of rotation

$$I_p = \frac{1}{2}MR^2 = \frac{1}{2}200kg(4m)^2 = 1600 \text{ kg m}^2$$

The initial moments of inertia of the man about the axis of rotation when he is at the rim=I_{mo} = mR^2

=50kg (4 m)^2=800 kg m^2

The total moment of inertia of the system when the man is at the rim

$$I_o = I_p + I_{mo} = 1600 + 800 = 2400 \text{ kg m}^2$$

When the man is at r = m his moment of inertia changes to

$$I_m = mr^2 = 50 \text{ kg } (1\text{ m})^2 = 50 \text{ kg m}^2$$

The moment of inertia of the rotating system when the man is at 1 m from its centre

$$I_p + I_m = 1600 + 50 = 1650 \text{ kg m}^2$$

Since the angular momentm is conserved and

$\omega_o = 5$ rad/sec we have

$$I\omega = I_o \omega_o$$

i.e., $1650\,\omega = 2400 x5$

$$\therefore \omega = \frac{2400x5}{1650} = 7\frac{3}{11} \text{rad/sec.}$$

The angular velocity of the system when the man is at a distance of 1 m from the centre of the rotating table is $7\frac{3}{11}$ rad/sec.

Worked Example-3:

A 5 kg disk witha radius of gyration of 0.5m and rotating with an angular velocity of 120 rpm engages with another disk of weight 10kg whose radius of gyrations is 1m. This disk is rotating initially in the same direction as the first with an angular velocity of 60 rpm. Find the final angular velocity of both the disks.

Solution :

Moment of inertia of

5 kg disk = $I_1 = 5 \text{ kg } (0.5)^2 = 1.25 \text{ kg m}^2$

Moment of inertia of

10 kg disk = $I_2 = 10 \text{ kg } (1)^2 = 10 \text{ kg m}^2$

$$\text{Angular velocity of 5 kg disk} = \frac{120}{60} x 2\pi = 4\pi \text{ rads}^{-1}$$

$$\text{Angular velocity of 10 kg disk} = \frac{60}{60} x 2\pi = 2\pi \text{ rads}^{-1}$$

Applying the principle of conservation of angular momentum to the system before and after engagement, we have

$$I_1\omega_1 + I_2\omega = (I_1 + I_2)\omega$$

$$\omega = \frac{[I_1\omega_1 + I_2\omega_2]}{I_1 + I_2} = \frac{1.25x4\pi + 10\,2\pi}{10 + 1.25}$$

$$\omega = \frac{25\pi}{11.25} = 2.22\pi = 2.22\pi x \frac{1}{2\pi} 60.\text{rpm}$$

$$\omega = 66.6\text{rpm}$$

5.4 APPLICATION OF THE PRINCPLE OF CONSERVATION OF ANGULAR MOMENTUM

In day to day life we come across with many situations where the principle of conservation of angular momentum is applicable. This law is useful to explain certain interesting features of rotating objects. For example a rapidly spinning bullet is shown in Fig.5.2a resists wobbling. If it were to wobble, then the direction of angular momentum vector should change. Since torques due to small deflecting forces acting on the bullet cannot produce appreciable changes in the angular momentum, the bullet remains oriented in a particular direction and hence no wobbling motion. Fig.5.2b shows a sleeping top. It will not wobble and fall until it spin rate becomes very small.

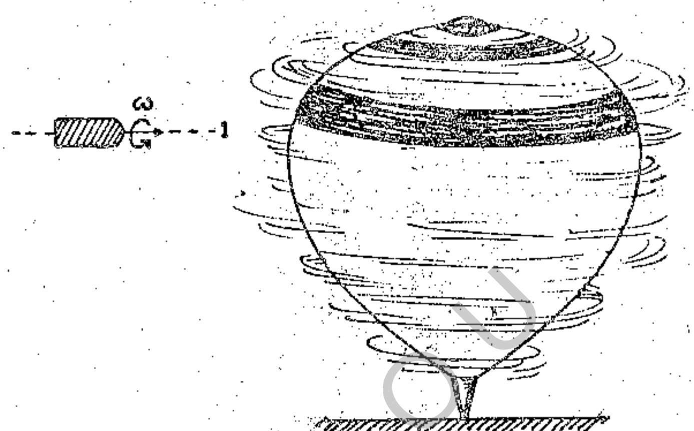


Fig. 5.2a Spinning projectile b) Spinning top

A gyrostat also works on the principle of conservation of angular momentum. The gyrostat consists of a heavy circular disc which spins with very high frequency about its symmetry axis. The axle of the disc is mounted on gimbals such that the disk along with axle can turn freely about any of the three mutually perpendicular axis. Hence no external torques can act on the spinning disc and in space. Its angular momentum remains constant. Therefore the spinning disc maintains it's orientation in space.

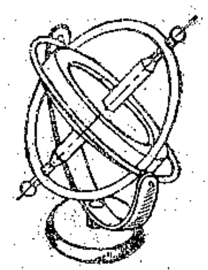


Fig. 5.3 Gyrostat

A navigational gyro is an air craft, relies on this fact. The wheel is kept spinning by directing an air steam at the periphery of the wheel. The direction of the wheel's axle remains fixed in space while the plane turns, banks and rolls about it. Thus regardless of external visibility the pilot always has a fixed reference direction available to him.

Acrobats, ice skaters, divers, ballet dancers etc. make use of the principle of conservation of angular momentum to perform feats. A dramatic illustration of the application of the principle of conservation of angular momentm can be seen in an ice skater's spinning motion. To start with an ice skater swings into a turn, then stands on the toe of one skate, rotating slowly about that point. The ice skater constitutes a torque-free system since, the contact between the skate toe and the ice is essentially frictionless. The skater usually begins the spin with arms and one leg extended and the torso horizontal as shown in Fig. 5.4a

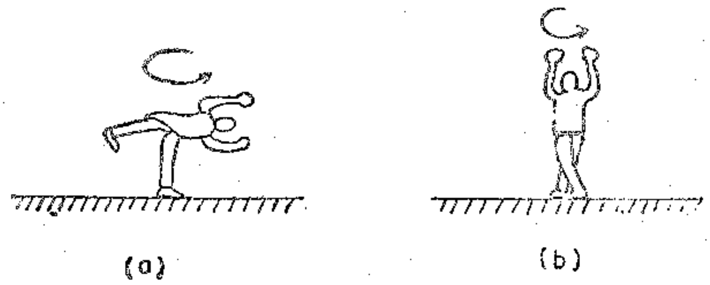


Fig. 5.4 a) Intial position of a skater b) Final position of a skater

This position makes most of the mass of the skater's body placed at maximum distance from the axis of rotation and thus maximizing the moment of inertia. While spinning about the reference axis, the skater slowly changes position of the body to that shown in Fig. 5.4b. At this position the bulk of the skater's mass is close to the axis of rotation and hence the moment of inertia is greatly reduced. As per the principle of conservation of angular momentum the product I ω must remain constat. Since I decreases ω should increase. Thus skilled skaters can achieve rotatinal velocities so large that they appear blurred to the observer.

Let us consider an acrobat who has just left a swing as shown in Fig. 5.5 with arms and legs extended and with a small clockwise angular momentum. When the acrobat pulls his arms and legs in., his moment of inertia I become smaller. Since I ω should be constant, the angular velocity of the acrobat increases and as a result he can easily take turn upside or down performing a feat.

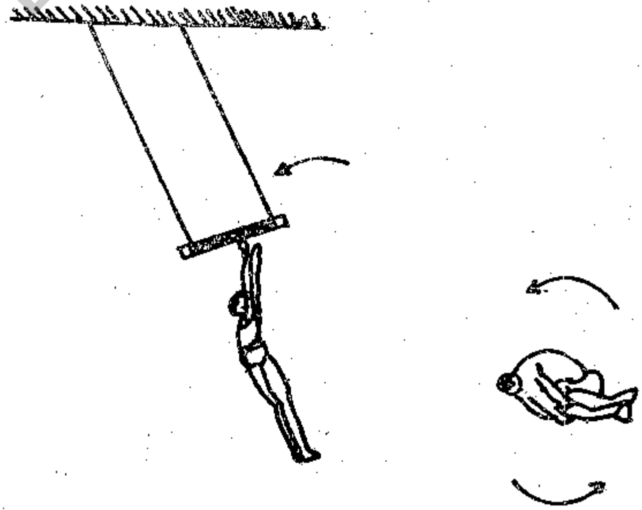


Fig. 5.5 Acrobat leaving the swing

Let us consider the motion of a diver as shown in Fig. 5.6 Let ω_o be the angular velocity of the diver about a horizontal axis passing through the centre of mass at the instant the diver leaves the diving pad so that he would rotate through half a turn before he strikes water. If the diver likes to make one and a half turn somersault before he strikes water, he has to triple his angular speed. Since there are no external forces acting on him except gravity (and gravity exerts no torque about his centre of mass) the angular momentum of the diver must remain constant.

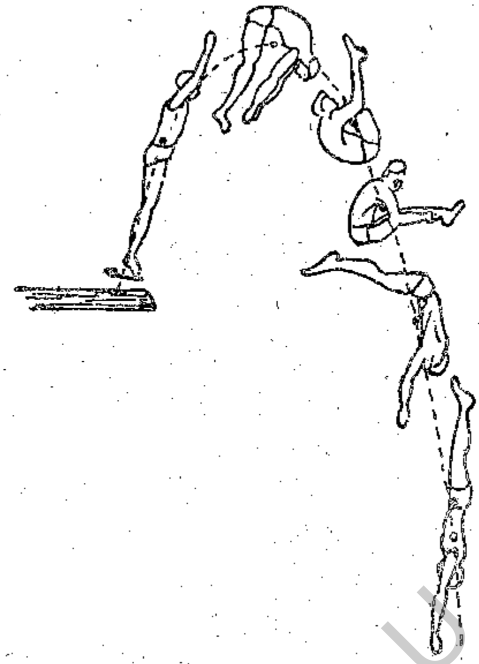


Fig. 5.6 Motion of a diver

That is $I_o\omega_o = I\omega$. Since the diver should change his moment of inertia about the horizontal axis passing through the centre of mass the initial value of Io to a value of $I=I_o/3$. This he does by pulling his arms and legs close to the centre of mass of the body. This result indicates that the greater his initial angular velocity and more he can reduce his rotational inertia (skill of the diver), the greater would be the number of revolutions he can make before he touches the waters i.e., in a given interval of time.

The rotational kinetic energy the diver changes as he rotates through. Since $I < I_o$ the rotational kinetic energy of the diver while executing the somersault will be greater than that initially he had at the instant of leaving the diving pad, i.e., $\frac{1}{2}I\omega^2 > \frac{1}{2}I_o\omega^2$ The increase in rotational kinetic energy is supplied by the diver who does work when he pulls the parts of his body together.

Another example of the application of the principle of conservation of angular momentum is illustrated in Fig. 5.7

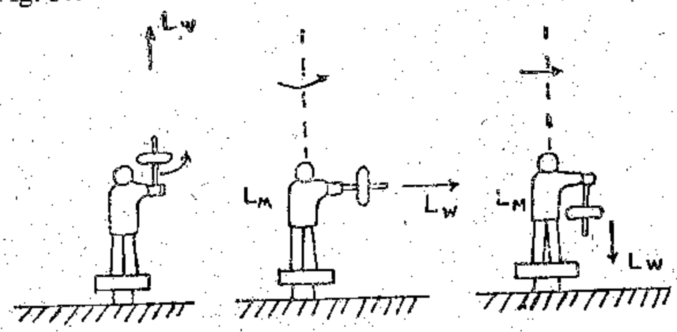


Fig. 5.7 Motion of a man with a wheel on a turn table

Here a man is standing on a turn table so that no torques can be transmitted to him about a vertical axis. If the man at rest holds a spinning wheel with its angular momentum pointing vertically, the man+the wheel system has a total vertical angular momentum equal to L wheel. If he turns the axis of the wheel away from the vertical, then the vertical component of the angular momentum of the wheel decreases. Its z-component will be $L_z \cos\theta$. Where θ represents the angle of the wheels axis with respect to the vertical. As per the principal of conseration of angular momentum, the total value of L_z must remain constant. Hence the man and the turntable begin to rotate making up for the reduced vertical component of the wheel's angular momentum. When the wheel is returned to its original direction, the rotation of the man and the turn table ceases. As long as no external torque acts on the system, the observations are true irrespective of the way how the z-component of the angular momentm of the wheel is changed. For example the man on the initially stationary turn table can stop the spinning wheel with his hand. Immediately the man and the turn table will turn so that L_z will be unchanged. By keeping the wheel axis vertically if the man uses his free hand to start wheel to spin, both the man and the turn table comes to rest when the wheel attains its original spinning speed.

5.5 APPLICATION OF THE PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM TO MICROSCOPIC AND MACROSCOPIC SYSTEMS

The principle of conservation of angular momentum is also applicable to systems involving of sub-atomic particles (microscopic systems) that we come across in atomic and nuclear physics. In atomic collision processes this principle is highly useful in understanding the interaction process. It is well known that Newtonian mechanics does not hold good in the atomic and nuclear domain, but the applicability of the law of conservation of angular momentum in this domain indicates that this principle is more fundamental than the Newtonian principles.

We have seen the law of conservation of angular momentum holds good for a system of particles. Given if the system consists of bodies with finite dimensions and the bodies have spin, the conservation of angular momentum still holds good provided we include in total angular moment the angular momentum associated with their spin. In atomic and nuclear physics we deal with elementary particles like electrons, protons, neutrons, mesons, baryons etc. which posses angular momentum associated with their spin motion and also due to their orbital motion about some external point. When we apply the law of conservation total angular momentum of the system. We should include the spin angular momentum also.

The principle of conservation of angular momentum is also applicable to macroscopic system such as the motion of planets, satellites etc. In applying this principle it is necessary that we should take into account the intrinsic spin motion of these bodies. The conservation of angular momentum plays a key role in the evaluation of theories put forward regarding the origin of solar system, contraction of gaint stars etc. For example let us understand the phenomenon involved in the evaluation of stars where the principle of conservation of angular momentm is taken care off.

Stars are formed by self-gravitation of clouds of dust and gas in interestellar space. Any small angular velocity possessed by the system of dust cloud, owing to the slow rotation of the entire galaxy increases tremendously as the star shrinks from a diffuse extended close to a relatively small dense object. This is because the moment of inertia of the system decreases and to satisfy the principle of conservation of. angular momentum, the angular velocity increases proportionately.

The process of gravitional contraction heats the star to the point of luminescence and beyond, eventually raising the internal temperature. As the temperature thus increase the self sustaining nuclear reaction begins to take place.

5.6 SUMMARY

The total angular momentum of the system remains constant when the resultant torque acting on a system of particles is zero. This is known as principle of conservation of angular momentum. Acrobats, ice skaters, divers, ballet dancers etc. Use the principle of conservation of angular momentum in performing the feats.

5.7 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following in 30 lines.

1. Explain the principle of conservation of angular momentum. Give some applications of this principle.

II. Solve the Following Problems.

1. A wheel of 1 m diameter starts from rest and accelerates uniformly to an angular velocity of 60 rad s^{-1} in 10 sec. Find the angular acceleration and angle turned through during this time.

   (Ans. α =6 rad s^{-2}, θ = 300 rad.)

2. A point on the rim of a wheel moves 4 m wen th wheel turns through an angle of 1/5 rad. Find the radius of the wheel.

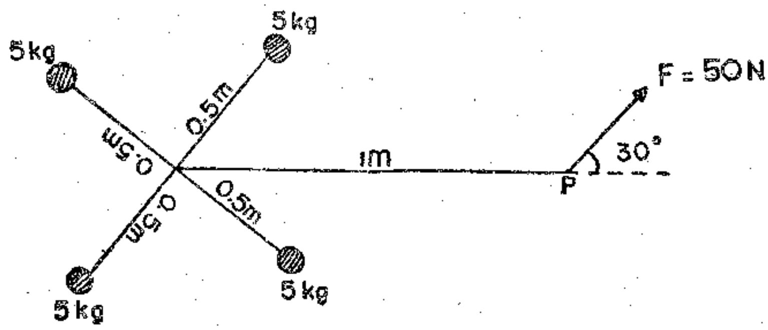
3. A wheel is initially rotating with an angular velocity of 5 rad s^{-1}. After 10 sec. its angular velocity is found to be 15 rad s^{-2}. Find the average angular acceleration. (Ans. 1 rad.s^{-2})

4. A motor shaft is turning at 500 rev/sec. When the motor is turned off. It stops after 50s. Find the acceleration. (Ans. 62.8 rad s^{-2})

5. A wheel starts from rest and in 10 sec reaches a speed of 500 rev/sec. Find its angular acceleration. (Ans. 314 rad s^{-2})

6. A weight is falling with an acceleration of 60 ms^{-2} from a pulley of radius 0.5m. Find the angular acceleration of the wheel. (Ans : 12 $rads^{-2}$)

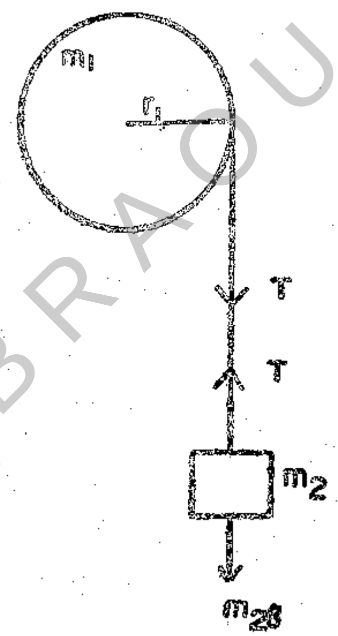
7. Four 5 kg mass are connected by 0.5 m spokes to an axle as shown in Fig.III-1. A force of 50 N acts on a lever of 1m long to produce an angular acceleration α. Find its magnitude.



Four 5 kg massess connected by spokes to the axel.

(Ans. 5 rad^{-2}).

8. A solid cylindrical wheel has a cord wrapped around its circumference from which is suspended a weight as shown below. Find the angular acceleration of the fly wheel if it has a radius of 0.2m, a mass of 4 kg and the weight has a mass of 10kg.

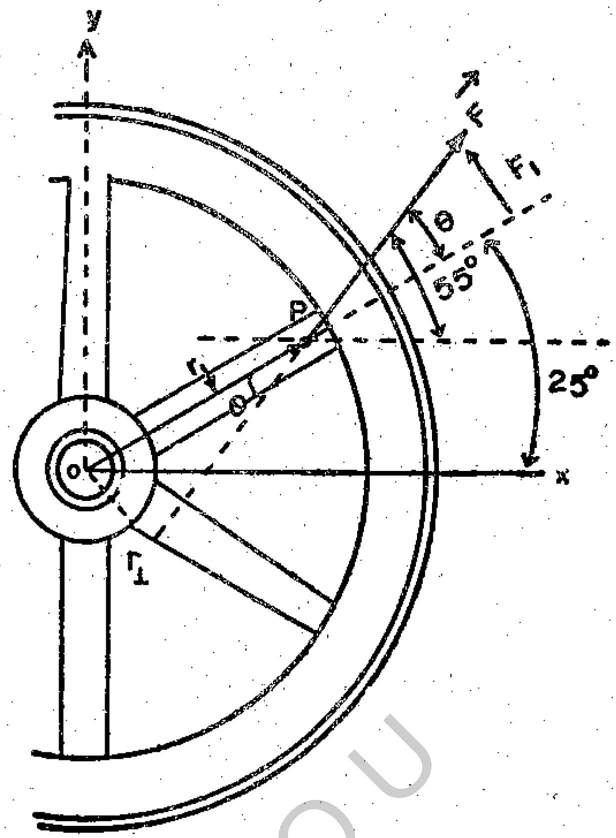


Weight attached to the cylindrical axle. (Ans 40.85 rads^{-2})

9. Consider a body consisting of two spherical mass of 10 kg each connected by a light rod of 1m length. Find the moment of inertia of the body about an axis normal to it and passing through the centre of the system. (Ans : 5 kg m^2)

10. A uniform of rod of length 1 m and mass 0.6 kg is free to rotate about one end of or a frictionless pivot. A point mass of 0.01 kg is connected to the lower end of the rod. Find the moment of inertia of the system about the axis passing through the pivot and perpendicular to the plane of the system. (Ans. 0.31 kg m^2).

11. A wagon wheel is free to rotate about a horizontal axis through o as shown below. A force of 50 N is applied to spoke at P which is at the distance of 0.5 m from the point o. OP is making an angle 25° with the horizontal axis and the force is in the plane of the wheel making an angle of 55° with the horizontal. Find the torque acting on the wheel.

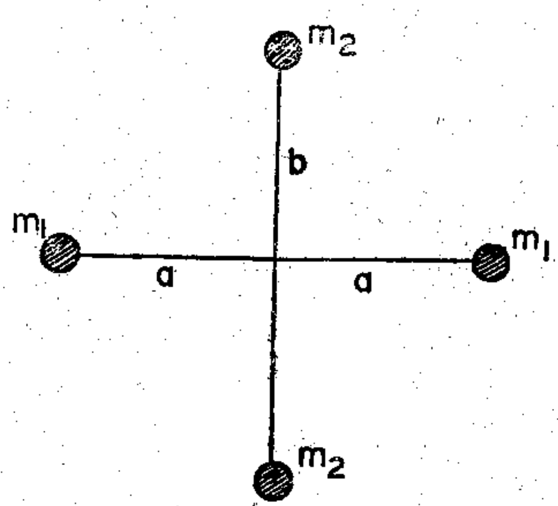


Wagon wheel rotating about a horizontal axle. (AnsT = 12J).

12. A rifle bullet wit a mass of 0.01 kg is fired with a velocity of 500 ms^{-1}. It strikes the rim of a wheel tangentially. The wheel is initially at rest. It has a radius of 0.5 m and a mass of 4kg and is in the form of a solid cylinder. If the bullet embeds itself in the rim of the wheel, find the angular velocity of the wheel after the impact.

(Ans. 5 rad s^{-1})

13. Four particles are located at the corners of a polygon as show in the figure below. Assuming that the particles are connected by light rigid rods. Calculate the moment of inertia of the system about O. If the system rotates about Z-axis with an angular velocity of 5 rad s^{-1}, calculate the kinetic energy of the system. If m_1 5 kg and m_2 = 10 kg and a=1m and b=2m, calculate the moment of inertia of the system.



Four particles at the corners of a polygon. (Ans. I=90 kg m^2, k=1125J).

14. An electric drill motor produces a 50N m torque. It is attached to 1 m diameter sanding disk with a moment of inertia of 0.05 kg m^2. Find the tangential velocity of a point on the rim of the initially stationary disc 0.05s after the motor is turned on. (Ans 25 ms^{-1})

15. A man stands at the centre of a turn table holding his arms extended horizontally with 0.5 kg weight in each hand. He is set rotating about a vertical axis with an angular velocity of 1 revolution per sec. Find the angular velocity of the system when the man drops his hands downwards. The moment of inertia of the man may be assumed constant 20 kg m^2. The original distance of the weights from the axis is 1 m and their final distance is 0.5m.m

(Ans $\omega = 2.4\pi$ rad s^{-1})

16. An ice skater whose initial moment of inertia is 5 kg m^2. is revolving about himself with an angular velocity of rotation per second. After sometime if the moment of inertia of the ice skater reduces to 1 kg m^2. find his angular velocity.

(Ans $\omega = 10\pi$ rad s^{-1})

17. A star in the form of dust cloud has a radius of 10^{25}m. Its angular velocity is 1 rotation per hour. Find the angular velocity of the star when it condenses to a radius of 5×10^3 m.

(Ans 1.11×10^6 rot s^{-1})

18. Suppose the earth suddenly contract so that its moment of inertia became one-third its present value. How long would be then the present 24 hr. day. (Ans 8 hours per day).

5.8 GLOSSARY

Free-fall	: A body dropped to the ground from a height with zero velocity falls to the ground under uniform acceleration. Such a fall is said to be a free-fall.
Rigid body	: A body is said to be rigid if all the particles of the body undergo the same translation as the body itself when subjected to motion.
Wobbling motion	: A body undergoing a change in its orientation as it moves is said to posses a wobbling motion.
Spin motion	: Rotation of a body about an axis passing through its centre as it moves round another axis.
periphery	: Outer surface of an object.
Somersault	: Fall in which one turns heels over head before landing on one's feet.
Subatomic particles	: Elementary particles like proton, electron, neutron, meson etc.
Luminescence	: Emission of light energy by a body
Nuclear reaction	: A reaction involving atomic nuclei.

Vector angular momentum	: Angular momentm is a vector. Addition of angular momenta is to be done vectorially.

5.9 RECOMMENDED BOOKS

1.	Resnick, R. and Halliday, D.New Delhi.	Physics, Part I	Wiley Eastern Pvt. Ltd.,
2.	Sears, F.W. and Zemansky, M.W.	College Physics	Addision Wesley Publishing Co. Inc. London.
3.	Zafiratos, C.D.	Physics	John Wiley and Sons Inc. New Delhi.
4.	Bueche, F.	Technical Physics	Harper and Row Publishers, New York.
5.	Harnwell, G.P. and Legge, G.J.F.	Physics, Matter, Energy and Universe	East West press Pvt. Ltd., New Delhi.
6.	Serway, R.A.	Concepts, Problems and solutions in general Physics, Vol.I	W.B. Saunders Co. London
7.	White, H.E.	Modern College Physics	East West Press Pvt Ltd. New Delhi.
8.	Subrahmanyam (Editor)	Mechanics	Telugu Akadami, Hyderabad.

[illegible]

BLOCK -3

GRAVITATION

UNIT-6 MOTION OF PLANETS AND SATELLITES KEPLERS LAWS.

Contents

6.1 AIMS AND OBJECTIVES

This unit explains the motion of planets and satellites. The motion of planets is explained by applying Kepler's laws to it (both qualitatively and quanlitatively). The orbital velocity of a satellite is evaluated quantitatively. After going through this unit you will be able to (1) derive the Kepler's laws of planetary motion from conservation laws; (2) evaluate the velocity of an orbiting satellite close to the earth.

6.2 INTRODUCTION

In this unit the motion of planets and satellites is discussed on the basis of kepler's Laws. With the help of kepler's laws the orbital velocities of the earth satellite is evaluated.

6.3 MOTION OF PLANETS AND SATELLITES-GENERAL DESCRIPTION

The universe is a dynamic system consisting of galaxies, stars, planetary systems etc. which are in a state of motion. There is no system within the universe which is at rest. It is the quest of man to gain an insight into his surroundings. Astronomy is a branch of science that has been developed dealing with motion of cellestial bodies. Earliest efforts were directed first to understand our solar system. Ptolemy belonging to 2nd century B.C. proposed geocentric theory in which he considered earth as stationary at the centre of the universe with the sun, moon, planets and stars-all revolving about the earth in complex orbits.

Nicolaus Copernicus assumed Earth as a plant like Venus, Mars and Jupiter. He proposed heliocentric theory wherein earth was considered a planet rotating on its axis and revolving about the sun which is considered to be stationary. The rotation of the earth about its axis was assumed to account for the apparent daily motion of the sun, moon and stars. These cellestial bodies are so far away, for any motion of the earth to show an observable change in their relative positions. The other planets are also assumed to have motion around the sun similar to that of the earth as shown in Fig.6.1.

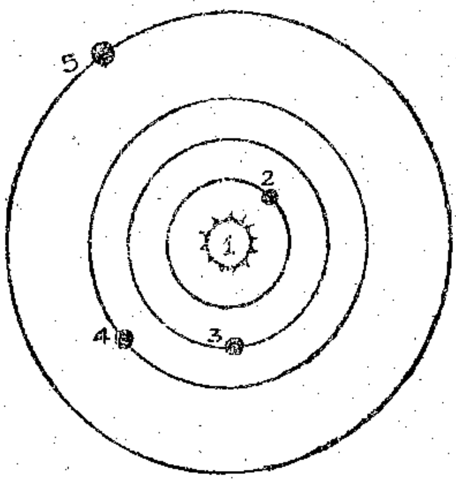


Fig. 6.1 Copernican view of the solar system.

Proper understanding of the dynamic nature of the solar system came into existence due to the untiring efforts of Tycho Brahe and Johannes Kepler. Tycho Brahe collected data on planetary position for a period of 20 years and kepler made a systematic analysis of the data. Kepler made a careful study of motions of mars. He tried to fit the different recorded positions of planet to concentric circular orbits for mass and the earth around the sn. Although the observations did not fit to a circular orbit, he noted that mars seemed to move faster when its distance from the sun was less than the average and to move more slowly when it was a little farther away. Kepler tried an ellipse with the sun at one of its foci and found that the data fitted well to the trajectory witin the limits of observational error. This analysis led Kepler to discovery that planet's move in elliptical orbits. After analysing further the data of Tycho Brahe, Kepler announced three laws regarding the motion of planets round the sun.

1. The path of a planet is ellipse, with the sun at one focus as shown in Fig.6.2 a.

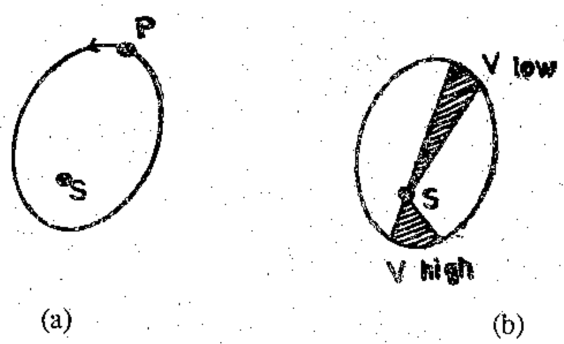


Fig. 6.2 (a) Path of a planet P, S is sun.
(b) Radius vector sweeps equal areas in equal intervals of time.

Copernicus concept of circular orbits for the planets was ruled out with this law. It may be said here that the orbits depart only slightly from circularity, particularly in the case of venus, earth and neptune.

2. A straight line joining a planet to the sun sweeps out equal areas in equal times as shown in Fig. 6.2b. This law implies that the planets do not move at constant speed. The speed is maximum when the planet is closest to the sun and a minimum when it is farthest from the sun.

3. The square of the period divided by the cube of the mean distance from the sun is the same quantity for all planets.

$$\text{i.e.,} \quad \frac{T^2}{r^3} = \text{constant} \qquad (6.1)$$

This can be seen from Table 6.1

Table 6.1 Measured characteristics of the planets.

	Planet	Mean distance between the planet and sun r (10^6 km)	Period T (Years)	T^2 / r^3
1.	Mercury	57.9	0.241	1.245
2.	Venus	108.1	0.615	1.252
3.	Earth	149.5	1.000	1.247
4.	Mars	227.8	1.881	1.249
5.	Jupiter	777.8	11.862	1.246
6.	Saturn	1426.0	29.458	1.247
7.	Uranus	2869.0	84.015	1.245
8.	Neptune	4496.0	164.790	1.246
9.	Pluto	5899.0	247.700	1.246

The motion of the planets round the sun in elliptical orbits based on Keplers' laws is shown in Fig. 6.3

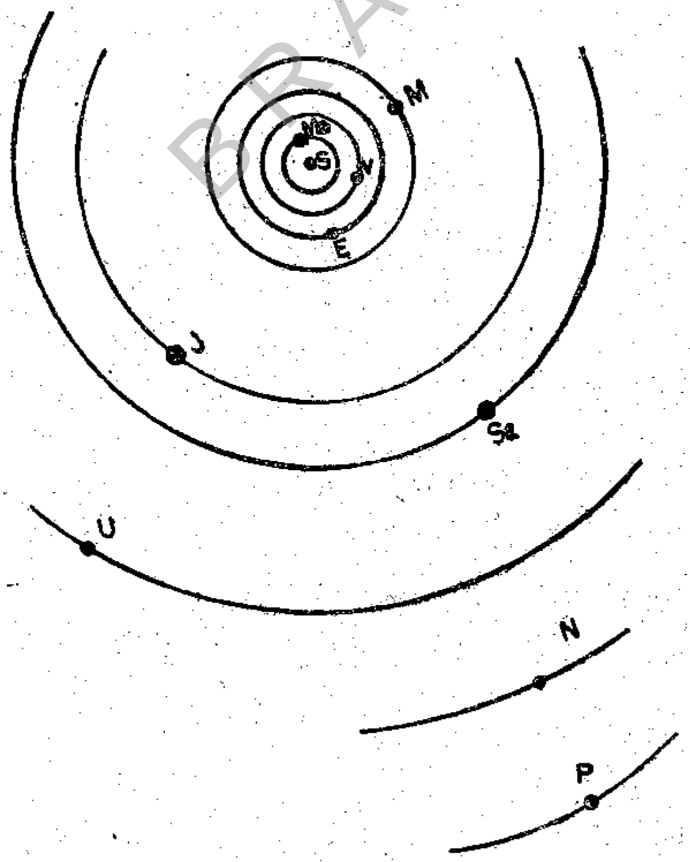


Fig.6.3 Solar System

The moon is the natural satellite of the earth. It revolves round the earth with a period of 27.3 days. Recent advances in science and technology and major break-throughs in space travel led to the introduction of artificial satellites in fixed orbits round the earth. The planet jupiter has 12 moons. The motion of these satellites either natural or artificial, is governed by the laws of motion and Newton's law of universal gravitation.

6.4 MOTION OF PLANETS-EXPLANATION BASED ON NEWTON'S LAW OF GRAVITATION AND CONSERVATION LAWS - DERIVATION OF KEPLER'S LAW

The Universal law of gravitation together with the conservation laws governing motion of objects provides a complete description of the motion of planets and satellites. Each planet in the solar system may be considered to rotate about the sun independently of the others. This is because the sun is a few million times as massive as each of the four planets close to him. The average distance between any two of these planets is comparable to the separation of a planet from the sun. Hence the perturbing effect of these planets of each other should be small in comparison with the effect of the sun by a factor of about 10^6. The perturbing effect of Jupiter on earth is only 10^{-4} times the influence of sun on earth.

The first law is known as law of orbit. The second law is known as the law of areas. The third law is known as law of periods,. The kepler's laws can be derived from Newton's laws and the law of universal gravitation.

Let a planet of gravitational mass 'm' be moving under the gravitational attractions of sun, sun being at the origin.

This force of attraction is given by,

$$F = -\frac{GMm}{r^2}\bar{r} \qquad (6.2)$$

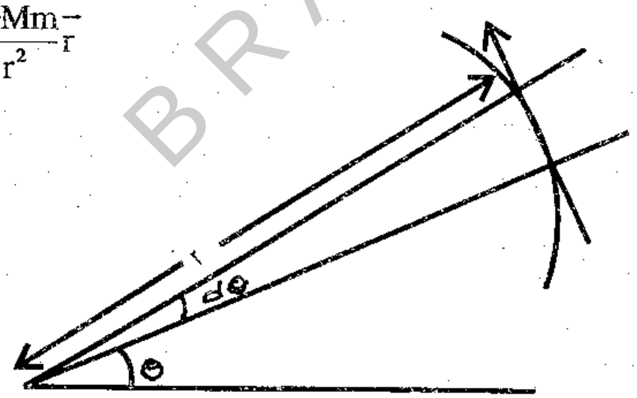


Fig. 6.4

Where G is the gravitational constant, M is the mass of the sun, r is the distance of the planet from the sun and $\vec{r}$ is must vector along r.

According to polar coordinates,

$$\vec{F} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{r} + m\left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\vec{\theta} \qquad (6.3)$$

Where $\vec{\theta}$ is unit vector along θ and perpendicular $\vec{r}$ Since gravitational force of attraction is radial, transverse acceleration of the particle is zero. Hence

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0 \tag{6.4}$$

on intergration,

$$r^2\frac{d\theta}{dt} = \text{constant (say p)} \tag{6.5}$$

Then the equation 6.3 can be written as,

$$\vec{F}=m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\vec{r} \tag{6.6}$$

With the knowledge of these expressions we can prove the kepler's laws.

6.4.1 Kepler's Ist Law:

since $F=-\frac{GMm}{r^2}\vec{r}$ which is also equal to

$$m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\vec{r}$$

$$\frac{-GMm}{r^2}\bar{r}=m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\bar{r} \tag{6.8}$$

or $$-\frac{GM}{r^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]$$

Let $r=\frac{1}{\mu}$ $\therefore$ $\frac{dr}{d\theta}=\frac{1}{\mu^2}\frac{d\mu}{d\theta}$

and $\frac{dr}{dt}=\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$

$$= -\frac{1}{\mu^2}\frac{d\mu}{d\theta}.\omega \qquad \because \omega=\frac{d\theta}{dt}$$

$$= -\frac{1}{\mu}\frac{d\mu}{d\theta}.\frac{P}{r^2}\omega \qquad \because p=r^2\frac{d\theta}{dt}$$

$$= -p\frac{d\mu}{d\theta}.$$

Similarly $\frac{d^2r}{dt^2}=\frac{d}{dt}\left(-p.\frac{d\mu}{d\theta}\right)=-p.\frac{d}{d\theta}\left(\frac{d\mu}{d\theta}\right)\left(\frac{d\theta}{dt}\right)$

$$= -p^2 \mu^2 \frac{d^2 v}{d\theta^2} \qquad \left[\because \frac{d\theta}{dt} = \omega = \frac{p}{r^2} = p\mu^2\right] \qquad (6.9)$$

Substituting these values, the equation (6.8) cna be written as

$$-GM\mu^2 = -p^2\mu^2 \frac{d^2 v}{d\theta^2} - \frac{p^2\mu^4}{\mu} \qquad (6.10)$$

Dividing with $p^2\mu^2$ and rearranging the terms, the equ.610 can be rewritten as

$$\frac{d^2\mu}{d\theta^2} + \mu = \frac{GM}{p^2} \qquad (6.11)$$

The solution of this equation can be written as

$$\mu = \frac{1}{r} = A\cos\theta + \frac{GM}{p^2} \qquad (6.12)$$

Where A is the constant Multiplying with $\frac{p^2}{GM}$ on both sides

$$1 + \frac{p^2 A}{GM}\text{Cos}\theta = \frac{p^2/GM}{r} \qquad (6.13)$$

This can be compared to $1 + e\ \cos\theta = \frac{1}{r}$ (6.14)

Where $\qquad 1 = \frac{p^2}{GM}$ and $e = \frac{p^2 A}{GM}$ (6.15)

Here both e & l determine the shape of the path. This can be an equation of ellipse or a circle or a parabola.

Since ellipse and circle are only closed figures it follows that, the plannets must move in ellipse about the sun with Sun at one of its foci. The above equations represents an ellipse with $\frac{p^2 A}{GM} < 1$ where l is the semilatus rectum of the elliptical orbit. If e=o Motion will be circular.

6.4.2 Kepler's 2nd Law:

The area d·A swept out by $\bar{r}$ in turning through angle $d\theta$ will be

$$dA = \frac{1}{2} r\ .rd\theta = \frac{r^2}{2} d\theta \qquad (6.16)$$

If $d\theta$ is the angle turned time dt, then

$$\frac{dA}{dt} = \frac{r^2}{2} . \frac{d\theta}{dt} = \frac{p}{2}$$

Which is constant. This is keplers second law.

6.4.3 Kepler's 3rd Law:

In case pf an ellipse, the relation between semilatus rectum 'l', semi major axis 'a' and semi minor axis 'b' is

$$l^{\Phi} = \frac{b^2}{a} = \frac{p^2}{GM} \qquad (6.17)$$

So the period of the planet $T = \frac{\text{Area of ellipse}}{\text{Areal velocity}} = \frac{\pi ab}{p/2}$ (6.18)

or $T^2 = \frac{4\pi^2 a^2 b^2}{-p^2}$ Taking Squares (6.19)

$$T^2 \text{ or } \frac{4\pi^2 a^2 p^2 a}{GM\, p^2} = \frac{4\pi^2 a^3}{GM} \qquad (6.20)$$

That is $T^2 \propto a^3$ (6.21)

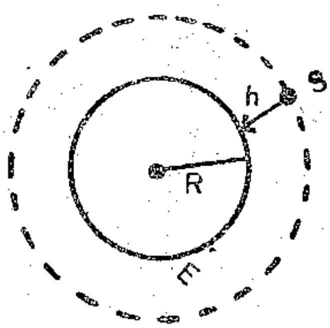
This is kepler's third law.

Worked Example -1:

A typical satellite of the earth moves in an orbit of 5×10^5 m above the earth. The radius of the earth is 6.37×10^6 m. Determine the period of the satellite. The mass of the earth is 5.98×10^{24} kg. $G=6.67 \times 10^{11}$ N m^2 kg^2

As shown in the Figure the distance between the centre of the earth and the satellite is

$$r = R + h$$
$$= 6.37 \times 10^6 m + 5 \times 10^5 m$$
$$= 6.87 \times 10^6 m$$



Satellite moving round the earth.

The period of the satellite T is given by

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$= 4\left(\frac{22}{7}\right)^2 \frac{1 \; (6.87 \times 10^6)^3}{6.67 \times 10^{-11} \times 5.98 \times 10^{24}} = 5.67 \times 10^3 \text{ sec.}$$

6.5 MOTION OF SATELLITES

We are living in space age. We are all familiar with satellite communication systems, interplanetary space travel establishment of space station etc. When space vehicle takes off from the ground to orbit the earth as a satellite, its initial take off direction is vertically upwards. As the rocket gains height control fins are set to make it turn slowly towards a horizontal trajectory. To find the velocity that a space vehicle should acquire to circle the earth let us understand the details presented in Fig.6.5.

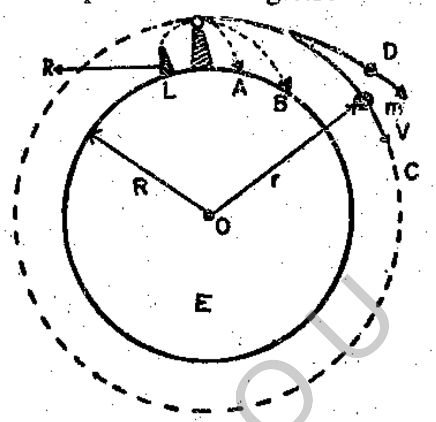


Fig.6.5 Launching of projectiles from the top of a tower.

It is assumed that projectile are launched horizontally from the top of a tower of several kilometers height. When the initial velocity is low the projectile takes a parabolic path indicated by A. At high initial velocity the trajectory would be B. But at very high velocities the projectile falling toward the earth takes the path of a circle of radius r. This velocity is called orbital velocity. At still higher initial velocities such as D the projectile may follow an elliptical path or even escape from the earth's gravitational field. When the satellite just takes a circular orbit, then the force on such a satellite will be directed toward the centre of the earth as shown in Fig.6.6

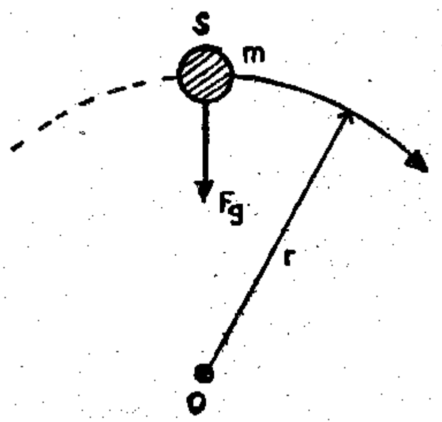


Fig. 6.6 Direction of the force acting on the satellite. Centre of the earth.

When the satellite takes a circular path than the required centripetal force is provided by gravity. Hence the centripetal force $F_c = F_g$ gravitational force.

$$F = \frac{mv^2}{r} = F_g = G\frac{M_e m}{r^2} \tag{6.22}$$

$$\text{Hence} \quad v^2 = \frac{GM_e}{r} \tag{6.23}$$

Eq.6.23 represents the orbital velocity of the satellite. This equation holds good for any satellite moving in a circular orbit around any astronomical body through out universe. If the satellite is revolving round the earth is close to its surface then we can replace r by R, the radius of the earth. Under these conditions the centripetal force is just mg where g is acceleration due to gravity. Hence

$$mg = \frac{GM_e m}{R^2} \tag{6.24}$$

Here R=6370 km and $g=9.8\ ms^{-2}$

Now $G\,M_o = gR^2$ (6.25)

Using Eq.6.25 in Eq.6.24 we get

$$v = \left[\frac{gR^2}{r}\right]^{1/2} \tag{6.26}$$

The above equation represents the orbital velocity of a satellite revolving round the earth.

Worked Example-2

A satellite orbits the earth 1000 km above the surface. Find its orbital speed given the radius of the earth as 6.37×10^6 m, mass of the earth as 5.98×10^{24} and the gravitational constant G as $6.673 \times 10^{-11}\ N\ m^2\ kg^{-2}$.

The orbital velocity of the satellite is given by

$$v = \left[\frac{GM}{r}\right]^{1/2} = \left[\frac{GM}{R+h}\right]^{1/2}$$

$$v = \left[\frac{6.6673 \times 10^{-11} x 5.98 x 10^{24}}{\left(6.37 \times 10^6 + 1 \times 10^6\right)}\right]^{1/2}$$

$$v = \left[\frac{6.673 x 10^{13}}{7.37 x 10^6}\right]^{1/2} = \left[\frac{6.673 x 10^7}{7.37}\right]^{1/2}$$

Worked Example - 3

One of jupiter's 12 moons (Europa) takes 3.5 days to mplete an approximately circular orbit with a radius of 7×10^8 m. The earths moon completes an orbit once in 27.3 days. The orbital radius of the earth's moon is 4.01×10^8m. Find the mass of the jupiter given the mass of the earth, $5.98\ x 20^{24}$ kg

Considering the moon revolving round the planet in circular orbit we have the gravitational force of attraction is equal to the centrifugal force i.e.

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 \qquad (1)$$

Let M_g, M_{EM}, M_J and M_{JM} respesent the masses of the earth's moon, jupitar and jupitar's moon respectively. Let r_{em} and r_{jm} represent the radii of earth's moon orbit and jupitar's moon orbit respectively.

Then

$$\frac{GM_e m_{em}}{r_{em}^2} = \frac{m_{em}}{r_{em}}\left(\frac{2\pi r_{em}}{T_{em}}\right)^2 \qquad (2)$$

$$\frac{GM_j m_{jm}}{r_{jm}^2} = \frac{m_{jm}}{r_{jm}}\left(\frac{2\pi r_{jm}}{T_{jm}}\right)^2 \qquad (3)$$

Dividing Eq. (3) by Eq. 2, we get

$$\frac{M_J}{M_E} = \left(\frac{T_{em}}{T_{jm}}\right)^2 \left(\frac{r_{jm}}{r_{em}}\right)^3$$

$$M_J = \left(\frac{27.3\,\text{days}}{3.5\,\text{days}}\right)^2 \left(\frac{7x10^8\,m}{4.01x10^8\,m}\right)^3 M_E$$

$Mj = (7.8)^2\,(1.74564)^2 \times 5.98 \times 10^{24}$ kg

$Mj = (60.84)\,(5.3194) \times 5.98 \times 10^{24}$ kg

$Mj = 1.935 \times 10^{27}$ kg

Worked Example -4

A satellite at a distance of 7×10^6m from the earth takes 90 minutes to complete one rotation. If this satellite is to be placed such that it remains stationary when viewed by an observer on earth find the distance of the satellite from the centre of the earth.

A fixed satellite must orbit the earth in an equitorial plane every 24 hours travelling west to east. Then it turns about the earth in the same sense and at the same rate as earth spins about its axis. Using Kelpler's third law, we have

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{R_1}{R_2}\right)^3$$

$R_1 = 7 \times 10^6$m, $T_1 = 90 \times 60$ s, $T_2 = 24 \times 60 \times 60$s

$$R_2^3 = (R_1)^3\left[\frac{T_2}{T_1}\right]^2$$

$$R_2^3 = \left(7x10^6\right)^3\left[\frac{24x60x60}{90x60}\right]^2$$

$$R_2^3 = (343)10^{18} x16[5488x10^{11}]$$

$$R_2 = \sqrt[3]{5488x10^6}m = 1.76x10^7 m$$

6.6 SUMMARY

Johannes Kepler proposed the laws governing the motion of planets round the sun.

1. The path of a planet is an ellipse with sun at one of its foci.
2. The radius vector joining the sun and the planet sweeps equal areas in equal intervals of time.
3. The square of the period of any planet about the sun is proportional to the centre of the planets mean distance from the sun.

A satellite is a body revolving round a planet. The orbital velocity of satellite is given by

$$v = \left(\frac{GM}{r}\right)^{1/2}$$

Where M is the mass, r the radius of the satellite.

6.7 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. Discuss briefly the basic developments in science that lead to a proper understanding of the motion of planets in our solar system.
2. Prove that the path traced by a planet round the sun is elliptical from energy considerations of the system.
3. Show how Newton's law of gravitation and conservation laws governing motion are useful in proving the second and third laws of Kepler.

II. Answer the Following Questions in About 10 Lines.

1. State Kepler's laws of planetary motion.
2. What is a satellite? Show that for a satellite moving close to the earth orbital velocity is given by $v=(gR^2/r)^{1/2}$.

III. Solve the Following Problems

1. One of the satellites of a planet revolves round it once in 5 days. The radius of the orbit of the moon is $5x10^6$ km. Determine the mass of the planet. (Ans: 39.67 x 10^{24}kg)
2. Determine the period of mercury for which the semi - major axis of the orbit traced by it round the sun is 5.79 x 10^{10} m. The mass of the sun is 1.97 x 10^{30} kg. (Ans: 7.64 x 10^6 sec)
3. An artificial satellite is to be placed in an orbit about the earth in the plane of its

equator so that it appears stationary to an observer on earth. Find the radius of the orbit.

(4.22×10^7m)

4. A satellite is placed in an orbit of the moon at an altitude of 100 km above the moon's surface. Determine the orbital velocity of the satellite and all its period. The mass of the moon is 7.355×10^{22} kg and the radius of the moon is 1.1738×10^6m.

(Ans: $1.634 \times 10^3 ms^{-1}$ 1.964 hrs)

5. The period of rotation of one of the moons of jupiter in 1.53×10^5 sec. The radius of the moon's orbit is 5 times the radius of the planet juppiter. Determine the density of jupiter.

(Ans: 1050 kg/m^2)

6. The distance between the planet mars and sun is 1.524 times larger than the distance of the earth from the sun. Determine the period of mars round the sun.

(Ans: 1.878 years)

7. Determine the period of a satelite revolving round the earth from a height of 900 km.

(Ans: 196 s)

UNIT 7 GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

Contents

7.1 AIMS AND OBJECTIVES

This unit discusses the concept of gravitational field and gravitational potential of a body at a given point.

It explains that the existence of gravitational field is due to the attraction between any two bodies., and the gravitational potential due to the point in gravitational field. After going through this unit you will be able to (1) evaluate the binding energy of a system of particles;

(2) learn that the gravitational field strength decreases as the distance increases.

7.2 INTRODUCTION

In this unit we shall discuss the concept of Gravitational field and gravitational potental. By using the concept of Newtons universal law of gravitation (17th Century), potential energy in a gravitational field.

7.3 GRAVITATIONAL FIELD: FIELD STRENGTH

Newton's Universal law of gravitation envisages attraction between any two masses. This is due to the fact that each mass exerts a force on the other. This may be considered to be due to direct interaction between the two mass particles i.e. action at a distance eventhough the particles are not in contact with each other. The attraction between any two mass particles can also be explained via the field concept which regards a mass particle as modifying the space arond it in some way and setting up a gravitational field. This field acts on any mass particle placed in it by exerting a force on it. The field concept is more conceptual than action at a distance. Hence the force of attraction between any two masses can be explained through the existence of gravitational field. To do this we have to first determine the field established by a given distribution of mass particles and then calculate the force that this field exerts on another mass particle in it.

The field concept associated with a mass particle may be given a quantative embodiment by defining a field strength at any point as the gravitational force that a unit mass would experience if placed at that point. The field strength is a vector quantity and the direction of field strength is the sáme as that of the force.

Let us now examine the gravitational field associated with earth. Whenever a body is brought in the vicinity of the earth, a force is exerted on it. The direction of this force is toward the centre of the earth. Its magnitude is given by mg. We can now associate with each point near the earth a vector g which is the acceleration that a body would experience if it were released at this point.

g defines the gravitational field strength

$$I = g = \frac{F}{m} \tag{7.1}$$

According to Newton's Universal law of gravitation

$$F = GMm/r^2 \tag{7.2}$$

Substituting for F from equation 7.2 in equation 7.1 we get

$$I = F/m = GM/r^2 \tag{7.3}$$

The gravitational field around the earth is shown schemetacally in Fig. 7.1. The arrows show the direction of the field as every where radially inward. The seperation of the lines show that the field is strongest at the surface. For every point at the same distance from the centre the field intensity I has the same magnitude. As the distance increases the field strength decreases.

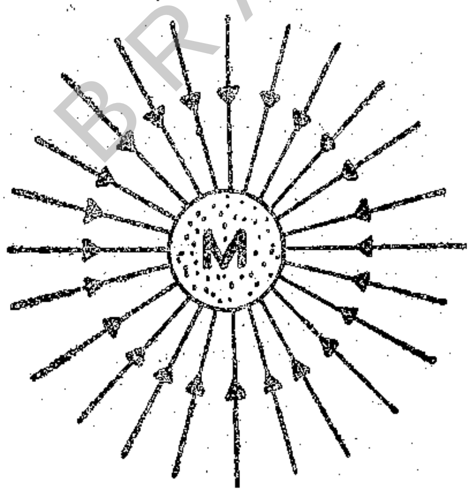


Fig. 7.1 Gravitational field around a spherical mass M.

The gravitational field is a vector field. Since the gravitational field due to a given mass does not change with time it is called stationary field.

The field concept is more conceptual and usefl than the action at a distance concept. It was first Faraday who used the field concept to describe the forces between moving electric charges. It was Albert Einstein who utilised the field concept to explain the phenomenon of gravitaion in developing his general theory of relativity.

7.4 POTENTIAL ENERGY IN A GRAVITATIONAL FIELD - GRAVITATIONAL POTENTIAL

Consider a particle of mass m situated at some point A acted upon by the gravitational field due to mass M situated at the point O as shown in Figure 7.2

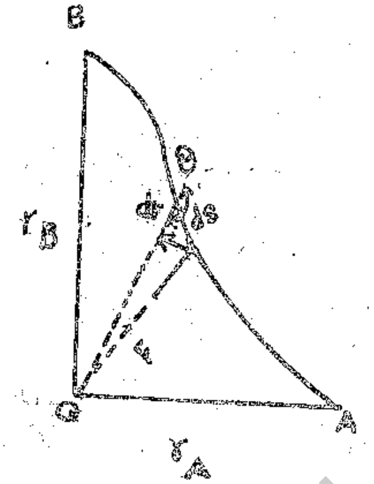


Figure 7.2 Motion of a particle in a gravitational field

The centre of the gravitational field due to the mass M lies at the point O itself. The total energy of this system is constant. When the mass m is at a distancer r_A from a centre O, it posessess some kinetic energy and some potential energy. To obtain an expression for the potential energy of the particle m let us evaluate the work done on the mass m to move it from A to B along the path AB.

The work done in moving the mass m along the path AB through a distance ds is given by

$$dw = \vec{F}.\vec{S} = FS\cos\theta \tag{7.4}$$

As per the figure 7.2

$$dr = ds\,Cos\,\theta \tag{7.5}$$

Hence dw = Fdr (7.6)

The total work done in moving the mass m from A to B is given by

$$\omega = \int_{r_A}^{r_B} F\,dr \tag{7.7}$$

The total work done on the system depends only on the positions of the particle A and B and not on the path traced by the particle. The work done in moving the particle from A to B results in the form of a change in the potential energy of the particle. If the potential energy of the particle at A is U_A and at B is U_B then

$$-w = U_B - U_A = -\int_{r_A}^{r_B} F\,dr \tag{7.8}$$

Since the force F is the gravitional force of attraction we can write

$$F=-GMm/r^2 \tag{7.9}$$

Hence

$$-w = U_B - U_A = -\int_{r_A}^{r_B} \left(-GMm / r^2\right)dr \tag{7.10}$$

$$-w = GMm \int_{r_A}^{r_B} \frac{dr}{r^2} \tag{7.11}$$

$$-w = GMm \int_{r_A}^{r_B} \left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{7.12}$$

The potential energy of a particle at some point P in a gravitational field can be given by the work done in moving the particle from some reference point to the point under consideration. Let the reference point A be at ∞. The $r_A = \infty$ $r_B = r$

Hence equation 7.12 can be written as

$$U_{(r)} = -GMm/r \tag{7.13}$$

$$U_{(r)} - U_{(\infty)} = U_{(r)} - o = -GMm\left(\frac{1}{r} - \frac{1}{\infty}\right) \tag{7.13a}$$

The minus sign indicates that the potential energy is negetive at any finite distance. The potential energy is zero at infinity and decrease as the separation distance decreases. This indicates that the gravitational force exerted on the particle by the earth is attractive. As the particle moves from infinity to r, the work done by the gravitational force on the particle is positive indicating U(r) to be negetive.

Equation 7.13 indicates that the potential energy of the particles M and m is a characteristic of the system. The potential energy is a poperty of the system as a whole. Either M moves or m moves as the potential energy of the system is changing. Usually if M is large [earth] we speak of the potential energy of mass m given by the equation 7.13. This is because when the potential energy changes it is reflected in a change in the kinetic energy of the larger mass M which is practically zero. A change in the potential energy of the system is reflected in a change in the kinetic energy of the lighter mass. It is in this sense we speak of the potential energy of small mass m as given by 7.13.

The potential energy of unit mass in a gravitational field is defined as gravitational potential. The gravitational gotential V is given by

$$V = U/m = -GM/r \tag{7.14}$$

The gravitational potential of a particle depends on the mass M responsible for the gravitational field and the distance of the particle from the centre of the gravitational field. Like gravitational field strength the gravitational potential also describes the nature of gravitational field. But there is a subtle difference between these two parameters. The gravitational field intensify is a vector and the gravitational potential is a scalar.

The gravitational force can be obtained from the potential energy. For a spherically symmetric potential energy function.

$$F = \frac{-dU}{dr} = -d\frac{\left(-GMm / r\right)}{dr} \tag{7.15}$$

$$\therefore F = \frac{-GMm}{r^2} \tag{7.16}$$

As usual the minus sign indicates that the force is attractive directed inward along a radius opposite to the radial displacement vector.

7.5 POTENTIAL ENERGY OF MANY PARTICLE SYSTEMS

Let us consider two particles separated by a distance r. Their potential energy is given by

$$U_{(F)} = -W_{\infty r} \tag{7.17}$$

In the above equation $W_{\infty r}$ represents the work done by the gravitational force in bringing the particles from infinite separation to seperation r.

Let us consider a system of 3 particles of mass m_1, m_2 and m_3.

Let the intial distance of separation between any two of the particles be infinity. Let us now calcaulate the work done in bringing three particles to the configuration shown in Fig. 7.3.

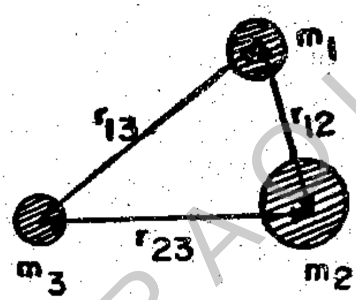


Fig 7.3 Configuration of m_1, m_2, m_3 brought from infinity.

The work done by the gravitational force in bringing the m_1 and m_2 from infinite separation to the separation r_{12} is given by

$$W_{12} = \frac{Gm_1m_2}{r_{12}} \tag{7.18}$$

Let the mass m_3 be brought from infinity to the separation r_{13} from m_1 and r_{23} from m_2. The work done by the gravitational force exerted by m_1 on m_3 is given by.

$$W_{13} = \frac{GM_1M_3}{R_{13}} \tag{7.19}$$

Similarly the work done by the gravitational force exerted by m_2 on m_3 is given by

$$W_{23} = \frac{Gm_2m_3}{R_{23}} \tag{7.20}$$

The total work done in assembling the system as shown in figure 7.3 is given by the total potential energy of the system.

$$-W = U = -\left\{\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right\} \tag{7.21}$$

Eq. 7.2 indicates that the work done in assembling a system to a particular configuration depends only on the configuration and not on the way how the configuration is achieved from and initial infinite separation. Hence the potential energy must be associated with the system as a whole and not with any one or two particles.

The energy required to isolate the masses of the system where the separation between each particle is infinity is given by the quantity

$$\frac{Gm_1m_2}{r_{12}}+\frac{Gm_1m_3}{r_{13}}+\frac{Gm_2m_3}{r_{23}} \qquad (7.22)$$

and this is called the binding energy holding the particles in the configuration shown in Fig. 7.3

Worked Example: 1

For example let us calculate the binding energy of the earth-sun system. For simplicity let us neglect the influence of other planets and satellites on the motion of earth round the sun. The work done against the gravitational force to bring the earth and sun from an infinite separation to a separation R_{ES} is

$$W_{ES}=\frac{GM_EM_S}{R_{ES}}$$

Using M_E=6 x 10^{24} Kg.M_2 = 198 x 6 x 10^{28}kg.and R_{ES} = 150 x 10^9 m

We get

$$W_{ES}=\frac{-6.673x10^{-11}x\,6\,x\,10^{24}x\,198\,x\,10^{28}}{150\,x\,10^9}$$

=-5.281 x 10^{33} J

The minus sign indicates the force is attractive and hence the work is done by gravitational force. The same amount of work is required to be done by an outside agent to separate these bodies to infinite distance when the earth-sun system will be rest. Since the earth is moving round the sun it has kinetic energy which will be half the magnitude of the potential energy of the earth-sun system. Hence only half of the work W_{ES} is required to break the system. Assuming that the earth-sun system will be at rest after break up, the effective binding energy is given by 2.64 x 10^{33}.

The gravitational potential energy is associated with the configuration of the system of mass particles held together by gravitational force. The gravitational potential energy is stored in the gravitational field of the system of particles. A change in the configuration of the system of particles or in the gravitational field results in the change in the potential energy of the system.

7.6 ENERGY CONSIDERATION IN THE MOTION OF PLANETS AND SATELLITES

Let us consider the motion of body of mass m say that of planet or satellite about a massive body of mass M say sun in the case of planets and planet in the case of satellite. Let us also consider that the mass M is at rest in an Inertial reference frame (sun in the case of planetary system and earth in the case of moon and artificial satellites). Let the small body m be moving about M in a circular orbit of radius r as shown in Fig.7.4

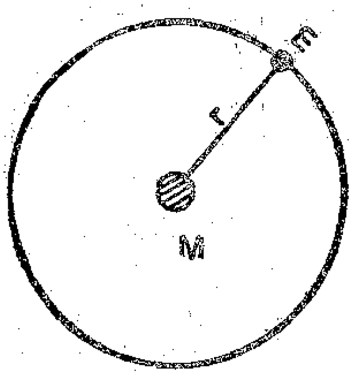


Fig.7.4 Motion of satellite (planet) of 'Mass m moving round a planet (sun) of mass M.

The potential energy of the system is given by

$$K = (1/2)m\omega^2 r^2 \qquad (7.23)$$

The kinetic energy of the system

$$GM/r = \omega^2 r^2 \qquad (7.24)$$

Since the mass M is at rest

$$K-(1/2)GMm/r \qquad (7.25)$$

Eq.7.24 can be written as

$$E=K + U= ½\ GMm/r-GMm/r \qquad (7.26)$$

The total energy E of the system is

$$E=-GMm/2r \qquad (7.27)$$

or

$$U_{(r)} =-GMm/r$$

Equation 7.26 indicates that the total energy is consistant and negative. The kineic energy can never be negative. At the most it becomes zero. The potential energy is always negative becoming zero at infinite separatio. These aspects are illustrated in Fig.7.5

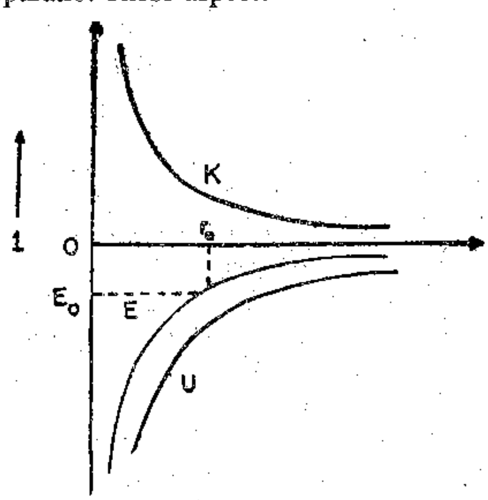


Fig.7.5 K,U and E versus r for a body of mass m moving in a circular orbit around a mass M

The total energy of the system being negative indicates that the system is found.

7.7 SUMMARY

The attraction between any two bodies can be explained as due to the existence of gravitational field, the gravitational field strength I of a body of mass M at any point r is given by $I=GM/r^2$. The gravitational field strength decreases as the distance increases. The gravitation potential at any point is given by $V=GM/r$. The binding energy of a system of particles is equal to the amount of energy required to isolate the particle of the system.

7.8 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. Derive an expression for the binding energy for a system of particles.

   Show that the total energy of a system of particles is constant taking the bound system like earth and sun.

II. Solve the Following Problems

1. Three particles of mass $m_1 = 10$ kg $m_2 = 20$ kg and $m_3 = 30$ kg are at the corners of a right angled triangle of sides 3m and 5m. Determine the binding energy of the system.

   (Ans : 11.1×10^{-11})

2. Determine the gravitational field strength of sun at 5×10^6 km from its centre. The mass of the sun is 198×10^{28} kg.

   (Ans : 2.64×10^{-13} N kg^{-1})

3. Determine the gravitational potential due to earth at a distance of 10^8 km from the centre of the earth. The mass of the earth is 6×10^{24} kg.

4. An artificial satellite of mass 100 kg is revolving round the earth at a distance of 104km from the earth. Determine the binding energy of the system. The mass of the earth is 6×10^{24} kg.

   (Ans: 3.125×10^9)

5. What will be the velocity of the projection of a body so that it reaches to a height equal to the turce three radius of the earth [9239M/Se]

6. Calculate the gravitational potential energy of the system of three masses each having a mass of 10kg and situated at the corners of a square of each side of length 0.25M.

   $\left[400\sqrt{2}\,G\left(1.12\sqrt{2}\right)\right]$

7.9 GLOSSARY

Telescope	:	Instrument used for observing distant objects.
Relativity Theory	:	A branch of Physics dealing with the motion of particles with velocities comparable to that of the velocity of light.
Quantum Theory	:	A branch of Physics dealing with the motion of microscopic particles.

Brownian motion	:	Zig zag motion of atoms and molecules of a gas or liquid.
Centripetal acceleration	:	Acceleration of a rotating body directed toward the centre of the orbit.
Radius vector	:	The line joining the particle in circular motion and the centre of the orbit.
Kinetic energy	:	Energy possessed by a body by virtue of its motion.
Potential energy	:	Energy possessed by a body by virtue of its position.
Binding energy	:	Energy required to bind a system of particles from seperation.

7.10 RECOMMENDED BOOKS

1. Resnick, R Halliday, D	Physics Part I	Wiley Estern Pvt Ltd., New Delhi
2. Serway,R.A	Concepts, Problems and Solutions in General Physics	W.B.Saunders Co. London
3. Buecha,F	Technical Physics	Harper and Plow Publishers, New York
4. White,H.E	Modern College Physics	East West Press Pvt.Ltd.,New Delhi.
5. Sears, F.W.amd Zemansky,M.W	College Physics	Addison - Wesley Publishing CO.Inc., London
6. Taylor.L.W.	Physics The Pioneer Science. Vol.I	Dover Publications New York
7. Zafirators,C	Physics	John Willey and Sons, New York
8. Harnwall, G.P.and Legge, G.J.F.	Physics - Matter Energy and the Universe	East West Press Pvt. New Delhi
9. Carr, H.Y.and Weidner, R.T	Physics from the ground up	Mc Graw - Hill Book Co. New York.
10. Subrahmanyam,S.V	Mechanics	Telugu Akadami, Hyderabad.

BLOCK -IV

COLLISIONS

UNIT-8 COLLISION CROSS SECTION AND APPLICATION TO ATOMIC COLLISIONS

Contents

8.1 AIMS AND OBJECTIVES

This unit discusses the term collision cross-section and the evaluation of it in different situations.

The term "Collision Cross Section" is explained quantitatively.

After going through this unit you will be able to explain collision cross section as the probability with which a particular collision can take place.

8.2 INTRODUCTION

A nuclear reaction may takes place whenever an incident particle collides with a target particle. The nature of collision between nuclear particles can be described by the impact parameter. In defining the impact parameter we can consider the target particle to be at rest. But in practice we cannot define that track of the incident particle or the location of the target particle precisely. It is not possible to isolate the incident particle and the target particle and study the collision process particularly when the particles involved are atoms and subatomic particles. The collision process has to be dealt with in a statistical way. To study nuclear reactions in detail it is necessary to have a quantitative measure of the probability of a given nuclear reaction. The quantity most often employed is the colliston cross-section. Let us now study how cross-section for a nuclear reaction can be estimated and its importance in the understaing of nuclear reaction processes.

8.3 COLLISION CROSS - SECTION

Before we go in to estimate the cross-section for atomic collisions let us analyse interaction between macroscopic particles and estimate the corresponding collision cross-section. Consider random firing of bullets from a machine gun on to a barn where a number of plates of area σ are hung in a random manner. Let the number of plates be n and the rate at which bullets strike the barn be R_o. The rate at which the plates are broken is given by

$$R = R_o(\sigma n)/A \quad (8.1)$$

Here 'σ n' represents the total area of all plates.

$$\text{Now } \sigma = RA/R_o n \quad (8.2)$$

σ can be determined by measuring R,A,R_o and n

σ represents the cross-section for the event consisting of the impact of a bullet on a plate.

If we consider a more restricted event like the bullet breaking the plate into 10 pieces then the rate R_{10} at which events of this type occur is much less than the rate R at which each plate is hit. If σ_{10} represents the cross-section for this type of event to occur then

$$\sigma_{10} = R_{10} A/R_o n \quad (8.3)$$

σ_{10} is obviously not equal to the geometrical area of the plate but will be much less than that σ_{10} is a measure of the probability of occurance of the event of a plate breaking into 10 pieces.

Depending upon the nature of the event we can assign a corresponding cross-section for the event to occur. None of these cross-sections necessarily has anything to do with the geometrical area of the plate. The cross-section simply represents the probability for the particular event to take place. The general relation for events of type - s take place is given by.

$$\sigma_s = \frac{R_s A}{R_o n} \quad (8.3a)$$

Where R represents the number events of type s and σ_s is the cross section for such events.

If N is the numerical density of nuclei (number of nuclei per cubic meter) in a thin sheet of target material of thickness t, the number of nuclei per unit area of target material is given by Nt.

If R represents the number of collisions per square meter per second and R_o represents the rate at which the projectiles strike unit and of the target

We can write

$$R = R_o (Nt)\sigma \quad (8.4)$$

The collision cross-section - σ - can be expressed as

$$\sigma = R/R_o Nt \quad (8.5)$$

σ can be measured if we know R, R_o, N and t. The cross-secion for collision is defined as the number of collisions per unit volume per second for unit incident flux and unit nuclear density. Cross-sections are commonly expressed in barns.

$$1 \text{ barn} = 10^{-28} \text{ m}^2$$

Check your Progress 1

What is the unit of a cross-section?

8.4 COLLISION CROSS-SECTIONS - APPLICATION TO ATOMIC COLLISIONS

In nuclear physics we generally have situations where the target materials are bombarded by nuclear particles accelerated to high energies in order to study the particle interactions or reaction processes. By measuring the rate at which a particular event takes place the cross-section for that particular event may be assigned. For example let us consider a thin foil of alluminium $_{13}Al^{27}$ bombarded by deutrons ($_1H^2$). Many events may take place depending upon factors line energy. These events may be (1) elastic scattering of deutrons into the forward hemisphere, (2) elestic scattering of the deutrons into the backward hemisphere (3) inelastic scattering of deutron between the angle 30° and 60° with direction of the incident beam and (4) the nuclear reactions of the type.

i) $_{13}Al^{27} + _1H^2 \rightarrow _{12}Mg^{25} + _2He^4$

ii) $_{13}Al^{27} + _1H^2 \rightarrow _{13}Al^{28} + _1H^1$

iii) $_{13}Al^{27} + _1H^2 \rightarrow _{14}Si^{28} + _0n^1$

iv) $_{13}Al^{27} + _1H^2 \rightarrow _{11}Na^{24} + _2He^4 + _1H^1$

Each of the above events has its own cross-section σx. Say $\sigma_o, \sigma_i, \sigma_a$, σ(d, α), σ(d,p), σ(d,n), σ(d, α p) and are called cross-sections for elastic scattering, cross-section for inelastic scattering, Cross-section for absorption, and cross-section for particular type of nuclear reaction etc. To determine the cross-section experimentaly, usually the number of events of a given kind is counted. The cross-section for a particular process is determined using the Eq.8.5. The collision cross-section corresponding to the effect of all possible processess is called the total cross section σ_t. It is given by the sum of the cross-section for all possible reactions.

A rough estimate of the cross section for nuclear reactions can be obtained as follows. The radius of a nucleus can be expressed as

$$r = 1.1 A^{1/3} \times 10^{-15} m \qquad (8.6)$$

Where A is the mass number.

$$\sigma = \pi r^2 = \pi (1.1)^2 A^{2/3} \times 10^{-30} m^2 \qquad (8.7)$$

for a nuclues of intermediate mass, A = 125. Hence.

$$\sigma = \pi (1.1)^2 (125)^{2/3} \times 10^{-30} m^2$$

$$= 0.95 \times 10^{28} m^2$$

or = 0.95 barns.

Eventhough it is simple to consider the cross-section as target area and get a rough estimate of its magnitude by calculating the geometrical cross-section, it will not represent the true meaning of cross-section pertaining to a nuclear rection process. The experimental meaning of the cross-section comes from its use as a measure of the number of nuclear events which occur under a given set of conditions. Nuclear cross-sections are found to have values ranging from a small fraction of a barn to hundreds of thousands of barns and these values many at times differ greatly from the geometrical cross-sections. A given nucleus may have different cross-sections for different nucle.. ractions and values represent the relative probabilities of those reactions. Under certain special conditions the scattering cross-section and geometrical cross-section can be related directly. The measured values of scattering cross-sections may be used to determine the radius of the nucleus of the target material. The elastic scattering of neutrons with energies greater than 10 Mev is an example of one such reactions. For intermediate nuclei (A ≅ 125) the total cross-section is given by.

$$\sigma_t = 2\pi r^2 \qquad (8.8)$$

σ_t can be determined by transmission measurements and hence r can be determined.

The results of an experimental study of a nuclear reaction can be expressed in terms of the number of processes that take place under a given set of experimental conditions. We may also express the results of a nucler reactions in terms of the cross-section for the reaction. The advantage of the use of the cross-section is that its value is independent of the flux of incident particles and density of the material used as targets. When σ_x is known it is possible to predict the number of reactions that will take place when a sample of the nuclide is exposed to a known flux of those particles for a given length of time. This kind of information is necessary in many practial problems like manufacture of artificial radio nuclides. Hence cross-sections are a valuable form of nuclear data.

Cross-sections always depend on the energy of the incident particle, often exhibiting sharp peaks as the energy is varied. This indicates that at certain specified energies certain types reactions are more likely to take place than at other energies. The cross-section as a function of the energy of well collimated neutron beam as they pass through cadium is presented in Fig.8.1

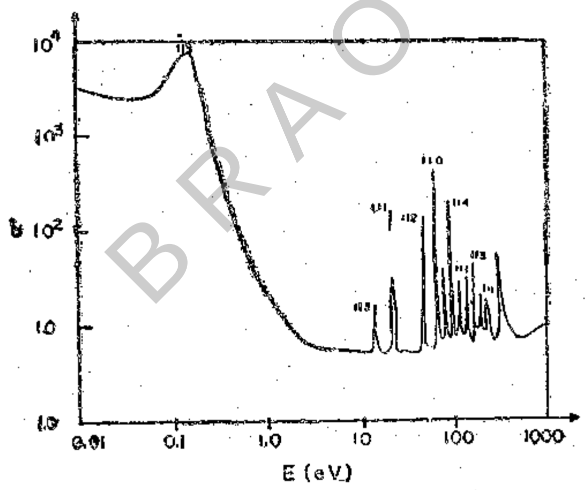


Fig. 8.1 Cross section versus energy of neutron beam, target being cadmium

The processes to which the cross-section - σ - in Fig. 8.1 refers are processes like elastic scattering, inelastic scattering or absorption leading to the removal of neutrons from the collimated beam. The member shown at the peaks represent the type of cadium isotope responsible for that peak. The strong peak labeled "113" that occurs at 0.17 ev energy of neutron beam is caused by the reaction.

$$_{48}cd^{113} + {}_0n^1 \rightarrow {}_{48}cd^{114} + \gamma \qquad (8.9)$$

Here γ represents gama ray. This reaction for which σ is 7600 barns is responsible for the very large absorbing power of cadium for slow neutrons. It is because of this high absorbing power cadium rods are used to control chain reaction in a nuclear reactor.

The discussion presented above indicates the applicability of nuclear cross-sections in many partical situations and hence important in the study of nuclear reactions.

Worked Example - 1

α particles from a polonium source are allowed to strike a gold foil 4 x 10^{-7} m thick. About 1 in 6.17x10^6 are scattered backward. The number of gold atoms per unit volume in the foil is 5.9 x 10^{28}/m^3 Determine the cross-section for backward scattering.

The fraction of α - particles scattered=Ntα

$$\therefore \frac{1}{6.17} = \left(5.9 \times 10^{28} m^{-3}\right)\left(4.0 \times 10^{-7} m\right)\alpha$$

$$\therefore \sigma = \frac{1}{6.16 \times 10^{6} \times 5.9 \times 10^{28} \times 4.0 \times 10^{-7}} = 6.9 \times 10^{28} m^{2}$$

= 6.9 barns.

Worked Example -2

A gold foil of 0.0005 m thick is irradicated for 10 minutes with a beam of thermal neutrons with a flux of 10^{16} neutrons/m^2 sec. The nuclide Au^{198} with a half life of 2.7 days is produced by the reaction Au^{197} (n, α) Au^{198}. The density of gold is 19.3x10^3 kg/m^3 and the cross-section for the reaction is 98.6x10^{-28} m^2. Determine the number of at oms of Au^{198} produced per square meter of the foil.

The cross-section for the nuclear reaction is given by $\alpha = \dfrac{R}{R_o Nt}$

Where R represents the number of collisions per square meter per sec.R_o is the flux of projectiles. N is the number of atoms per unit volume and t the thickness of the target.

As per date given in the problem

α =98.7 x 10^{-28} m^2, R_o = 10^{16} neutrons/m^2 - sec

t = 0.0005 m.

$$\text{No. of atoms per unit volume} = \frac{6.02 \times 10^{23} \times 19.3 \times 10^{3}}{197 \times 10^{3}}$$

=5.9 x 10^{23} / unit vol.

$\therefore R = R_o Nt\sigma$

R = 10^{16} x 5.9 x 10^{23} x 98.7x10^{-28} x 0.0005

R=0.2912 x 10^{11}

Total number of Au^{198} produces [= 0.29 x 10^{11} x 10 x 60

When irradicated by neutron flux for 10 minutes [=1.75 x 10^{13} atoms

8.5 SUMMARY

The collision cross-section defines the probability with which a particular collision can take place. The collision cross-section for nuclear reaction is expressed as

$$\sigma = \frac{R}{R_o Nt}$$

Where R represents the collision rate per unit area and R_o represents the rate at which projectiles strike the target per unit area N and t represent the nuclear density and thickness of the target.

8.6 MODEL ANSWERS

Check your Progress 1

Cross section are expressed in barns which is equal to 10^{-28} m^2.

8.7 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. Explain the term "Collision Cross-Section". Derive an expressior for cross section for atomic collisions.

2. Discuss the importance of cross-section in the study nuclear reactions.

II. Answer the Following Question in About 10 Lines.

Define collison cross-section and impact parameter

III. Solve the Following Problems.

1. A beam of fast neutrons imports on a 5.0 mg sample of Cu^{65}

A possibility may exist that a copper nucleus may capture a neutron to form Cu^{66} which is radioactive and decays to Zn^{66} which is stable. From a study of the electron emission of the copper sample it is found that 4.6×10^{11} neutron captures occur each second. The intensity of the neutron beam striking the target is 1.1×10^{18} neutrons/m^2 sec. Determine the neutron capture cross section in barns for the process involved.

(Ans: $\sigma\, n_o$ capture = 90 barns)

2. A beam of slow neutrons strikes an aluminium foil. Some neutrons are captured by the aluminum which becomes radioactive and delays by emitting an electron as per the following reaction.

$$_0n^1 + {}_{13}Al^{27} \rightarrow {}_{28}Si^{28} +$$

If the neutron flux is $3.0 \times 10^{-16}/m^2$ sec. and the transmutations taking place are $4.2 \times 10^{11}/m^2$ determine the nuclear surface density of the target if the cross-section for the above reaction is 0.23 barms.

(Ans : 6.08×10^{23} atoms/m^2)

3. A nitroget molecule moving in the X-direction with a velocity of 800 ms^{-1}makes a head on collision with an oxygen molecule.

After collision the nitrogen molecule comes to a dead stop. Determines the initial and final velocities of the oxygen molecule. The mass of nitroget molecule is 28 a.m.u. and the mass of oxygen is 32 a.m.u.

(Ans: V_i=50 ms^{-1} ,V_r=750 ms^{-1})

8.8 GLOSSARY

Nuclear reaction	:	A nuclear reaction involves the Interaction of a nuclear particle with an atom leading to a product particle and product nucleus. The product particle may be the incident particle or a new particle. The product nucleus Conversation of momentum and energy hold good for the nuclear reaction.
Nuclear force	:	It is a short range attractive force responsible for binding nuclides inside the nucleus
Compound Nucleus	:	A compound nucleus forms when the incident particle is absorbed by the target nucleus. This takes place in all nuclear reactions. The compound nucleus decays in a short time of the order of 10^{-8} sec into product nucleus and product particle.
Moderator Material	:	Material used in a nuclear reactor to slow down the fast neutrons.
Fission	:	Disintegration of heavy nuclei when bombareded by neutrons into fragments with the release of energy and nuclear particles. eg. ${}_0n^1 + {}_{92}U^{235} \rightarrow {}_{51}Sb^{133} + {}_{31}yt^{95} + {}_2He^4 + 4\left({}_0n^1\right) + 4$
lev (Electron Volt)	:	Amount of energy required to rise the potential of an electron by 1 volt. It is equal to $1.6x10^{19}$ J

8.9 RECOMMENDED BOOKS

1. Resnic, R & Halliday, D	Physics Part I	Wiley Eastern Pvt Ltd. New Delhi.
2. Bueche. F	Techinical Physics	Harper & Row Publishers New York.
3. White, H.E.	Modern college Physics	East West Press Pvt. Ltd. New Delhi.
4. Kaplan, I	Nuclear Physics	Addision-Wesley Publishing Co., London.
5. Serway, R.A.	Concepts, Problems and solutions in General Physics.	W.B.Saunders Co., London.
6. Subrahmanayam, S.V.	Mechanics	Telugu Akadami, Hyderabad

BLOCK - V

OSCILLATIONS

UNIT-9 SIMPLE HARMONIC MOTION

Contents

9.1 AIMS AND OBJECTIVES

This unit discusses the characterstics of simple harmonic motion. In order to make you understand the concept and its characteristics. The potential energy of a particle undergoing Harmonic motion is evaluated quantitatively and the equation of a simple harmonic oscillator and its solution treated analytically. After going through this unit you will understand that the resultant motion of two simple harmonic motions will have the shape of a straight line or circle or an ellipse. The shape of the resultant curve depends upon their frequencis, phase difference and amplitudes of the harmonic motion.

9.2 INTRODUCTION

Any type of motion which repeats itself at equal intervals of time is called a periodic motion or harmonic motion. Motion of planets round the Sun and of the Moon round the earth are examples of periodic motion. If a particle in periodic motion moves to and fro over the same path, it is called oscillatory motion. We have innumerable examples of oscillatory motion in our daily life. The motion of the bob of a simple pendulum or the motion of weight attached to a vertical spring and pulled down are examples of oscillatory motion. These are examples of mechanical oscillations. We also have other types of oscillations. Radio waves and visible light are oscillatory magnetic and electric field vectors. We also have oscillations of atoms in molecules.

9.3 HARMONIC MOTION

As mentioned earlier any repetitive motion is called harmonic motion. The time required for one oscillation or to complete one cycle is called time period (T).

The number of oscillations per unit time is called frequency (f)

$$f = \frac{1}{T} \tag{9.1}$$

In MKS system the unit of frequency is cycle per second or Hertz (not Hertz per second).

Let us consider the case of mechanical oscillatory motion. Let a particle oscillate between fixed limits along a straight line as shown in Fig. 9.1

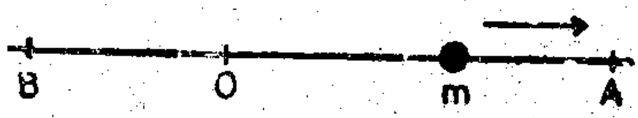


Fig. 9.1 A particle of mass 'm' oscillating between A and B

The position at which the net force acting on the particle is zero is the equilibrium position.

The velocity and acceleration of the particle and the force acting on it vary periodically in magnitude and direction.

The potential energy function of a particle undergoing harmonic motion is minimum at its equalibrium position where the net force is zero.

The force acting on the particle at any positions is derivable from a potential energy function (U) and is given by

$$F = -\frac{dU}{dx} \tag{9.2}$$

The force is a restoring force in the sense that it always tends to accelerate the particle in the direction of its equilibrium position.

The total mechanical energy for the oscillating particle.

$$E = K + U \tag{9.3}$$

Where K is kinetic energy and U is potential energy. The total energy E remains constant if no non-conservative force such as frictional forces act.

One form of potential energy U as a function of x is plotted in Fig.9.2

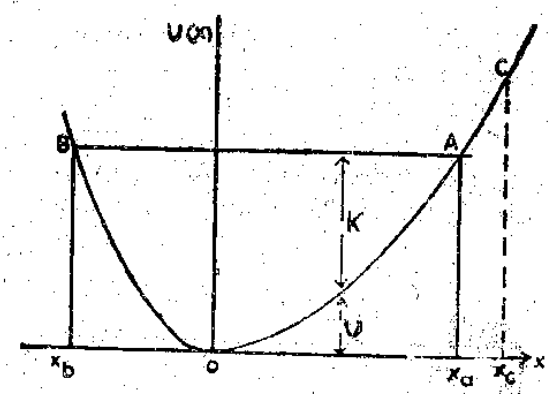


Fig.9.2 Potential energy U as a function of distance x from equilibrium position for harmonic motion

The slope of the curve at any point gives numerically the force acting at that point

$$\left(\text{since } F = -\frac{dU}{dx}\right)$$

At the equilibrium position, O, the slope is zero. This is as it should be, since the force is zero at the equilibrim position.

Let total energy be E and let the ordinate corresponding to this energy cut the potential energy curve at A and B. Then the abscissae corresponding to A and B (x_a and x_b) are the limits and the particle cannot go beyond these limits. This is seen as follows:

Suppose the particle is at some point C where $x_c > x_a$. Then the Potential energy corresponding to this point is greater than the total energy E.

This would mean (from Eqn. 9.3) that kinetic energy is negative which is physically impossible. The limiting positions X_a and X_b are determined by the total energy E and different energies will correspond to different lmiting points. These limiting points are called turning points because at these points the particle comes to rest and then turns back to travel along the path along which it has come. Further in the most general case X_a and X_b need not necessarily be equal.

Check your Progress 1

A Hertz is a unit of________________

9.4 THE SIMPLE HARMONIC OSCILLATOR

In the previous section we treated U in general without giving a specific functional form to it. In this section we consider a particular case which is of great importance in physics.

Let an oscillating particle be subjected to a potential which is a function of x given by

$$U(x) = \frac{1}{2}kx^2 \tag{9.4}$$

where k is a consant

$$F(x) = -\frac{dU}{dx} = -kx \tag{9.5}$$

The particle in this case is called a simple harmonic oscillator and the motion is called simple harmonic motion.

The potential energy curve in this case is symmetric about the y-axis. This is obvious from the form of U(x) Eqn. (9.4). The limiting positions are equally distant from the equilibrium position i.e, ($x_a = x_b$)

The potential energy function as given by Eqn. (9.4) is the potential of an ideal spring of force constant k compressed or extended by a distance x.

A body of mass m attached to an ideal spring and free to move over a frictionless horizontal surface is an example of simple harmonic oscillator.

A simple pendulum with a small angle of oscillation an electrical circuit comprising an inductance L and a capacitance C are other example of simple harmonic oscillators.

It is possible to analyse many complicated motions as combinations of individual simple harmonic motions. Hence a detailed study of simple harmonic motion is basic for the understanding of many phenomena in both classical and quantum physics.

9.5 EQUATION OF A SIMPLE HARMONIC MOTION (ANALYTICAL TREATMENT)

Let us go back to Eqn. (9.5) and apply Newton's second law to it.

$$\text{i.e., } \vec{F} = \text{mass x acceleration} = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = O \qquad (9.6)$$

This is the differential equation governing the simple harmonic motion.

To find the position of the particle at a given time, we must know x as function of time which satisfies Eqn. (9.6)

Let us try a general solution of the type

$$x = A\,\text{Cos}\,(\omega t + \delta) \qquad (9.7)$$

Here A, ω and δ are constants whose physical significance will become clear shortly.

$$\text{Then} \quad \frac{d^2x}{dt^2} = -\omega^2 A\text{Cos}(\omega t + \delta) = -\omega^2 A \qquad (9.8)$$

Comparing Eqns. (9.6) and (9.8) we find that if we choose the constant such that

$$\omega^2 = \frac{k}{m} \qquad (9.9)$$

then x=A Cos $(\omega t + \delta)$ is a solution of Eqn. 9.6 i.e., the equation of a simple harmonic oscillator.

9.5.1 Physical Significance of ω

Let the displacement at some time t be x_1

$$x_1 = A\,\text{Cos}\,(\omega t + \delta)$$

If the time is increased by $\frac{2\pi}{\omega}$ and if the corresponding displacement is x_2, then

$$x_2 = A\,\text{Cos}\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \delta\right]$$

$$= A\,\text{Cos}\,[\omega t + 2\pi + \delta]$$

$$= A\,\text{Cos}\,[\omega t + \delta]$$

That is the function repeats itself after a time $\frac{2\pi}{\omega}$ therefore is is the period of the motion T. But since

$$\omega^2 = \frac{k}{m} \tag{9.9}$$

we have $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

All motions given by Eqn. (9.6) have the same period of oscillation and is determined only by the mass of the vibrating particle and the force constant (k). Th frequency of oscillation.

$$f = \frac{1}{T} = \frac{1}{2\Pi}\sqrt{\frac{k}{m}} \quad \text{or} \quad \omega = 2\pi f = \frac{2\pi}{T} \tag{9.10}$$

9.5.2 Physical Significance of A

The maximum displacement from the equilibrium position is given by the maximum value of A cos $(\omega t + \delta)$. Since the maximum value that a cosine function can take is one, we have ($A=(x_{max})$ Which is the amplitude of the motion. As a cosine function varies between the limits of + 1 and - I the displacement x varies between the limits of + A and -A. It is possible to have motions of various amplitudes as solution to the Eqn. (9.8), but all of them have the same frequency.

It is important to note that the frequency of a simple harmonic solution is independent of the amplitude of the motion.

9.5.3 Significance of δ

The quantity of $(\omega t + \delta)$ is called the phase of the motion and determine the state of motion. Two motions may have the same amplitude and frequency but may differ in phase. If δ =0; x = A cos ωt. Displacement at t=0 is maximum and is equal to A

If $$\delta = -\frac{\pi}{2};\ x = A\cos\left(\omega x - \frac{\pi}{2}\right) = A \sin \omega x$$

Displacement at t = 0 is zero i.e., (A=0).

The Quantity δ is called the phase constant. Thus we see that the amplitude A and phase constant are determined by the initial position and speed of the particle.

9.6 VARIATION WITH TIME OF THE BASIC QUANTITIES OF SIMPLE HARMONIC MOTION.

(i) Displacement : $X = A\cos(\omega t + \delta)$; x_{max}=A (9.11)

(ii) Velocity $$\vec{v} = \frac{dx}{dt} = -\omega A[\text{Sin}(\omega t + \delta)] \tag{9.12}$$

Magnitude of $\vec{v}_{MAX} = \omega A$

(iii) Acceleration $\vec{a} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \delta)$ (9.13)

Magnitude of $\vec{a}_{MAX} = \omega^2 A$

(iv) Potential energy = $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \delta)$ (9.14)

$U_{MAX} = \frac{1}{2}kA^2$

(v) The potential energy varies between zero and the maximum value.

(vi) Kinetic energy : $K = \frac{1}{2}m\vec{v}^2$

$= \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \delta) = \frac{1}{2}kA^2\sin^2(\omega t + \delta)\ (\because k/m = \omega^2)$ (9.15)

$K_{MAX} = \frac{1}{2}kA^2$

The kinetic energy varies between zero and the maximum value.

(vii) Total mechanical energy $E = K + U = \frac{1}{2}kA^2\cos^2(\omega t + \delta) + \frac{1}{2}kA^2\sin^2(\omega t + \delta)$

$E = \frac{1}{2}kA^2$ (9.16)

Thus the total mechanical energy is constant. At the maximum displacement kinetic energy is zero but potential energy is maximum [(1/2) KA^2]

At the equilibrium position potential energy is zero but kinctic energy is maximum [(1/2)KA^2]

At other positions kinetic energy and potential energy will have values depending upon the position but their sum is always equal to [(1/2) KA^2]. An important point to note is that the total energy is proportional to the square of the amplitude of the motion.

The variation with time of the above quantities is shown in Fig. 9.3 The variation of potential and kineic energy with displacement is shown if Fig.9.4.

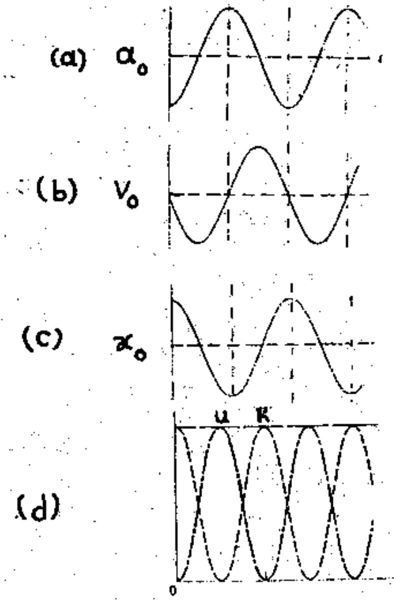


Fig.9.3 Time Variation of (a) acceleration (b) velocity (c) displacement and (d) energy of simple harmonic oscilator. The broken line represent kinetic energy (k) and solid line represents the potential energy(U).

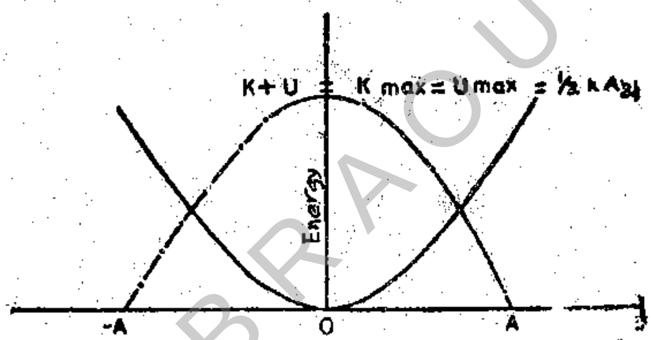


Fig.9.4 Energies of a simple harmonic oscillators

The boken line represents the kinetic energy K (x) = [(1/2)mv^2] and the solid represents the potential energy U (x) = [1/2 Kx^2]

Physical variable displacement	At maximum	At equilibrium position
1. Displacement	(x) Maximum (A)	Zero
2. Velocity	($\vec{v}$) Zero	Maximum (ω A)
3. Acceleration	(a) Maximum (ω^2 A)	Zero
4. Potential energy	(U) (1/2) KA^2	Zero
5. Kinetic energy	(K) Zero	(1/2 KA^2
6. Total mechanical energy	(E) (1/2 KA^2	(1/2) KA^2

The physical content of Eqn. (9.8) can be expressed in words as follows :

Simple harmonic motion is the motion of a particle whose acceleration is always proportional to its displacement from a fixed point and is directed towards that point. As has been impressed earlier this is a mechanical example of **Simple harmonic motion**. But it is important to note that any physical quantity (not necessarily only displacement) whose variation with time is expressible by an equation of the type (9.6) will execute simple harmonic oscillation.

As an example consider an electric circuit containing an inductance (L) and a capacitance (C) (Fig. 9.5).

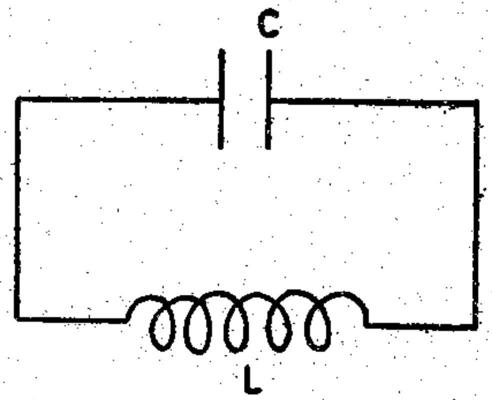


Fig. 9.5 Oscillatory discharge of a condenser

Suppose the condenser has been charged to a particular value and is discharging through the inducatance. The differential equation governing the discharge is the following.

$$\frac{d^2Q}{dt^2}+\frac{Q}{LC}=O$$

Where Q is the charge at any instant.

We see at once that the above equation is of the form (9.6) with

$$Q \rightarrow x$$

and $$\frac{1}{LC} \rightarrow \omega^2$$

$$\omega = \frac{1}{\sqrt{LC}} = \text{or frequency } f = \frac{1}{2\pi\sqrt{LC}}$$

The condenser gets charged and discharged periodically with the above frequency. This phenomenon is called oscillatory discharge of a condenser. This is an example of electrical oscillations.

9.7 SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

(A Geometrical Interprectation of Simple Harmonic Motion)

Imagine a point P to move with uniform angular velocity ω on a circle of radius A. Let the origin of co-ordinates be the centre of the circle. The x-axis will be one of the diameters (Fig. 9.6)

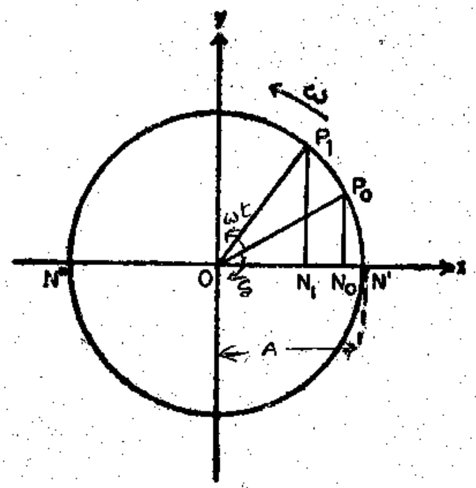


Fig. 9.6 Simple harmonic motion as the projection of a uniform circular motion on a diameter.

Initially at time t=0, let the position of the point be P_o and let the radius make an angle δ with the x-axis. Let N_o be the projection of P_o on the x-axis. After a time t let the position be P_1 and let the projection be N_1.

$\underline{|P_oOP_1}$ is the angle swept in a time t is = ω t.

(i) The displacement of N_1 from 0=ON_1 = OP_1 cos $\underline{|P_oOP_1}$

=A cos $(\omega t + \delta)$

(ii) Velocity of N_1 = x component of the velocity of P_1

= v Sin $(\omega t + \delta)$

But the uniform circular motion we know $\vec{v} = A\omega$

Magnitude of the velocity of N_1 = $\omega A \sin(\omega t + \delta)$

(iii) Acceleration of N = x component of the acceleration of P. But

The acceleration of $P = \frac{v^2}{A} = \omega^2 A$ towards the centre

Magnitude of the acceleration of N_1 = $\omega^2 A \cos(\omega t + \delta)$

The directions of the various quantities are shown in fig.9.7.

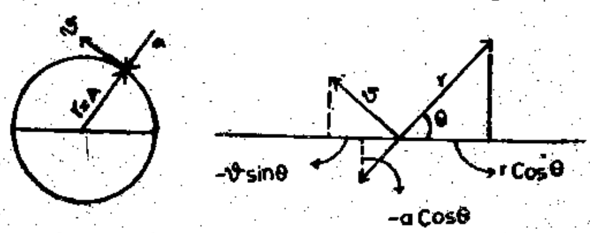


Fig.9.7 Directions and magnitudes of displacement, velocity and acceleration of a Simple Harmonic Motion $(\theta = \omega t + \delta)$

As the point P executes uniform circular motion, the projection N executes a to and fro motion with the limiting positions ON' and ON'' (ON' = ON'' = A)

Comparing the expressions for the displacement, velocity and acceleration of N obtained above with the corresponding expressions in section 9.6 Eqns. 9.11, 9.12 and 9.13. We see at once that the motion of N is simple harmonic. Instead of the projection on the x-axis if we had taken the projection in the y-axis we would have obtained an equation

$$y = A \,\text{Sin}\, (\omega t + \delta) = A \cos (\omega t + \delta - \pi/2)$$

This is also a Simple Harmonic Motion differing in phase from the earlier one by $\pi/2$. Hence the projection of a uniform circular motion on one of the diameters is simple harmonic motion.

This can be demonstrated in a simple manner as follows :

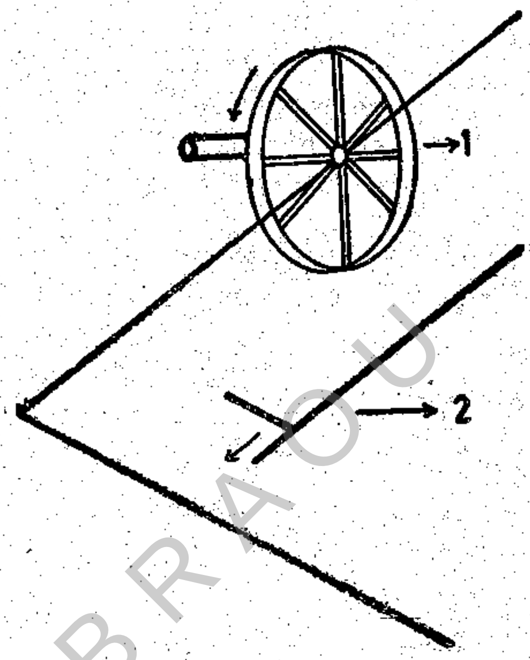


Fig. 9.8 Projecton of a point on a wheel moving with constant angular Velocity
1. Wheel with handle 2. Shadow from a light directly overhead.

Place a wheel relative to a distant source of light such that its shadow on a plance forms a straight line as shown in Fig 9.8. The shadow of an extension fixed to the circumference of the wheel will represent a projection of a point on the wheel in the plane of the shadow. When the wheel rotates with constant velocity the shadow of the extension will oscillate back and forth along the linear shadow of the wheel executing Simple Harmonic Oscillations.

The circle over which P moves (in Fig.9.6) is called the reference circle and the point P is called the reference point. We see that

(i) The radius of the reference circle is equal to the amplitude of the Simple Harmonic Motion and

(ii) The angular velocity of the reference point is ω where $\omega = 2\Pi f$ and f is the frequency of the Simple Harmonic Motion.

9.8 COMBINATION OF TWO SIMPLE HARMONIC MOTIONS AT RIGHT ANGLES TO EACH OTHER.

Consider two Simple Harmonic Motions one along the x-direction and the other along the y-directio. Let us assume the frequencies of both to be the same.

They can be represented as

$$x = A\cos(\omega t + \delta_1)$$

$$y = B\cos(\omega t + \delta_2) \qquad (9.7)$$

Case (i) If the phase constants are the same, $(\delta_1 = \delta_2 = \delta)$

We have $x = A\cos(\omega t + \delta)$

$$y = B\cos(\omega t + \delta)$$

$$\therefore \frac{y}{x} = \frac{B}{A}$$

or $y = (B/A)x$

This is the equation to a straight line with slope B/A. Hence the motion is linear along a line whose slope is determined by the ratio of the amplitudes. If in addition the amplitudes are equal (A=B) the motion is along a line equally inclined (making an angle of 45°) to either axis.

case (ii) If the phase constants differ by $\pi/2$ be$(\delta_1 - \delta_2 = \pi/2)$

We have $x = A\cos(\omega t + \delta_1)$

$$y = B\cos(\omega t + \delta_2)$$

or $y = B\cos(\omega t + \delta_1 - \pi/2)$

or $y = B\,\mathrm{Sin}(\omega t + \delta_1)$

$$\therefore \frac{x^2}{A^2} = \cos^2(\omega t + \delta_1)$$

$$\frac{y^2}{B^2} = \cos^2(\omega t + \delta_2)$$

$$\therefore \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

Here x_{MAX} occurs when y=0 and vice versa. The motion will be along an ellipse. If in addition the amplitudes are equal (A = B) the resulting motion is circular.

(i.e) $x^2 + y^2 = A^2$

In the most general case all possible combinations of two Simple Harmonic Motions of the same frequency and at right angles to each other will correspond to elliptical paths. The circle and straight a line are special cases of an ellipse.

The ratio of the amplitudes B/A and the difference is phase between the two $(\delta_1 - \delta_2)$ will determine the shape and orientation of the ellipse. The direction of actual motion (whether clockwise or anti clockwise) will depend upon which component leads in phase. If the frequencies of the components are also different we get different patterns for the path traced out by the point.

These patterns are called Lissajou's figures. Given the amplitude and frequency of each component of vibration and the phase relation between them, the resulting figure, can be geometrically constructed. Some of these figures are shown in Fig. 9.9

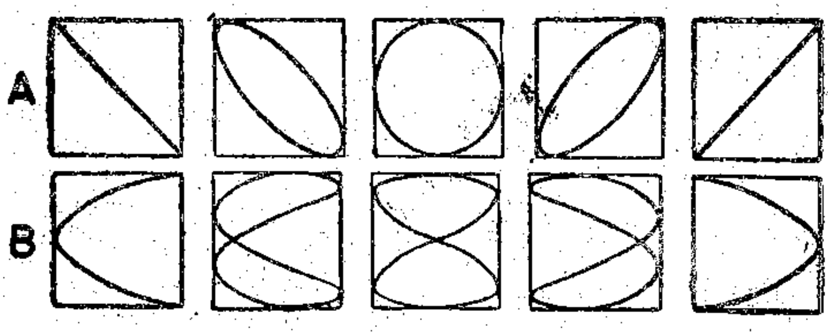


Fig. 9.9 Lissajou's figures
(A) Frequencies equal
(B) Frequencies in the ratio 2 : 1

A simple way to demonstrate these figures is by means of an oscilloscope as shown in Fig. 9.10

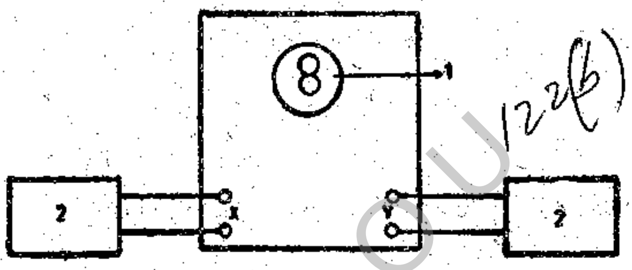


Fig. 9.10 Block diagram of a C.R.O.
1. Fluorescent screen 2. Audio Oscillator, X, Y - Plates.

In this electrons are deflected by each of two electric fields at right angles to each other (X-plates and Y-plates). The two Simple Harmonic Motions in the form of electrical oscillations are fed to the X and Y plates respectively of the oscilloscope. It is possible to vary the amplitude and phase. By doing this electrons can be made to trace out various patterns (Lissajou's figures) on a flourescent screen. (Fig.9.10).

These patterns can also be produced mechanically by swinging a pendulum whose oscillations are not confined to one vertical place.

Combination of two Simple Harmonic Motions of the same frequency at right angles to each other are very important in the study of polarized light.

Worked Example 1 :

A body is vibrating with simple harmonic motion of amplitude 0.15 metre and frequency 4 Hz. Compute (a) the maximum value of the velocity and acceleration (b) the velocity and acceleration when the displacement is 0.09 metre (c) the time required to move from the equilibrium position to a point 0.12 metre distant from it.

Solution

$$x = A\cos(\omega t + \delta)$$

In the present case A = 0.15 metre, $\omega = 2\pi f = 2\pi \times 4$

$= 8\,\pi$ radians/sec^2.

$$x = 0.15 \text{ Cos } (8\pi t + \delta)$$

a) $v_{MAX} - \omega A = 3.77$ metre/sec.

$a_{MAX} = \omega^2 A = (8\pi)^2 x15 = 94.7$ metre/sec^2

b) Displacement x = 0.09 metre

$0.09 = 0.15 \cos (\omega t + \delta)$

$\text{Cos } (\omega t + \delta) = 0.6$

$\text{Sin } (\omega t + \delta) = 0.8$

Magnitude of velocity = $\omega A \text{ Sin } (\omega t + \delta) = 3.77 \times 0.8 = 3.01$ m/sec.

Magnitude of acceleration = $\omega^2 A \text{ Cos } (\omega t + \delta) = 94.7 \times 0.6 = 56.80$ m/sec^2.

c) Let the time at which the body is at the equilibrium position (x=0) be t and the time at which it is at distance of 0.12 m. from the equilibrium position be t_2.

$$\text{Them Cos } (8\pi t_1 + \delta) = 0; (8\pi t_1 + \delta) = 90^\circ = 0.5\pi$$

$$\text{Cos } (8\pi t_2 + \delta) = \frac{0.12}{0.15} = 0.8; (8\pi t_2 + \delta) = 36 = 0.2\pi$$

$$(t_2 - t_1) = \frac{0.3}{8} = 0.038 \text{ sec.}$$

Time required to move from the equilibrium position to a point 0.12m from it 0.038 sec.

Worked Example 2 :

In a simple harmonic motion if the displacement is one half the amplitude. Show that the potential energy is 1/4 (E) and the kinetic energy 3/4(E) where E is the total energy.

Solution :

Let $x = A \text{ Cos } (\omega t + \delta)$

Velocity $v = -\omega A \text{ Sin } (\omega t + \delta)$

when the displacement(x)is $\frac{A}{2}$ then $\text{Cos } (\omega t + \delta) = \frac{1}{2}$

and $\text{Sin } (\omega t + \delta) = \frac{\sqrt{3}}{2}$

Velocity=$v = \frac{\omega A \sqrt{3}}{2}$

Potential energy $= \frac{1}{2} kx^2 = \frac{1}{2} k \left(\frac{A}{2}\right)^2 = \frac{KA^2}{8}$

$$\frac{1}{4}\left(\frac{KA^2}{2}\right)=\frac{1}{4}E$$ Since E is the total energy and is equal to $E=\frac{kA^2}{2}$

Kinetic energy $=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2A^2x\frac{3}{4}=\frac{3}{8}kA^2$ $\left(\text{Since } \omega^2=\frac{k}{m}\right)$

$$=\frac{3}{4}\left(\frac{1}{2}kA^2\right)=\frac{3}{4}E$$

Worked Example 3:

Electrons in an oscilloscope are deflected by two mutually perpendicular electric fields, in such a manner that the displacement at any time t is given by $x=A\,\text{Cos}\,\omega t;\ y=A\,\text{Cos}\,(\omega t+\delta)$.

Describe the path of the electron and determine its equation when (a) $\delta=0^\circ$ (b) $\delta=30^\circ$ and (c) $\delta=90^\circ$

Solution: $\text{Cos}\,\omega t=\frac{x}{A}\ \text{Sin}\,\omega t=\sqrt{1-\frac{x^2}{A^2}}$

$$\frac{y}{A}=\text{Cos}(\omega t+\delta)=\text{Cos}\,\omega t.\text{Cos}\,\delta-\text{Sin}\,\omega t\,\text{Sin}\,\delta$$

$$=\frac{x}{A}\text{Cos}\delta-\sqrt{1-\frac{x^2}{A^2}}.\text{Sin}\delta$$

$$\therefore\sqrt{1-\frac{x^2}{A^2}}.\text{Sin}\delta=\frac{x}{A}\text{Con}\delta-\frac{y}{a}$$

Sqaring and rearranging we get $x^2+y^2-2xy\,\text{Cos}\,\delta=A^2\sin^2\delta$

(a) $\delta=0^\circ;\ \text{Cos}\,\delta=1,\ \text{Sin}\,\delta=0$

$\therefore x^2+y^2-2xy=0;(x-y)^2=0$ i.e $x=y$

It is a straight line bisecting the angle between the x and y axes.

(b) $\delta=30^\circ;\ \text{Cos}\,\delta=\frac{\sqrt{3}}{2}\ \text{Sin}\delta=\frac{1}{2}$

$$\therefore x^2+y^2-2xy.\frac{\sqrt{3}}{2}=\frac{A^2}{4}$$

$$4x^2+4y^2-4\sqrt{3}\,xy=A^2$$

It is an ellipse.

(c) $\delta=90^\circ$ $\text{Cos}\,\delta=0,\ \text{Sin}\,\delta=1$ $\therefore\ x^2+y^2=A^2$

It is a circle of radius A.

Worked Example 4 :

When 1kg of mass is hung from a spring of length 3 meters, the spring streaches to 40cms. Now the mass is pulled down to 10 cm and left free without taking the mass of the spring into consideration find the (a) spring constant (b) period of oscillation, (e) frequency (d) phase angle.

Ans:-

Mass of 1 kg = 9.8 N

Due to the stretch in the spring = 40 cm = 0.4 M

a) Force const k $= \frac{9.8}{0.4} = 24.5$ N/M

b) Period of oscillation $T = 2\pi\sqrt{\frac{1}{24.5}}$

$= \frac{2\pi}{4.95} = 1.27$ Sec

c) Frequency $f = \frac{1}{T} = \frac{1}{1.27} = 0.787$ oscillations/Sec.

d) If t=o x=-10 cm.

(i e) $\cos(\omega t + \delta) = -1$

$\therefore \cos \varepsilon = -1$ or $\delta = \pi$

Worked Example : 5

For a body moving in SHM the maximum speed is 0.5 M/Sec. Its natural frequency is 1.66 oscillations/Sec. Find (a) its Amplitude of oscillation (b) maximum acceleration.

Ans:

(a) Maximum speed $= \omega A = 0.5$ M/Sec

Natural frequency $= f = 1.66$ oscillations/Sec

$\therefore$ Amplitude of oscillation $A = \frac{0.5}{2\pi \times 1.66} = 0.048$ Meters

(b) Maximum Acceleration $= \omega^2 A = (2\pi f)^2 A$

$= [2\pi(1.66)]^2 \, [0.048]$

$= 5.21$ M/Sec^2

9.9 SUMMARY

The potential energy function of a simple harmonic oscillator is of the form $U = \frac{1}{2}kx^2$

Where k is a constant and x is the displacement. The solution of the equation is given by

$$x = A \cos(\omega t + \delta)$$

A represents the amplitude, ω the angular velocity δ the phase of the motion. At the position of maximum displacment kinetic energy is zero but the potential energy is maximum. At equillibrium position the potential energy is Zero but the kinetic energy is maximum. When two simple harmonic motions at right angles to each other are combined the resultant motion will be along a curve whose shape will depend upon their frequencies, phase difference and amplitudes of simple harmonic motions. The different figures are called lisajou's figures. Straight lines circles and ellipse are special case of these figures.

9.10 MODEL ANSWERS

Check your Progress 1

A Hertz is a unit of frequency and is equal to number of cycles per second.

9.11 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. Formulate the differential equation governing simple harmonic motion and obtain its solution.
2. How do the following quantities vary with time in the case of simple harmonic motion? (a) Displacement (b) Velocity (c) Acceleration (d) Potential Energy (e) Kinetic Energy. Depict the variations graphically.
3. Discuss the resultant effect of combining two simple harmonic motions of the same frequency acting at right angles to each other.

II Answer the Following Questions in About 10 Lines.

1. Distinguish between harmonic motion and simple harmonic motion.
2. Explain the physical significance of A, ω and δ in the equation $x = A \cos(\omega t + \delta)$
3. Assuming that no nonconservative forces act, show that the total mechanical energy of a Simple harmonic Oscillator is constant and is proportional to the square of the amplitude of motion.
4. Explain the correspondence between uniform circular motion and Simple harmonic motion.
5. What are Lissajou's figures?

IV Solve the Following Problems.

1. (a) Write an equation for the position of a particle of mass m = 15 gms. moving along the X-axis at any time t if it is executing simple harmonic motion. Let its equilibrium position be at x_o = 10cm and the amplitude of motion be 5 cms. It

takes 2 secs, for the particle to go through a complete cycle of the motion.

(b)Write an equation for the force that must act on the particle to cause this motion.

(c)For what values of 'x' does the particle have maximm velocity?

(d)For what values of 'x' does the particle have maximum acceleration?
Ans. (a) $x=10+5 \sin \pi t$ (b)$F = - \sin \pi t$
(c) 10 cms. (d) 5.15 cms

2. The balance wheel of a watch vibrates with an angular amplitude of radians and with a period of 0.5 sec. Find 9a) the maximum angular velocity (b) the angular velocity when the displacement is half of its amplitude (c) the angular acceleration when its displacement is 45°

[(Ans. (a) 40 rad/sec. (b) 34 rad/sec. (c) rad/sec^2)]

3. The scale of a spring balance reading from zero to 32 1bs. is 6 in long. A body suspended from the balance is observed to oscillate vertically at 1.5 vib/sec. What is the weight of the body?

[(Ans. 23 1bs.)]

4. The displacement equation of a particle executing simple harmonic motion is $x=0.01 \text{Sin } \pi (t+0.005)$ meters. Calculate (a) amplitude (b) periodic time and (c) maximum velocity.

[(a) 0.01M (b) 0.02Sec (c) 3.14M/S]

5. A particle of mass 5 gm. executing SHM has amplitude of 8cm. If it makes 16 vibrations per sec, calculate its max. velocity and energy at mean position.

[8 M/sec, 0.16 Joules]

6. The dislacement of a particle executing simple harmonic motion is given by $x=0.1 \text{ Cos } \pi \ (t+\frac{1}{3})$ Calculate the maximum velocity and maximum acceleration of the particle

UNIT-10 DAMPED HARMONIC OSCILLATIONS

Contents

10.1 AIMS AND OBJECTIVES

This unit discusses the effect of damping force on the motion of a simple harmonic oscillator.

This unit explains the effect of damping force on the motion of the harmonic oscillator qualitatively and quantitatively and the phenomena 'Resonance'. After going through this unit you will be able to evaluate the period of oscillation of compound pendulum and torsional pendulum.

10.2 INTRODUCTION

An ideal simple harmonic oscillator was discussed in the unit 9. We saw that the amplitude (A) remained constant throughout. Thus a pendulum or a mass on a spring would oscillate idefinitely with the same amplitude. But we know that in actual practice this does not happen. After sometime the oscillations die down. This is due to frictional forces acting on the oscillator.

10.3 DAMPED HARMONIC OSCILLATOR

Let us Consider a Harmonic oscillator

When frictional forces act, the motion of the oscillator is damped. The frictional force is due to air resistance and depends on the velocity. Usually it is proportional to the negative of velocity, i.e., $b\vec{v}$ where b is a postitive constant. Hence the resultant force is the sum of the restoring force and the damping force

$$\text{Resultant force} = -kx - b\vec{v}$$

$$= -Kx - b\frac{dx}{dt}$$

$$\left(\therefore \text{Velocity } \vec{v} = \frac{dx}{dt}\right)$$

By Newton's second law resultant force

$$\overline{F} = \text{mass x acceleration} = m\frac{d^2x}{dt^2}$$

$$\therefore -Kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = 0 \qquad (10.1)$$

The method of finding the solution is given at the end of this unit.

If b is small the solution of this equation can be shown to be

$$x = A\exp\left(-\frac{bt}{2m}\right)\text{Cos}(\omega' t + \delta) \qquad (10.2)$$

$$\text{Where } \omega' = 2\Pi f = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \qquad (10.3)$$

If we set b=0 in the above equations (i.e. when there is no damping)

$\omega = \omega' = \sqrt{\frac{K}{m}}$ which is the frequency of undamped motion, and $x = A\ \text{Cos}\ (\omega' t + \delta)$ which is the solution for an undamped oscillator.

The effect of friction is to alter both the frequency and amplitude.

1. The frequency is reduced since $\omega' < \omega$ from Eqn. 10.3 - $\left(\frac{bt}{2m}\right)$
2. The amplitude decreases with time. It is given by A_e as we can see from Eqn. 10.3

The interval (τ) during which the amplitude drops to 1/e of its initial value is called the mean life time of the oscillation. If the friction is so great that $(b/2m)^2 > \frac{K}{m}$ then ω' becomes imaginary as can be seen from Eqn. 10.3. In this case the motion is not oscillatory and is said to be 'Over damped'.

If $(b/2m)^2 = \frac{K}{m}$ the motion just fails to be oscillatory and is said to be critically damped. In damped motion the energy of the oscillator is gradually dissipated by friction. Fig 10.1 shows damped harmonic motion plotted against time. An ohmic resistance (R) in an L-C circuit plays a similar role to that of friction in mechanical oscillations.

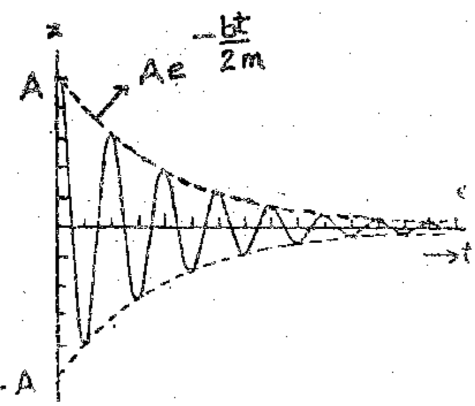


Fig. 10.1 damped harmonic motion.

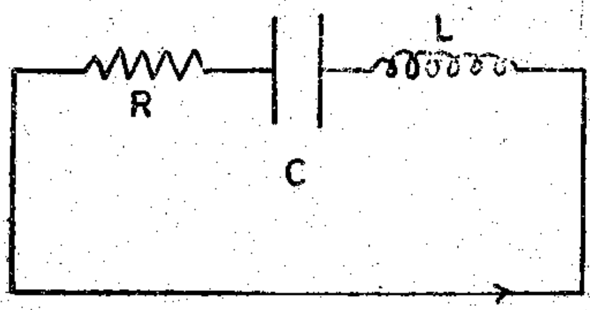


Fig. 10.2 LCR Circuit

In the LCR circuit shown in Fig 10.2 the differential equation governing the variation of charge (Q) is

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0 \qquad (10.4)$$

The equation is of the same form as Eqn. 10.1 with following correspondence

$$Q \rightarrow x\, m \rightarrow L.,\, b \rightarrow R \text{ and } K \rightarrow \frac{1}{C}$$

Therefore the solution of this by analogy with the solution for a mechanical oscillators is

$$Q = Q_m \exp\left[\frac{-Rt}{2L}\right]\text{Cos}(\omega' t + d) \qquad (10.5)$$

$$\text{Where } \omega' = 2\pi f = \sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2} \qquad (10.6)$$

Check your Progres - 1

What is meant by mean life lime of the damped oscillation?

10.4 FORCED OSCILLATIONS AND RESONANCE

We have so far considered natural oscillations (both mechanical and electrical) with and without dampings.

Let us now sec what happens if an oscillatory external force acts on the body. The external force is called the driving force. Let the angular frequency of the driving force be ω_e so that the driving force is given by $F_e \cos \omega_e t$ and $\omega_e = 2\omega f_e$

In the most general case where damping is also present the net force acting on the particle is

$$-Kx-b\frac{dx}{dt}+F_e \text{Cos}\,\omega_e t$$

By Newton's second law the net force must equal to $\left[m\left(\frac{d^2x}{dt^2}\right)\right]$

Equating the two and rearranging the terms we get

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+Kx=F_e\cos\omega_e t \qquad (10.7)$$

The solution of this equation can be shown to be

$$x=\frac{F_e}{G}\sin(\omega_e t-\delta) \qquad (10.8)$$

$$\text{Where } G=\sqrt{m^2\left(\omega_e^2-\omega^2\right)+b^2\omega_e^2} \qquad (10.9)$$

$$\text{and } \cos\delta=\frac{b\omega_e}{G} \qquad (10.10)$$

The system vibrates with the driving frequency (ω_e) and not with its natural frequency (ω) as is clear from Eqn (10.8). Such oscillations are called forced Oscillations. In musical instruments like Violin, Sitar or Veena sounds produced by the vibrating string are amplified by coupling it to a sound board which is a hollow wooden chamber containing air. The air inside vibrates with whatever frequency that is produced in the string though the natural frequency of the air filled chamber may be quite different. This is an example of forced vibrations.

Let us first consider the simple case where there is no damping (i.e.b=0)

Then we see from Eqn.(10.9) that $G=m(\omega_e^2-\omega^2)$ (10.11)

If the difference between ω_e and ω is large, G is large and F_e/G is small.

i.e.The amplitude of the resultant motion (F_e/G in Eqn.(10.8) is small)

As the driving frequency ω_e approaches the natural frequency ω G becomes smaller and smaller and approaches Zero (see Eqn.10.11)

The amplitude (F_e/G) approaches infinity.

In actual practise there is always some damping present. Hence the amplitude becomes very large but remains finite. There is a particular driving frequency for which the amplitude is maximum. This condition is called Resonance and the particular value of the driving frequency at which resonance occurs is called Resonant frequency.

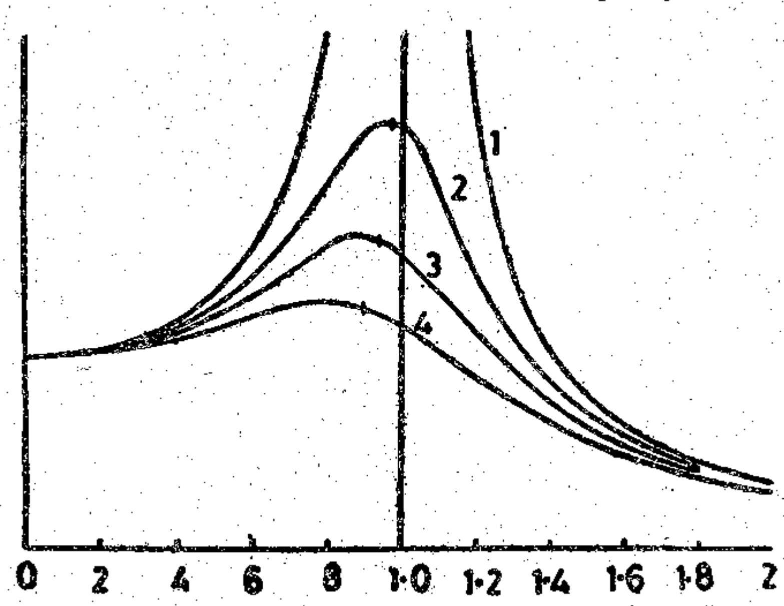


Fig.10.3 Amplitude for forced harmonic motion as a function of the driving frequency.

Fig.10.3 is a plot of driving frequency (x-axis) versus amplitude of the resultant vibration, (Y-axis) for four different values of b. For curve (1) b=o, and there is no damping.

The value of b increases through the curve (2), (3) and (4) damping being maximum for curve (4). The natural frequency in the absence of damping is shown by a vertical line.

We notice that the resonant peak corresponding to maximum amplitude moves nearer and nearer the vertical line(for which $\omega_e/\omega=1$) as b becomes smaller and smaller, we see that

(i) the resonant frequency is in general different from the undamped natural frequency.

(ii) For small damping the resultant frequency is close to the undamped natural frequency. In most cases of interest the damping is small enough so that we can take the resonant frequency to be the same as the natural undamped frequency.

We come across forced Oscillations and Resonance in electrical circuits also. If an LCR circuit contains a sinusoidally varying emf, given by

$E=E_m \cos \omega_e t$ (as shown in Fig 10.4)

The differential equation of the circuit is given by

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=E_m\cos(\omega_e t-\phi)$$

This equation is similar to Eqn. (10.7) and by analogy with Eqns.(10.8).(10.9)and(10.10) we can write down

$$Q=\frac{E_m}{G}\text{Sin}(\omega_e t-\phi) \qquad (10.13)$$

where $$G=\sqrt{\left(\omega_e^2 L-\frac{1}{C}\right)^2+R^2\omega_e^2} \qquad (10.14)$$

and $$\cos\phi=\frac{R\omega_e}{G} \qquad (10.15)$$

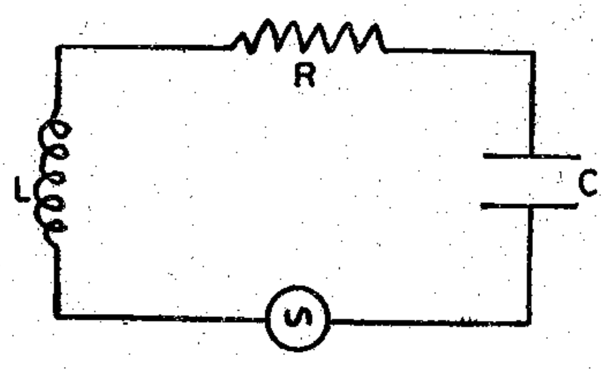


Fig.10.4 LCR.Circuit subject to an impressed sinusoidally varying emf.

10.5 ANGULAR HARMONIC MOTION

Suppose a body pivoted about an axis experiences a restoring torque proportional to anagular displacement θ is given by

$\tau = K\theta$ while K is a constant (10.16)

This relationship is similar to the one we had for lienar displacement

F = - Kx which gave rise to linear SHM.

In place of restoring force F we have restoring torque and in place of linear displacement (x) we have angular displacement θ. There is a correspondence between linear motion(or translational motion in a straight line and angular motion (or Rotational motion about a fixed axes as shown in the Table below.

Linear Motion in a straight line	Angular motion about a fixed axis
1. Linear displacement x	Angular displacement θ
2. Linear Velocity $v = \frac{dx}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
3. Linear acceleration $a = \frac{d^2x}{dt^2}$	Angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$
4. Mass M	Rotational inertia or I Moment of merita
5. Force $F = m\frac{d^2x}{dt^2}$	Torque $\tau = I\frac{d^2\theta}{dt^2}$

Equation 10.16 can be rewritten as

$$I\frac{d^2\theta}{dt^2} = -k\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{K}{I}\theta = -\omega^2\theta \qquad \text{When } \omega^2 = \frac{K}{I} \qquad (10.17)$$

This equation for angular motion is of the same form as the equation for linear harmonic motion (Equation 9.8).

The equation 10.17 is the equation for angular harmonic motion.

The period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{K}} \qquad (10.18)$$

10.6 THE TORSIONAL PENDULUM

Let a disc be suspended by a wire attached to the centre of the disk and let the wire be fixed securely to a solid support as shown in Fig. 10.5

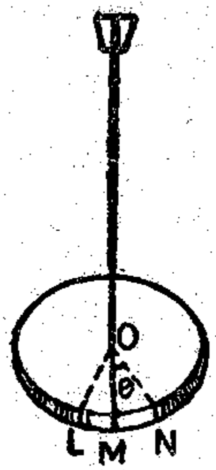


Fig. 10.5 A torsional pendulum

At the equilibrium position a radial line is drawn from its centre '0' to M. If the disc is rotated in a horizontal plane to the radial position N the wire gets twisted. The twisted wire will exert a torque on the disc trying to restore it to the equilibrium position M. This restoring torque is proportional to the angular displacement θ/for small twists.

$$\tau = -K\theta$$

Here K is a constant depending upon the elastic properties of the wire and is called torsional constant. It is the couple required to produce a twist of one radian. The above condition is the same as Eqn. (10) which is the condition for angular SHM.

In analogy with the solution for linear SHM we can write down the solution.

$$\theta = \theta_m \cos(\omega\tau + \delta) \quad (10.19)$$

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{K}}$ where θ_m is the amplitude

An arrangement like the above one is called a torsional pendulum. Here K refers to the torsional constant of the wire and I refers to the rational inertia or moment of intentia of the oscillating disc about a vertical axis passing through 0.

By noting down the time period of oscillation (T) of a torsional pendulum we can determine experimentally.

(i) the torsional constant of a given wire if the rotational inertia of the disc about vertical axis is known or

(ii) The rotational interia of the disc about the vertical axis if the torsional constant of the wire known. It is not necessary that the oscillating body should be a circular disc. It could be of any shape. All that we need to know is its rotational inertia about the axis.

10.7 THE COMPOUND PENDULUM

Suppose a rigid body is pivoted so that it can swing in a vertical plane about an axis passing through 0 as shown in fig.10.6

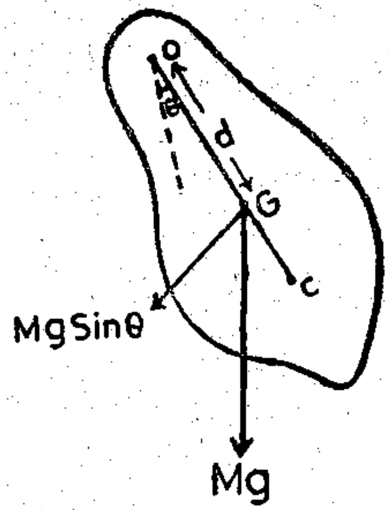


Fig. 10.6 A physical pendulum or compound pendulum

This is called a physical pendulum or a compound pendulum.

The equilibrium position is that in which the centre of mass of the body (G) lies vertically below. The distance between 0 and G is d and the rotational inertia of the body about an axis through the pivot is I and the mass of the body is M.

When the body is displaed from its equilibrium position a restoring torque tries to restore it to its equilibrium position. In this case the restoring torque is provided by the tangential component (Mg Sin θ) of the force of gravity acting at G.

The restoring torque $\tau = -(Mg \sin\theta)d$

For small angular displacements θ, $\sin\theta \rightarrow \theta$

$$\therefore \tau = -Mgd\,\theta = -K\theta \text{ by putting } K=Mgd \qquad (10.20)$$

This is the condition for angular S H M (Eqn. 10.16)

Condition (10.20) is valid only for small angles or small amplitudes. For larger amplitudes τ is proportional to Sin θ and the motion will not be simple harmonic.

Consider a point mass (m) suspended at the end of a weightless string of length l

$$I = ml^2;\ M = m : d = l$$

$$\because T = 2\pi\sqrt{\frac{I}{K}}$$

$$\therefore T = \sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{ml^2}{mgl}} = 2\pi\sqrt{\frac{L}{g}} \qquad (10.21)$$

This is the period of a simple pendulum for small amplitudes. Hence simple pendulum is a special case of the physical pendulum. It is possible to find an equivalent simple pendulum whose period is equal to that of a given physical pendulum.

if l_o is the length of the equivalent simple pendulum

$$\therefore T = 2\pi\sqrt{\frac{l_o}{g}} = 2\pi\sqrt{\frac{R}{Mgd}} \qquad (10.22)$$

$$l_o = \frac{L}{Md}$$

Hence as far as its period of vibration is concerned the mass of a physical pendulummay be considered to be concentrated at a point whose distance from the point of support is $\frac{\ell}{Md}$. This point shown as C in figure is called the centre of Oscillation.

It can be shown that if the pendulum is pivoted about new axis through C its period remains unchanged and point O becomes the new centre of oscillation. The point of support also known as centre of suspension and the centre of oscillation are inter changeable and are said to be congjugate to one another. Notice that the centre of oscillation and the length of the equivalent simple pendulum depend upon the location of the piovot. This is obvious from Eqn. (10.22) where ℓ_0 depends on ℓ which inturn depends on the position of the point of support.

Check your Progress-2

In a compound pendulum the point of support and the centre of oscillation are said to be conjugate to one another why?

Worked Example 1 :

A spring has a force constant of 2.5N. per metre. A mass of 0.025 kg. is hung on the lower end. Examine the type of motion for the following values of the damping force constant.

(i) 1 N.sec. per metre (ii) 0.5 N.sec. per metre (iii) 0.I.N.sec. per metre. If the motion is oscillatory find the frequency of oscillation.

Solution : Force constant = k = 2.5 N.per metre

mass = 0.025 kg.

(i) Damping force constant = b = I N.sec. per metre

$$\left(\frac{b}{2m}\right)^2 = 400\text{ Sec}^{-2}\quad \frac{k}{m} = 100\text{ Sec}^{-2}$$

$$\left(\frac{b}{2m}\right)^2 > \frac{k}{m}$$

Hence the motion is over damped.

(ii) b = 0.5 N.sec per metre $\left(\frac{b}{2m}\right)^2 = 100\text{ Sec}^{-2}$ $\left(\frac{b}{2m}\right)^2 = \frac{k}{m}$

The motion is critically damped.

(iii) b = 0.2 N.sec per metre $\left(\frac{b}{2m}\right)^2 = 16\text{ Sec}^{-2}$ $\left(\frac{b}{2m}\right)^2 < \frac{k}{m}$

The motion is oscillatory ; Argular frequency $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$

$$=\sqrt{84}=9.17\,\text{Sec}^{-1}$$

Worked Example 2 :

If in case (iii) of the above example the spring is subjected to an external periodic force F_e of 0.125N. having a frequency corresponding to ω_e as 8 radians per second. Find the displacment of the mass from its equilibrium position at any time t.

Solution :

$F_e = 0.125$ N. $\omega_e = 8$ rad. per sec. b = 02 N.sec per metre.

$$G=\sqrt{\left(m\omega_e^2-K\right)^2+b^2\omega_e^2}=1.836$$

$$\cos\delta=\frac{b\omega_e}{G}=0.87\quad\therefore\delta=0.512\text{ radians}$$

$$x=\frac{F_e}{G}\text{Sin}(\omega t-\delta)$$

$$x=0.061\text{ Sin }(8t-0.512)$$

Worked Example 3 :

A uniform rod of length 1 metre is mounted so as to rotate abot a horizontal axis perpendicular to the rod and at a distance 'x' from the centre of mass. Find the value of x for which the period is a minimum.

Solution:

Moment of inertia of a rod of mass M and of length l about an axis through the centre of mass and perpendicular to the rod

$$\frac{Ml^2}{12}=\frac{M}{12}\text{ Since }l=1\text{ metre}$$

Moment of inertia about a parallel axis through 0 which is at a distance of 'x' from G is $\left(\frac{M}{12}+Mx^2\right)$ (by the theorem of parallel axis)

$$\therefore T=2\pi\sqrt{\frac{\frac{M}{12}+Mx^2}{x}}\text{ (by equation 10.21)}=2\pi\sqrt{\frac{M}{12x}+Mx}$$

If the period is to be a minimum then $\frac{dT}{dx}=0$

Differentiating the expression for T we obtain

$$\frac{dT}{dx}=2\pi\frac{1}{2}\frac{1}{\sqrt{\frac{M}{12x}+Mx}}\left(-\frac{Mx^{-2}}{12}+M\right)=0$$

$$x^2 = \frac{1}{12} x = 0.29 \text{ Meters}$$

Distance from centre of mass 0.29 metre.

Worked Example 4 :

A solid cylinder having mass of 5 kg with 12cm diameter is suspend like a torsion pendulum by a wire passing through its center. Its period of oscillation is 4 sec. Find torsional content of the wire.

Moment of Inertia of the cylinder $I = \frac{MR^2}{2} = \frac{1}{2} x5(0.06)^2 = 9x10^{-3}$ kg M^2

Torsinal contant of the wire $K = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2 x9x10^{-3}}{42}$

$= 2.2x10^{-2}$ meter N/Radian

Worked Example 5:

A disc with 10.18 cm radus is oscillating on its edge. Its period of oscillation is 0.784 S. Find the acceleration due to gravity at that place.

The moment of of the disc about an axis
Passing through its centre is

$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Then the period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{MgR}} = 2\pi\sqrt{\frac{2}{3}\frac{MR^2}{MgR}} = 2\pi\sqrt{\frac{3}{2}Rg} \quad \because d = R$$

$$\therefore g = \frac{6\pi^2 R}{T^2} = 980.4 \text{ cm/Sec}^2 \qquad [R=10.18\text{cm}, \ T=0.784 \text{ Sec.}]$$

Annexure-1 : Damped Harmonic Oscillator - Solution of the Differential Equation.

The differential equation governing the motion of a damped harmonic oscillator is given by equation 10.1

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = 0 \qquad (10.A.1)$$

This is a linear homegeneous second-order differential equation with constant coefficients. This can be solved as follows:

Let $x = ce^{pt}$ (10.A.2)

Where C and P are constants.

$$\frac{dx}{dt} = cpe^{pt} \text{ and } \frac{d^2x}{dt^2} = P^2Ce^{pt}$$

Substituting these values of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in equation 10.A.1

we get $mp^2 + b\,p + K = 0$

$$\therefore p = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{K}{m}}$$

Substituting the value of p we obtain the most general solution as

$$x = \frac{-bt}{e^{2m}}\left[ce^{j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t} + De^{-j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t}\right] \qquad (10.A.3)$$

[Where & D are constant]

If the damping force is small compared to the restoring force then $\frac{b^2}{2m} < \frac{K}{m}$

The quantity $\sqrt{\frac{b^2}{4m^2} - \frac{K}{m}}$ is imaginary and can be written as

$$j\sqrt{\frac{K}{m} - \frac{b^2}{4m^2}} \quad \text{where } j = \sqrt{-1}$$

We can write equation 10.A.3

$$x = e^{\frac{-b}{2m}t}\left\{Ce^{j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.t} + De^{-j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.t}\right\}$$

$$= e^{\frac{-bt}{2m}}(Ce^{j\omega' t} + De^{-j\omega' t}) \qquad (10.A.4)$$

Where we have put $\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \omega'$ (10.A.5)

Making use of Eluer's theorm $(e^{\pm j\theta} = \cos\theta \pm j\sin\theta)$ and writing for $e^{j\omega' t}$ as $(\cos\omega' t + j\sin\omega' t)$

We can write equation 10.A.4 as $x = e^{\frac{b}{2m}t}[(C + D)\cos\omega' t + j(C - D)\sin\omega' t]$

Letting (C+D) =A Cos δ and j (D-C)=A Sin δ

We get $x = \exp\frac{-b}{2m}t[A\cos\delta\cos\omega' t - A\sin\delta\sin\omega' t]$

$$\gamma = A\exp(-bt/2m)\ \cos(\omega' t + \delta)$$

Forced oscillations of a Damped oscillator. Solution of the Differential equation.

The differential equation governing the motion of a damped harmonic oscillator subjected to an external oscillating driving force of angular frequency ω_e is given by equation 10.7

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+Kx=F_e \cos\omega_e t$$

Let $x = A \cos \omega_e t + B \sin \omega_e t$

$$\frac{dx}{dt} - \omega_e A \cos\omega_e t + \omega_e B \cos\omega_e t$$

$$\frac{d^2x}{dt^2} = -\omega_e^2 A \cos\omega_e t - \omega_e^2 B \sin\omega_e t$$

Substituting for x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in the above equation, we get

$$-m\,\omega_e^2[A \cos\omega_e t + B \sin\omega_e t] + b[\omega_e B \cos\omega_e t$$

$$-\omega_e A \sin\omega_e t]$$

$$+K(A \cos\omega_e t + B \sin\omega_e t) = F_e \cos\omega_e t$$

Combing the sine and cosine terms we get

$$\left[\left(K - m\,\omega_e^2\right)A + \omega_e B\right]\cos\omega_e t$$

$$+\left[-\omega_e bA + (K - m\omega_e^2)B\right]\sin\omega_e t$$

$$= Fe \cos\omega_e t$$

Equating coefficients of sine and cosie terms on both sides

$$\left(k - m\omega_e^2\right)A + \omega_e b\,B = F_e$$

$$-\omega_e bA + \left(k - m\omega_e^2\right)B = 0$$

$$A = \frac{F_e\left(k - m\omega_e^2\right)}{\left(k - m\omega_e^2\right)^2 + \omega_e^2 b^2}, B = \frac{F_e\omega_e b}{\left(k - m\omega_e^2\right)^2 + \omega_e^2 b^2}$$

If we write $\sqrt{\frac{\sqrt{k}}{m}} = \omega_e$

$$A = \frac{F_e m\left(\omega_o^2 - \omega_e^2\right)}{m^2\left(\omega_e^2 - \omega_e^2\right)^2 + \omega_e^2 b^2}; B = \frac{F_e\omega_e b}{m^2\left(\omega_o^2 - \omega_e^2\right)^2 + \omega_e^2 b^2}$$

writing A = Cos δ and B= Sin δ we get x = C cos(ω_et- δ)

Where $c = \sqrt{A^2 + B^2} = F_e\Big/\sqrt{m^2\left(\omega_o^2 - \omega_e^2\right) + \omega_e^2 b^2}$

and $\tan\delta = \omega_e b / m\left(\omega_o^2 - \omega_e^2\right)$

10.8 SUMMARY

The frequency and amplitude of a simple harmonic oscilator will be altered by the effect of a damping force. If an oscillating external force acts on a body it vibrates with the frequency of the driving force. These are forced oscillations. As the frequency of the driving force approaches the natural frequency of the body resonance occurs. At resonance the amplitude becomes very large. The period of oscillation of a torsional pendulum $T = 2\pi\sqrt{\frac{I}{K}}$ and the period of oscillation of a compound pendulum.

$T = 2\pi\sqrt{\frac{I}{Mgd}}$ Where I is the moment of inertia of a body, k Torsional constant of the supporting wire, m mass of the body d is the distance of the centre of mass of the body from the point of support.

10.9 MODEL ANSWERS

Check your Progress-1

The interval during which the amplitude drops to 1/e of its initial value.

Check your Progress-2

In a compound pendulum the point of support and the center of oscillation are conjugate because they are inter changeable. The compound pendulum oscillates with same period when suspended from center of oscillation.

10.10 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. For a damped harmonic oscillator derive the condition for the resultant motion to be oscillatory. How are the amplitude and frequency affected?

2. Write down the differential equation for angular harmonic motion. Derive expressions for the time period of (a) a torsional pendulum (b) a physical pendulum.

II. Answer the Following Questions in About 10 Lines.

1. Distinguish between over damped, critically damped and oscillatory motions of a damped harmonic oscillations.

2. Explain what is meant by resonant frequency. It is exactly equal to the undamped natural frequency? If not when is it closer to the undamped natural frequency?

3. The point of support and centre of oscilaltion for a physical pendulum are conjugate to one another Explain.

III. Solve the Following Problems

1. A solid sphere of mass 2.0 kg. and diameter 0.30 metre is suspended on a wire. Find the period of angular oscillation for small displacements if the torque constant of the wire is 6.0×10^{-3} N-m/radian. (Rotational inertia of a solid sphere of mass M and radius R about a diameter is $(2/5)\ MR^2$

   (Ans. 2.0 Sec)

2. A uniform rod of length 1.5 metre swings freely in a vertical plane about a horizontal axis at one end find its time period of oscillation.

   [Ans. 2.0 sec].

3. A condenser of capacitance 5 μF is discharged through a 0.5 Ω resistance and an inductance of 2 mH. calculate the frequency.

   [1592Hz]

BLOCK - VI

WAVES IN ELASTIC MEDIA

UNIT-11 PROGRESSIVE WAVES

Contents

11.1 AIMS AND OBJECTIVES

Thi unit introduces the concept of wave motion. In order to make you understand the conce different types of waves are explained qualitatively and quantitatively and the equati of a wave is worked out. It, explains the principle of superposition of waves and interfe nce. Principle of superposition is explained qualitatively. After going through this unit you will be able to evaluate the velocity of a transverse wave travelling in a string and the principle of superposition and interference of waves.

11.2 INTRODUCTION

We now consider waves in deformable elastic media. These are called mechanical waves. Ordinary sound is an example of mechanical wave. The mechanism of the motion of mechanical wave is as follows: Suppose some portion of the elastic medim is displaced from its normal position and oscillates about its equilibrium position. The disturbance is then transmitted from one layer to the next one because of the elastic properties of the medium. This layer then starts oscillating. In this manner the disturbance is transmitted from layer to layer in succession and the disturbance progresses. The medium itself does not move as a whole. Various parts of the medium oscillate about their respective equilibrium itself does not move as a whole. Various parts of the medim oscillate about their respective equilibrium positions. Since the transmission of disturbance is due to the elastic properties of the medim it follows that a material medium is necessary for the propagation of mechanical waves. It is well known that sound which is propagated in the form of mechanical waves cannot be heard if the medim is vaccuum.

The speed of the wave through the medim is determined by (i) the elastic properties of the medium and (ii) its inertia. We emphasise that a material medium is necessary only for mechanical waves. For example light waves are not mechanical in nature because the disturbance that travels is not due to a motion of material particle but due to oscillating electric and magnetic fields. They can be transmitted even through vacuum. Here the principle of superposition of two or more waves and the properties of standing waves is explained.

11.3 TYPES OF WAVES

Waves can be classified into transverse or longitudinal. Their characteristics are explained below.

11.3.1 Transverse Waves

In this type of wave motion the motion of the particles of matter are perpendicular to the direction of propagation of the wave. When one end of a vertical string is set into Oscillation a wave travels down the string as shown in fig. 11.1 Here the particles of the string vibrate perpendicular to the string whereas the wave travels along the string. Thus this is an example of transverse wave. In the case of light waves oscillations of the electric and magnetic fields are perpendicular to the direction of propagation. Hence light waves though not mechanical are transverse in nature.

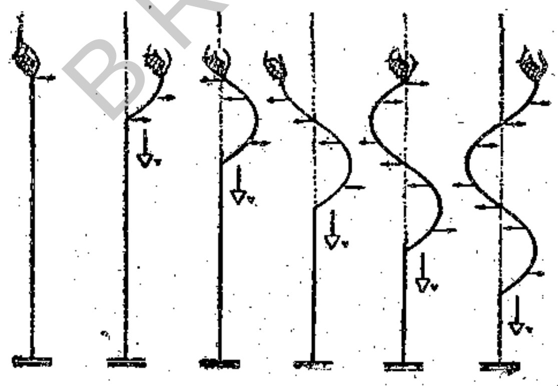


Fig. 11.1 In transverse wave the material (thread) particles vibrate in the perpendicular direction to that of the propagation of the wave direction.

11.3.2 Longitudinal Waves

In this type of wave motion, the motion of particles conveying the wave is in the same direction as that of wave propagation.

As an example consider the oscillations of a vertical spring set into oscillations up and down. Here the coil vibrates back and forth in the direction in which the disturbance travels. This is shown in Fig. 11.2.

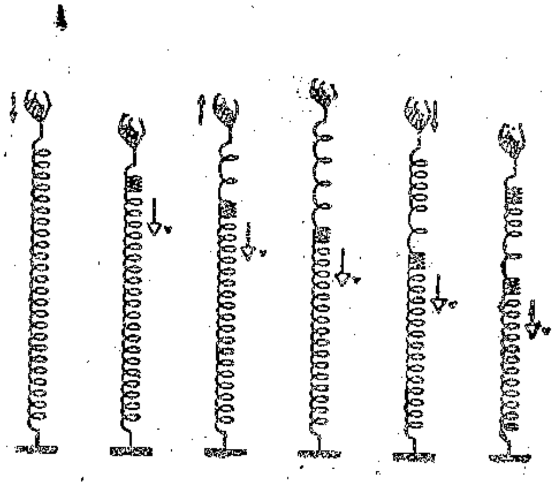


Fig. 11.2 In longitudinal wave material (spring) particle vibrate along the direction of the wave propagation.

Sound waves in a gas are longitudinal waves. Many aspects of wave motion and their general properties are common to both longitudinal and transverse waves (for e.g., Superposition Principle, Interference, diffraction).

There is one aspect of wave motion which is peculiar only to transerse waves and that is polarization. Longitudinal waves cannot be polarized. Sometimes we come across waves which are neither purely transverse nor purely longitudinal. But they can be resolved into 'Transverse Components' and 'Longitudinal Components'.

11.3.3 Travelling Waves

Let us consider a transverse pulse travelling along a string stretched in the x-direction. At the instant of time t = 0, let the shape of the string be represented by the equation.

$$y = f(x) \qquad (11.1)$$

Here y represents the transverse displacement of the string at the position x and f(x) is some function of x. This is shown in Fig. 11.3(a)

At later time t the pulse has advanced without changing its shape. The distance travelled by the wave is vt where, v is the magnitude of the wave velocity which we assume to be constant.

The equation of the curve at time t is

$$y = f(x^1) \qquad (11.2)$$

This shown in Fig. 11.3 (b)

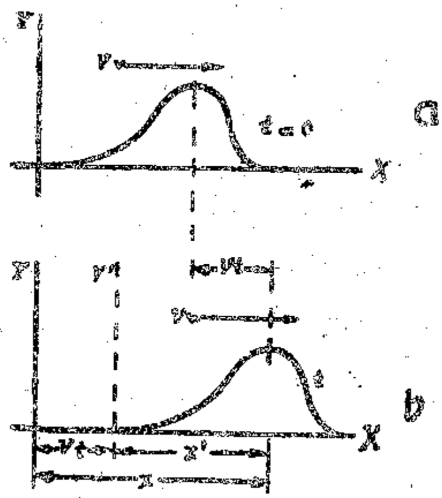


Fig. 11.3(a), (b) Transverse pulse travelling along a string.

This equation is of the same form as Eqn. (11.1) because the wave form remains the same.

But as we can see from the Fig. 11.3

$$x^1 = x - vt$$

$$\therefore y = f(x - vt) \qquad (11.3)$$

A wave travelling to the left will be represented by the equation

$$y = f(x + vt) \qquad (11.4)$$

The above equations represent the general equations to waves of any shape.

Although to fix out ideas we had chosen transverse waves on a string, the treatment is quite general and is valid for any type of wave including longitudinal waves.

We will now consider a wave of particular shape which is of great importance in Physics. The wave shape at time t=0 is given by the

$$y = f(x) = y_m \text{ Sin } \frac{2\pi}{\lambda} x \qquad (11.5)$$

The wave shape is a sine curve. The maximum displacement (y_m) is the amplitude of the sine curve.

The significance of the constant (y_m) will become clear soon.

Let the wave travel to the right with a velocity V

Then the equation of the wave shape at time t is given by

$$y = f(x - vt) = y_m \text{ Sin } \frac{2\pi}{\lambda}(x - vt) = y_m \text{ Sin } 2\pi\left(\frac{x}{\lambda} - \frac{vt}{\lambda}\right) \qquad (11.6)$$

The above equation represents the dependance of the transverse displacement with (i) distance (x) and (ii) time (t).

11.3.4 Variation of Displacement with Distance

Here we want to know the displacement of the various particles of the string at a particular time t.

If we treat t as constant in Eqn. (11.6) and plot y as a function of x the curve will be a sine curve as shown Fig. 11.4

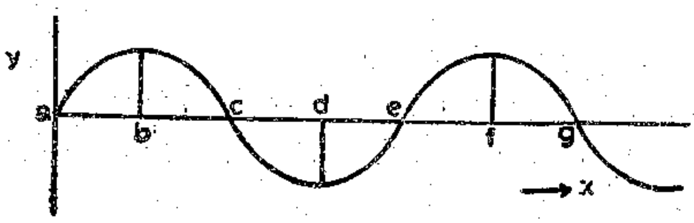


Fig.11.4 Sine curve

Let the displacement at a point x_1 be y_1

Then $y_1 = y_m \text{ Sin } \frac{2\pi}{\lambda}(x - vt)$

The displacement y_2 at a point $(x_1 + \lambda)$ will be

$$y_2 = y_m \text{ Sin } \frac{2\pi}{\lambda}(x_1 + \lambda - vt)$$

$= y_m \sin\frac{2\pi}{\lambda}(x_1 - vt) = y_1$ since the amptitude will be same after every wavelength.

The displacement is the same as at x_1

Similarly we can see that the displacement at distances $(x_1 + 2\lambda)$, $(x_1 + 3\lambda)$ will also be the same as y_1

Thus λ represents the distance between adjacent points which are in the same phase and is called the wavelentgh.

The time required for the wave to travel a distance equal to the wavelength λ is the period T

$$\lambda = vT \tag{11.7}$$

Substituting this in Eqn. (11.6) we get

$$y = y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \tag{11.8}$$

11.3.5 Variation of Displacement with Time (t)

Here we fix our attention on a particular particle at a distance say x from the end of the string. Let the displacement at time t_1 be y_1

$$\text{Then } y_1 = y_m \sin 2\pi\left[\frac{x}{\lambda} - \frac{t_1}{T}\right]$$

The displacement at a time (t_1+T) is

$$y_2 = y_m \sin 2\pi\left[\frac{x}{\lambda} - \left(\frac{t_1 + T}{T}\right)\right]$$

$$= y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t_1}{T}\right) = y_1$$

The displacement is the same as at time t_1. Similarly we can see that the displacement at times $(t_1,+2T)$, $(t_1,+3T)$ will also be the same (y_1).

That is the displacements of a particular particle will be the same at time intervals separated by T, 2T, 3T,...

In other words the particle will oscillate witha time period T. This is true of all particles, all of them oscillating with the same period T. That is, as the wave travels along the string each particle of the string moves up and down at right angles to the direction of wave motion.

As we can see from Fig. 11.4 when particle at *a* is in the equilibrium position, particle at b has maximum displacement in the positive direction and particle at *c* is in equilibrium position, particle at d has maximum displacement in the negative direction and so on.

We say different particles are in different phases of vibration. Particles at a and e are in the same phase similarly particles at b and f, c and g etc. As already mentioned distance between these points (which are in the same phase) is the wavelength

we define two quantities

i) Wave number $k = \frac{2\pi}{\lambda}$ and

ii) Velocity $\frac{\omega}{k} = \frac{\lambda}{T} = v$

Thus

In terms of these quantities Eqn. 11.8 becomes

$$y = y_m \text{ Sin } (kx - \omega t) \quad (11.9)$$

Let us find out the velocity and acceleration of a particular particle of the string situated at a point x.

$$\text{Velocity } \bar{u} = \frac{dy}{dt} y_m \omega \text{ Cos } (kk - \omega t)$$

(We use partial differentitation since x is to be regarded as a constant. We are fixing our attention on 3 particular value of x. Note that velocity of the particle u is not the same as the velocity of the wave)

$$\text{Acceleration of the particle } \bar{a} = \frac{d^2y}{dt^2}$$

$$= -y_m \omega^2 \text{ Sin } (kx - \omega t)$$

$$= -y\omega^2 \quad (11.10)$$

i.e., Acceleration is proportional to the displacement (y) and is oppositely directed.

This as we know is the characteristic of a S.H.M.

Hence for the particular case of a sinusoidal wave represented by Eqn. (11.9) the different particles execute S.H.M's about their equilibrium positions all of them with the same frequency. However they will differ in phase, the difference in phase depending up on the position of the particles.

11.4 VELOCITY OF TRANSVERSE WAVE IN A STRING

If we know the characterstics of the medium we can calculate the speed of the wave in the medium. As an example we will calculate the speed of the transverse wave travelling in a stretched string.

Let us consider a wave proceeding from left to right in the string with speed v. Let the tension of the string be F.

As we know the string as a whole does not move and it is only the wave pulse that moves along the string. But the situation will be the same if we imagine the whole string to be moved to the right with the speed v. The wave pulse will remain fixed in space and the particle of the string successively pass through the pulse.

Consider a small section of the pulse of length $\Delta\ell$. Let it form an arc of a circle of radius R. If μ is the mass per unit length of the string the mass of the element of the string $\mu\Delta\ell$

The tension in the string which is tangential can be resolved into verticle and horizonal components as shown in Fig. 11.5

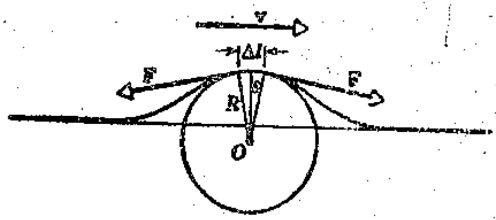


Fig. 11.5 Determination of wave velocity by considering the force on the elemental length Δl of the string.

The horizontal components F Cos θ being equal and oppositely directed cancel.

The vertical components F sin θ act in the same direction. The total vertical force 2F sin θ

For small values of θ we can approximate Sin θ with θ

The total vertical force $= 2\,F \sin\theta = 2\,F\,\theta = 2F\left(\frac{\Delta l/2}{R}\right) = F\frac{\Delta l}{R}$

This is the force giving rise to the centripetal acceleration of the string particles directed towards O.

But the centripetal force acting on a mass $(\mu\Delta l)$ moving in circle of radius R with velocity is-

$$\frac{\mu\,\Delta l v^2}{R}$$

Equating the two expressions for the force we get

$$F\frac{\Delta l}{R} = \frac{\mu \Delta l v^2}{R}$$

$$\therefore \sqrt{\frac{F}{\mu}} = v \qquad (11.11)$$

Thus we see that the velocity depends upon the two factors, tension and mass per unit length. These are quantities that are determined by the elasticity and inertia of the medium (as was mentioned in Section 11.1

The correctness of the form of Eqn. (11.11) can also be verified from diamensional consideration.

Dimensions of the LHS of Eqn. (11.11)

Dimensions of velocity $=LT^{-1}$

Dimensions of Tension $F=MLT^{-2}$

Dimensions of mass per unit length$=ML^{-1}$

Dimensions of $\sqrt{\frac{F}{\mu}} = \sqrt{\frac{MLT^{-2}}{ML^{-1}}} = \sqrt{L^2T^{-2}} = LT^{-1}$

Dimensions of the RHS of Eqn. 11.11

Dimensions of velocity - LT^{-1}

Thus dimensions of LHS and RHS are equal.

11.4.1 Wave Equation For a Strectched String

We shall now derive the differential equation of a travelling wave in a string. Let the x axis be the equilibrium position of the string. The distorted shape of the string is shown in Fig.11.6a

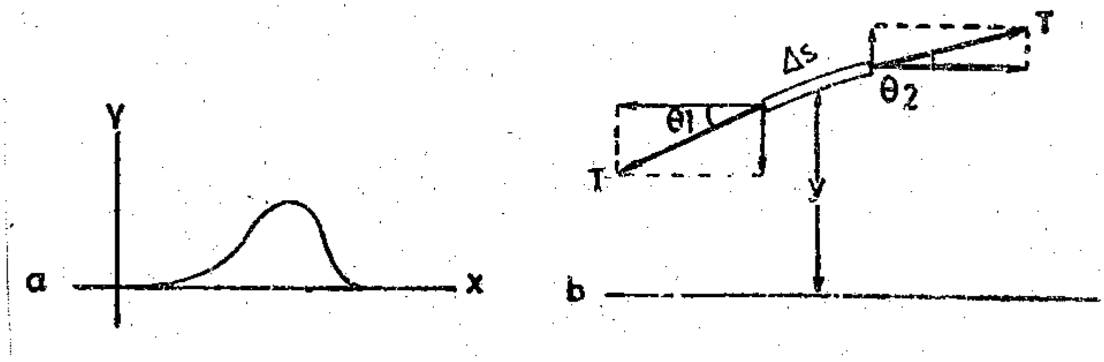


Fig 11.6 (a, b)

The displacement is assumed to be very small. Consider an element of length Δl shown magnified in Fig. 11.6 (b). We resolve the tension (F) at each end into x and y components.

Components

Resultant x component of the tension = $F\cos\theta_2 - F\cos\theta_1$

Resultant y component of the tension = $F\sin\theta_2 - F\sin\theta_1$

If the angles (θ_1 and θ_2) are small the resultant x component will be zero

Further for small angles, $\sin\theta \cong \tan\theta$

Hence the resultant y component = $(F\sin\theta_2 - F\sin\theta_1) \cong (F\tan\theta_1 - F\tan\theta_2)$

$$= F\Delta(\tan\theta)$$

$$= F\Delta\left(\frac{dy}{dx}\right)\left(\because \text{since } \tan\theta = \frac{dy}{dx}\right)$$

here $\Delta\left(\frac{dy}{dx}\right)$ means difference in slope between the ends of the element.

Let the mass per unit length of the string be μ and let us approximate the length by its components. Mass of the element = $\mu\Delta x$

We apply newton's Law (Force=mass x acceleration)

$$F\Delta\left(\frac{dy}{dx}\right) = \mu\Delta x\frac{d^2y}{dt^2}$$

$$F\frac{\Delta}{\Delta x}\left(\frac{dy}{dx}\right) = \mu\frac{d^2y}{dt^2}$$

$$= \frac{d^2y}{dx^2}$$

$$\therefore F\frac{d^2y}{dx^2} = \mu\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{\mu d^2y}{F\,dt^2} \qquad (11.12)$$

This is the differential equation of wave motion.

11.4.2 Plane Wave Solution Of The Wave Equation

Any plane wave disturbance which moves with a constant velocity (v) in the x direction has the form φ (x-vt). We will now show that the plane wave represented by

$$y = \varphi\ (x - vt) \qquad (11.13)$$

is a solution of the wave Eqn. (11.12)

Let (x - vt) = u

then

$$\frac{dy}{dx} = \frac{d\varnothing}{dx} = \frac{d\varnothing}{du}\frac{du}{dx}$$

$$= \frac{d\varnothing}{du}\left(\therefore \frac{du}{dx} = 1\right)$$

$$\frac{d^2y}{dx^2} = \frac{d^2\varnothing}{du^2}$$

$$\frac{dy}{dt} = \frac{d\varnothing}{du}\frac{du}{dt} = -v\frac{d\varnothing}{du}\left(\because \frac{du}{dt} - v\right) \qquad (11.14)$$

$$\therefore \frac{d^2y}{dt^2} = v^2\frac{d^2\varnothing}{du^2} \qquad (11.15)$$

substituting Eqn. (11.13) and (11.14) in the wave Eqn. 11.12 we get

$$\frac{d^2\varnothing}{du^2} = \frac{\mu}{F}v^2\frac{d^2\varnothing}{du^2}$$

$$\text{That is } v = \sqrt{\frac{F}{\mu}} \qquad (11.16)$$

This is just the expression for velocity of a transverse wave in a string that we had obtained earlier. (Eqn... (11.11). We can see that $y = \varnothing(x - vt)$ is also a solution of the wave equation (11.12)

As a particular form of the function we had

$y = y_m$ Sin (kx - ω t)

Let us now check that this is a solution of Eqn. (11.12)

we have

$$\frac{d^2y}{dx^2} = -k^2\, y_m\, \text{Sin}(kx - \omega t)$$

$$\frac{d^2y}{dt^2} = -\omega^2 y_m \text{ Sin}(kx - \omega x)$$

Substituting in Eqn. (11.12) we get

$$-k^2 y_m \text{ Sin }(kx - \omega t) = \frac{-\mu}{F}\omega^2 y \text{ Sin }(kx - \omega t)$$

$$\frac{\omega}{k} = \sqrt{\frac{F}{\mu}}$$

But $\frac{\omega}{k} = v \therefore v = \sqrt{\frac{F}{\mu}}$

11.5 GENERAL WAVE EQUATION

We have so far considered the particular case of the wave equation of a one dimensional wave in a string. But this can be generalised to include all types of waves travelling in three dimensions.

If a quantity U which is a function of the space of the space coodinates (x, y, z) and time 't' satisfies an equation of the type

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = \frac{1}{v^2}\frac{d^2U}{dt^2}$$

then this is a wave equation and V is the velocity of the wave. An example is the electronagenetic wave equation (derived from Maxwell's equations). In this case U is the Electric Vector (E) or the magnetic vector (H). The Velocity of the wave turns out to be the velocity of light in vacum.

11.6 THE SUPER POSITION PRINCIPLE

If two or more waves traverse the same space, then particle will experience displacements due to the different waves. The resultant displacement of a particle at a given time will just be the vector sum of the displacements due to the individual waves. In other words each wave will produce its effect as if the other waves were not present. This is called the Principle of Superposition.

It is every important to note that this principle holds only when the equations governing the wave motion are linear. The principle does not hold whenever non linear effects are present.

In the case of elastic media if we are within elastic limits, Hooke's law is valid. That is, the deformation is proportional to the restoring force. This is expressed mathematically by a linear equation. Hence Superposition principle holds. But if we go beyond the elastic limits Hooks law is not valid and the Superposition principle does not hold good. For electromagnetic waves the mathematical relations between electric and magnetic fields are linear. Hence they obey the Superposition principle.

In the case of shock waves the Superposition principle does not hold good because the equations governing the wave propagation are quadratic in nature.

The importance of this principle is, that when it holds, any complicated wave motion can be analysed as a combination of simple waves. Fourier, a French mathematician showed that any periodic motion can be representated as a combination of SHM's. This gives us a very useful analytical method of studying complicated periodic motions by the technique know as Fourier analysis.

11.7 INTERFERENCE OF WAVES

Superimposing two or more wave trains leads to Interference of waves. Let us consider two one dimensional waves travelling in the same direction. We assume the waves to travel with the same speed, to be of the same frequency and amplitude but differing in phase. The equations of the two waves may be expressed as

$$y_1 = A\sin(kx - \omega t - \varphi) \quad (11.17)$$

$$y_2 = A\sin(kx - \omega t) \quad (11.18)$$

In the two equations we have used the same values for A, k and because amplitudes, velocities and frequencies of both the waves are the same.

The resultant wave is represented by the equation

$$y = y_1 + y_2 = A[\sin(kx - \omega t - \varphi) + \sin(kx - \omega t)]$$

$$= A\left[2\sin\left(kx - \omega t - \frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\right]$$

$$= \left(2A\cos\frac{\varphi}{2}\right)\sin\left(kx - \omega t - \frac{\varphi}{2}\right) \quad (11.19)$$

Thus the resultant wave corresponds to a new wave which

i) is also sinusoidal

ii) has the same frequency (ω) as the original waves

iii) has an amplitude $\left(2A\cos\frac{\varphi}{2}\right)$

iv) differs in phase by $\frac{\varphi}{2}$ from the original waves.

We now consider two extreme cases :

Case I : $\varphi = 0$. The two waves are said to be in phase every where. The resultant amplitude is 2A The waves interfere constructively. The resultant wave at a particular time is as shown in Fig.(a) 11.7(a)

Case 2: $\varphi = 180$. The waves are said to be out of phase everywhere. The resultant amplitude is zero (since $2A\cos\phi/2 = 0$, since $\varphi = 180$)

The waves interfere destructively. The situation is as shown in Fig. 11.7 (b)

Even if the amplitudes of the two original waves are not equal it can be shown that we will still get a sinusoidal wave. The resultant amplitude will be

1) The sum of the amplitudes of the two waves $(A_1 + A_2)$ when $\varphi = 0$

2) The difference between the amplitudes $(A_1 - A_2)$ of the two waves when $\varphi = 180$

These are as shown in Fig. 11.7 (c) and 11.7(d).

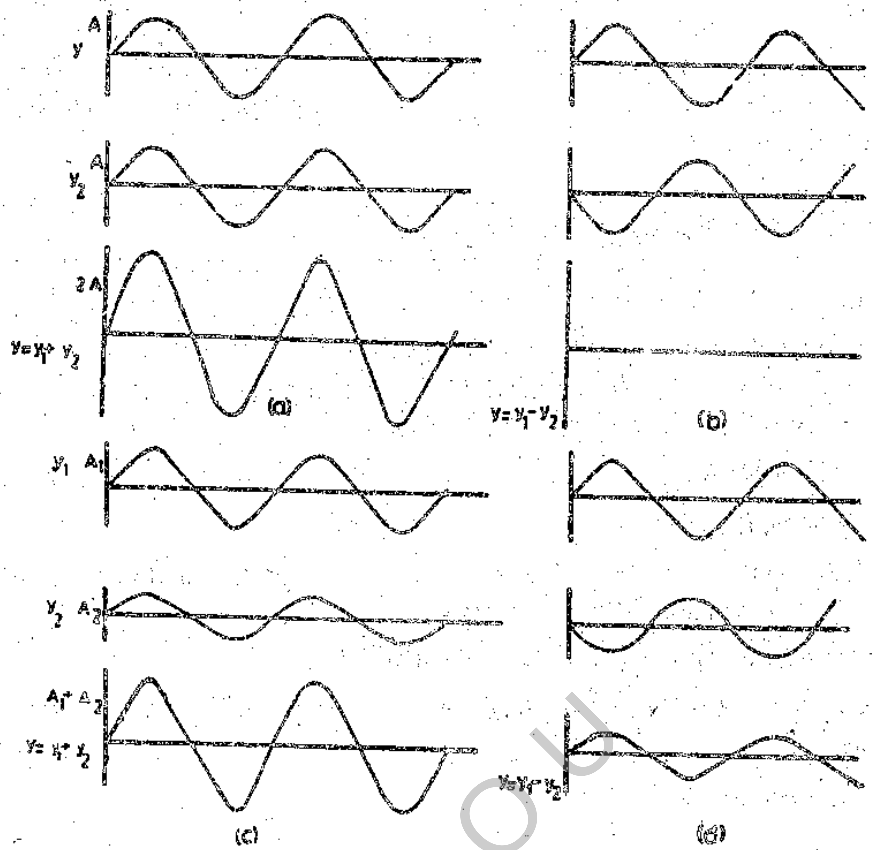


Fig.11.7 (a), (b), (c), (d)

Thus we see that the resultant amplitude is determined not only by the amplitudes of the individual waves but also the phase difference between the individual waves. The phenomenon of interference of waves is of importance in light.

11.8 STANDING WAVES

Let us consider a one dimensional wave (like the one travelling in a string) that gets reflected from a point. This will give rise to a reflected wave. The incident and reflected waves will interfere according to the principle of superposition. Let us now study the physical effects of this interference. Let the incident wave proceeding in the positive x direction (left to right) be represented by the equation

$$y_1 = A \sin (kx - \omega t) \qquad (11.20)$$

The reflected wave (of the sample amplitude, velocity and frequency) proceeding in the negative x direction (right to left) will be given by the equation.

$$y_2 = A \sin (kx + \omega t) \qquad (11.21)$$

We have assumed that there is no phase difference between the two waves.

The resultant wave can be written as

$$y = A \sin(kx - \omega t) + A \sin (kx + \omega t) \qquad (11.22)$$

$$y = (2A \text{ Sin } Kx) \cos \omega t \qquad (11.22)$$

Equation 11.22 the equation of a standing wave.

An examination of equation 11.22 indicates that

i) a particle at a particular point (x) executes SHM with a frequency (ω)

ii) all particles vibrate with the same frequency and

iii) the amplitude is not the same for all particles, If depends upon the location of the particle (given by 2A sinkx)

The resultant amplitude (2A Sin kx) has a maximum value of 2A at positions given by the condition

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots\ldots \text{ etc. (Since Sin } \frac{2\pi}{2} = I)$$

$$\text{i.e., } x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots\ldots \text{ etc. (since } k = \frac{2\pi}{\lambda})$$

These points are called antinodes.

The amplitude has a minimum value of zero at positions given by the condition

$$kx = \pi, 2\pi, 3\pi \ldots\ldots\ldots \text{ (since Sin } \pi = 0)$$

$$\text{or } x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots\ldots \text{ etc.}$$

These points are called nodes.

The distance between susccessive nodes or successive antinodes is one half wavelength (fig.11.8)

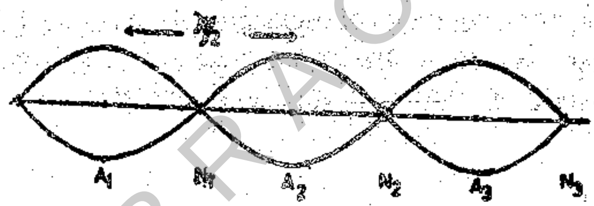


Fig. 11.8 Wave length

Thus we see that

in the case of travelling waves

i) All particles vibrate with the same amplitude

ii) all particles do not attain their maximum displacements at the same time

iii) the disturbance (crests and troughs)travel with velocity (v)

in the case of stationary or standing waves

i) different particles vibrate with different amplitudes, the amplitude being determined by the position. (Particles at the antinodes have maximum amplitude and particles at the nodes have zero amplitude).

ii) all particles attain their maximum displacements at the same time

iii) the wave as such is stationary.

In writing down the equation for the incident and reflected waves Eqns. 11.20 and 11.21 we had assumed that the amplitudes of the two waves are the same and that there is no phase difference between them.

The amplitudes will be the same only when the reflection is total. But in actual practice there will be partial reflection and partial transmission. Hence the amplitude of the reflected wave will be less than that of the incident wave.

It can be shown that whenever there is reflection of a wave at a fixed end the reflected wave is allways 180° out of phase with the incident wave. That is in reflection from a fixed end a wave undergoes a phase change of 180°. As an example of standing waves we now consider the case of a string fixed at both ends. Standing waves can be established in the string. The end points which are fixed cannot have any displacement and hence must correspond to nodes. But there may be any number of nodes between them.

(Hence the wave length associated with the standing waves can have different values). Since the distance between adjacent nodes is $(\lambda/2)$ a string of length l must be exactly an integral number (n) of half wavelengths

$$\frac{n\lambda}{2} = l \text{ i.e., } \lambda = \frac{2l}{n}$$

but $\lambda = vT$ and $v = \sqrt{\frac{F}{\mu}}$ (11.23)

$\therefore$ Natural frequencies of the osciallation are

$$f = \frac{1}{T} = \frac{n}{2l}\sqrt{\frac{F}{\mu}}, \; n=1, 2, 3$$

If a system capable of oscillating is acted on by periodic impulses having a frequency equal to one of the natural frequencies of oscillation of the system it is set into oscillations with a large amplitude. We say the system resonates with the applied impulse. Hence by giving a periodic impulse (such as with the help of a vibrator)

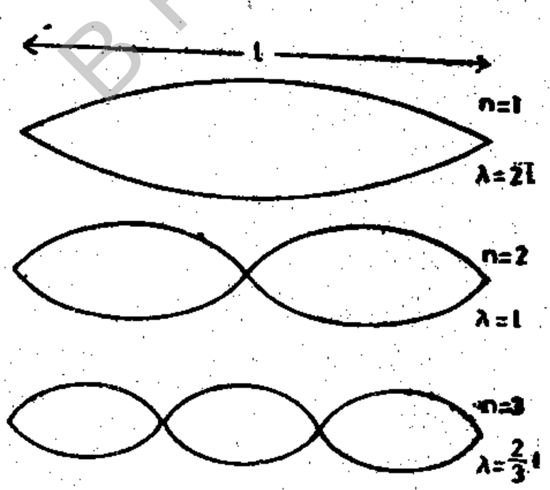


Fig. 11.9 Modes of vibration of a spring fixed at both ends

The frequency corresponding to n=1 is called the fundamental frequency and those correspoiding to higher values of n are called harmonics.

From equation 11.23 we see that the frequency of oscillation of a stretched string is

i) inversely proportional to the vibrating length if the tension in the string (F) and mass per unit length (μ) are constant

ii) directly proportional to the square root of tension (F) if the length (l) and mass per unit length (μ) are constant

iii) inversely proportional to the square root of the mass per unit length of the string (μ) if the length (ℓ) and tension (F) are constant.

The above three are the laws of transverse vibrations of strings and they can be experimentally verified.

11.9 BEATS

Suppose two wave trains of slightly different frequencies travel in the same region and in the same direction. At a particular point (x) let the displacements produced by the two waves be represented by

$$y_1 = A\cos 2\pi f_1 t \qquad (11.24)$$

$$y_2 = A\cos 2\pi f_2 t$$

(f_1 and f_2 are the frequencies)

We have assumed the amplitudes of the two waves to be the same. By the principle of superposition the resultant displacement is

$$y = y_1 + y_2 = A\left[\cos 2\pi f_1 t + \cos 2\pi f_2 t\right]$$

$$= \left[2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\left[\cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t\right]$$

The above equation shows that the resultant vibration has a frequency given by $f' = \frac{f_1 + f_2}{2}$ the average frequency of the two waves

What is more important is the amplitude itself varies with a frequency given by $f'' = \frac{f_1 - f_2}{2}$

If f_1 and f_2 are nearly equal this term is small and the amplitude fluctuates slowly.

A maximum of amplitude occurs whenever cos $2\pi\left(\frac{f_1 - f_2}{2}\right)t = \pm 1$

This Phenomenon is called beats. Since each of these values occurs once in each cycle the number of beats per second is twice the frequency f'' i.e., number of beats per second = $f_1 - f_2$ = difference of the frequencies of the two waves.

Beat frequencies of about seven per second can be detcted by the human ear.

The following points of similarity between standing waves and the phenomenon responsible for beats are worth emphasising. In both the cases the amplitude of the resultant wave is not constant. In the first case (standing waves) particles at different points in space have different amplitudes.

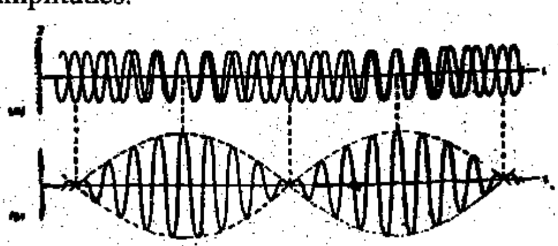


Fig.11.10(a) & (b)

Amplitude is a function of the location in space (given by 2A Sin kx).

Hence this is called interference in space. In the second case (beats) the amplitude of a particle at a particular location in space varies with time $\left(\text{given by } 2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right)$. Hence this is called interference in time.

Worked Example 1 :

The equation of transverse travelling wave on a string is y=2 sin π (0.5x-200t) Where 'x' and 'y' are in centimeters and 't' is in seconds. (a) Find the amplitude, wavelength, frequency and velocity of propagation of the wave. (b) Find the maximum transverse speed of a particle in the string.

Solution :

(a) $y=2\,\text{Sin}\,\pi\,(0.5x-200t)$

Comparing this with equation 11.9 $y=y_m\,\text{Sin}\,(kx-\omega t)$

we find $y_m = 2$cms., $k=0.5\pi$ and $\omega = 200\pi$

$\therefore$ Amplitude = y_m = 2cms.

Wavelength = $\lambda = \frac{2\pi}{k} = 4$ cms.

Frequency = $\frac{\omega}{2\pi} = 100$ Hz.

Velocity of propagation = f λ = 400 cms per sec.

(b) Transverse speed of the particle $= u = \frac{dy}{dt} = -y_m\omega\cos(kx-\omega t)$

Magnitude of the maximum value of the speed $= \omega y_m = 400\pi = 1257$cms. per sec.

Worked Example 2 :

A steel wire 6 metres long has a mass of 60 gms. and is streched with a tension of 1,000 newtons. What is the velocity of propagation of a transverse wave in the wire?

Solution: Length of the wire=6 meteres

Mass of the wire = 60 gms

Mass per unit length of the wire μ = 0.01 kg. per metre

Velocity of propagation of a transverse wave = $\sqrt{\frac{F}{\mu}} = 316$ meteres per sec.

Worked Example 3 :

Show that the slope of a string at any point is numerically equal to the ratio of the particle speed to the wave speed at that point.

Solution :

$$y = y_m\,\text{Sin}\,(kx-\omega t)$$

Wave speed = $v = f\lambda = \frac{\omega}{2\pi} x \frac{2\pi}{k} = \frac{\omega}{k}$

Particle speed $= \frac{dy}{dt} = -\omega y_m \cos(kx - \omega t)$

Ratio of particle speed to wave speed = $- ky_m \cos(kx - \omega t)$

Stope of the string = $\frac{dy}{dx} = ky_m \cos(kx - \omega t)$

Ratio of particle speed to wave speed is numerically equal to the slope of the string.

Worked Example 4:

The amplitude of three sinusoidal waves of the same period are in the ratio 1:1/3: 1/4. Their phase angles are 0, $\frac{\pi}{2}, \pi$ respectively.

What will be the resultant displacement at any point (x) and time (t)?

Solution : Let the amplitude of the first wave be A. Then amplitudes of the second and third waves are $\frac{A}{3}$ and $\frac{A}{4}$

The phases of the three waves are $\varphi_1 = 0.\ \varphi_1 = \frac{\pi}{2}$ and $\varphi_3 = \pi$

The displacements are $y_1 = A \sin(kx - \omega t)$

$$y_2 = \frac{A}{3}\sin(kx - \omega t - \pi/2) = -\frac{A}{3}\cos(kx - \omega t)$$

$$y_3 = \frac{A}{4}\sin(kx - \omega t - \pi) = -\frac{A}{4}\sin(kx - \omega t)$$

Resultant displacement $y = y_1 + y_2 + y_3$

$$= A \sin(kx - \omega t) - \frac{A}{3}\cos(kx - \omega t)$$

$$- \frac{A}{4}\sin(kx - \omega t)$$

$$= \frac{3}{4} A \sin(kx - \omega t) - \frac{A}{3}\cos(kx - \omega t)$$

Worked Example 5 :

A string vibrates according to the equation

$y = 8\sin\left(\frac{\pi x}{4}\right)\cos(30\pi)$ where x and y are in cms. and t is in seconds. Find out

(a) the amplitude and velocity of the component waves whose superposition can give rise to this vibration. (b) the distance between the nodes. (c) velocity of a particle of the string at the position x=3 cms and t=1.5.

Solution:

The two waves

$$y_1 = y_m \sin(kx - \omega t)$$

$$\text{and } y_2 = y_m \sin(kx - \omega t)$$

When superposed gives rise to the standing wave $y = 2y_m \sin kx \cos \omega t$

Comparing this with the given equation we find $y_m = 4, k = \pi/4$

$$\omega = 30\pi$$

(a) Amplitude of the component waves $= y_m = 4\text{cm}$

Velocity of the component waves $v = \frac{\omega}{k} = 120 \text{ cm/sec.}$

(b) Distance between the nodes $= \frac{\lambda}{2} = \frac{\pi}{k} = 4\text{cm}$

(c) Velocity of the particle $= \frac{dy}{dt} = -(2y_m \sin kx)(\omega \sin \omega t)$

Magnitude of the velocity of the particle at x=3 and t = 1.5

$$= \left(8 \sin \frac{\pi}{4} 3\right) \sin\left(30\pi \frac{3}{2}\right) x\, 3\pi = 0$$

Worked Example 6:

Two identical piano wires when stretched with the same tension have a fundamental frequency of 440 Hz. By what fractional amount must the tension be increased in one wire so that 6 beats per second can be heard when both the wires vibrates simultaneously?

Solution:

The fundamental frequency $f = \frac{1}{2}\sqrt{\frac{F}{\mu}}$ Since the wires are identical μ - is the same for both.

When the tension of one wire is increased of F^1 the frequency of the note emitted by that wire will increase to

$$f = \frac{1}{2l}\sqrt{\frac{F}{\mu}}$$

We are given that f=440 Hz

Then f^1=446Hz(since the number of beats heard is 6 per sec.)

$$440 = \frac{1}{2l}\sqrt{\frac{F}{\mu}}$$

$$446 = \frac{1}{2l}\sqrt{\frac{F'}{\mu}}$$

$$\frac{F}{F'} = \left(\frac{446}{440}\right)^2 = 1.027$$

Fractional increase in Tension= $\frac{FF^1}{F} = 0.027$

Percentage increase in tension =2.7

11.10 SUMMARY

The transfer of energy from one point to the other in a material medim is called a wave motion. The wave motion has periodic character. In transverve wave motion, the particle vibration is perpendicular to the wave propagation direction, while in longitudiual wave motion the particle vibration is parallel to the wave propgation direction. The velocity of the wave is given by product of frequency and wave length of the wave. The distance between two successive crests or troughs gives the wave length. The number of waves generated in uint time gives the frequency.

If two or more waves are superimposed each will produce its effects as if the other waves were not present. This principle is known as the principle of super position. Standing waves are formed when a travelling wave gets reflected. Points of maximum amplitudes are called antinodes and the point of zero amplitude are called nodes. The distances between succesive nodes or anti nodes is one half the wave length.

11.11 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. Explain the characteristics of travelling waves and obtain an expression for the displacement (y) at any point (x) and time (t)

2. Obtain an expression for the velocity of transverse waves in a string in terms of tension in the string and linear density of the string. From dimensional considerations check the correctness of the expressions.

3. Discuss the mods of transverse vibrations of the string.

4. Explain the phenomenon of beats and explain the beat frequency.

II. Answer the Following Questions in About 10 Lines.

1. Explain the mode of presentation of waves in a medium

2. Distinguish between transverse waves & longitudind waves.

3. Is the principle of superpostion applicable in all cases? If not what is the condition to be satisfied by the waves for this principle to hold good.

4. Distinguish between progressive waves & stationary waves.

5. Explain the constructive interference and distructive interference.

6. Explain phenomenon of beat is refered to interference in time.

III. Solve the Following Problems:

1. A wave of frequency 500Hz has a phase velocity of 350 m/sec. (a) How far apart are

two points 60° apart) (b) what is the phase difference between two displacements at a certain point at times 10^{-3} seconds apart.

[Ans: (a) 12 cm (b) 180]

2. Determine the amplitude of the resultant motion when two sinusoidal motions having the same frequency travelling in the same direction are combined. If their amplitudes of 6cm. and 8 cm. and they differ in phase by $\frac{\pi}{2}$ radians. [Ans 10 cm]

3. A travaling wave propagates according to the equation

$$y = 0.03 \sin (3x-2t)$$

Determine the (a) amplitude (b) wavelength and (c) frequency

[0.03M; 2.09M, 0.31H_z]

4. A string vibrates following the equation

$$y= 5 \sin\frac{\pi x}{3} \text{Cos } 40\pi t$$

Find the distance between the two successive nodes and the speed of the particle of the string at a position x=1.5 cm when t = 9/8 sec. [3 cm, 0]

5. The speed of a transverse wave in a streched string is 340 m/s when the tension in the string is 2.5 kg. ωt. If the tension is changed to 3.6 kg. wt calculate its speed of the transverse wave in the same string

[408 M/s]

6. Find the resultant displacement at x = 18M and time $t = \frac{1}{18}$ sec in the superposition of the two waves $y_1 = 10 \text{ Sin } (3\pi t-4x)$

$$y_2 = 10 \text{ Sin } (3\pi t+4x)$$

[9.675M]

UNIT-12 DOPPLER EFFECT

Contents

12.1 AIMS AND OBJECTIVES

This unit explains the phenomenon of Doppler effect. After going through this Unit you will be able to :

1) explain the change in the observed frequency when the observer and sound source are in relative motion.
2) evaluate the velocity of earth satellites with the help Doppler shift in the frequency of the radio waves transmitted by them.

12.2 INTRODUCTION

In this unit we shall study the phenomenon of Doppler effect. It was first noticed by C.J.Doppler (1803-1853) in sound waves. Doppler effect is not confined to only sound waves. It applies to waves in general. Doppler effect can be observed even in light.

12.3 WHAT IS DOPPLER EFFECT?

When we move towards a stationary source of sound the pitch (frequency) of the sound heard is higher than the one we hear at rest. If we move away from the source the pitch is lower. Similarly if we are at rest and the source of sound is approaching us the pitch is higher than when the source is stationary. If the source of sound is receding from us the pitch is lower. It is a matter of common experience that the pitch of the whistle of an approaching locomotive is higher than when it is receding.

If a vibrating tuning fork is moved rapidly towards a wall we hear two notes of different frequencies. One is the note heard directly from the tuning fork. The pitch of this is lowered as the source (tuning fork) is receding from us. The other note is due to the wave reflected from the wall. The pitch of this note is raised. Since the two notes are nearly of the same frequency due to the superposition of these waves we hear beats.

This phenomenon of a change in observed frequency when the observer and the source are in relative motion is called Doppler Effect.

12.4 CHANGE IN FREQUENCY

Whe shall now calculate the change in frequency due to Doppler Effect.

Let us consider both the source and observer to be in motion. We treat the velocities of the observer (V_o) and that of the source (V_s) to be along the line joining them. This is a special case.

We adopt the following convention of signs for the velocities

We take the positive direction of V_o and V_s as that from the position of the observer to the position of the source. The velocity of propogation sound waves (v) is always considered positive.

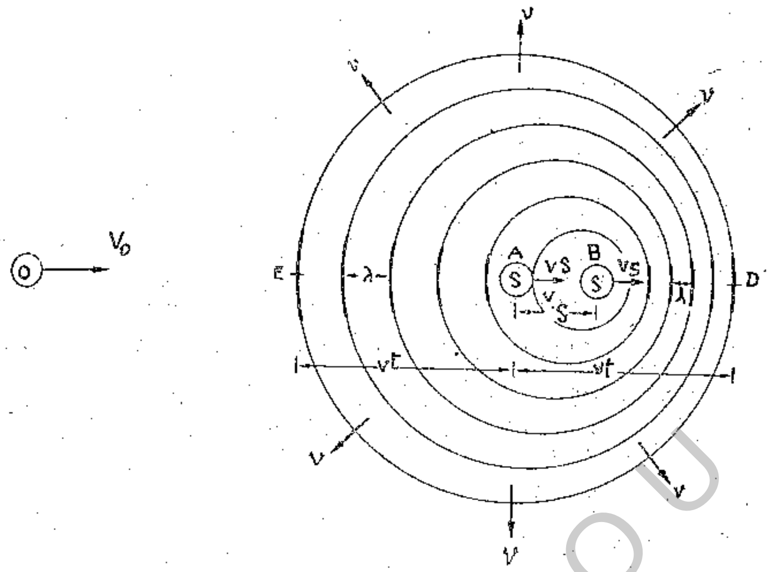


Fig.12.1 The propogation of Sound waves

In Fig.12.1 the observer is to the left of the source. The positive direction is from left to right and both V_o and V_s are positive.

The source is at a point A at time t = o and is at the point B at time t. The outer most circle represents the wave surface emitted at time t = 0. This is a sphere with centre at A and radius vt. The velocity of the wave is a property of the medium and is not affected by the fact that the wave emanated from a moving source. Once the wave leaves the source, the subsequent state of the source has no effect on the motion of the wave fronts.

AB = Distance moved by the source in time t=$V_s t$

EA = AD = Distance advanced by the wave front which left the source at t = 0 is vt

From the figure we see that

$$EB=EA+AB=(v+V_s)t \tag{12.1}$$

$$BD=AD-AB=(v-V_s)t \tag{12.2}$$

The number of waves emanated in the time interval t (between t=0 and t=t is N=ft- where f is the frequency of the note emanated by the source.

In front of the source these waves are crowded into the distance BD while behind the source they are spread over a distance EB:

The wavelength in front of the source

$$\lambda' = \frac{BD}{N} = \left(\frac{v-V_s}{ft}\right)t = \left(\frac{v-V_s}{f}\right) \tag{12.3}$$

The wavelength behind the source=

$$\lambda'' = \frac{EB}{N} = \frac{v+V_s}{ft}t = \frac{v+V_s}{f} \qquad (12.4)$$

The velocity of propagation of the waves approaching the moving observer is $(v+V_o)$ relative to him.

$$\text{The observed frequency } f' = \frac{\text{Velocity of propagation}}{\text{wavelength}}$$

$$= \frac{v+V_o}{(v+V_s)/f}$$

$$f' = \left(\frac{v+V_o}{v+V_s}\right)f \qquad (12.5)$$

All the special cases can be derived from the above equation if we make consistent use of the convention of signs as shown in the Table 12.1

let us take a specific case and calculate the observed frequencies.

The frequency of sound emitted f = 1000 Hertz.

Velocity of sound waves v = 1000 ft/sec.

i) observed frequency when the observer approaches a stationary source with a velocity of 100 ft/sec. $f' = \left(\frac{v+V_o}{v}\right)f = 1100\text{ Hz.}$

ii) observed frequency when the source approaches a stationary observer with a velocity of 100 ft/sec. $f' = \left(\frac{v}{v-V_s}\right)f = 1111\text{ Hz.}$

iii) observed frequency when the observer recedes from a stationary source with a velocity of 100 ft/sec. $f' = \left(\frac{v-V_o}{v}\right)f = 900\text{ Hz.}$

iv) observed frequency when the source recedes from a stationary observer with a velocity of 100 ft/sec. $f' = \left(\frac{v}{v+V_s}\right)f = 909\text{ Hz.}$

Table 12.1

Case	Description	Velocity of observer Vo	Velocity of the source Vs	Figure	observed f' as calculated from formula 12.1
1.	Source statinary, observer approaching the source	Positive	0		$\left(\frac{v+V_o}{v}\right)f$
2.	Observer stationary, source approaching the observer	0	Negative		$\left(\frac{v}{v-V_s}\right)f$
3.	Source stationary, observer moving away from the source	Negative	0		$\left(\frac{v-V_o}{v}\right)f$
4.	Observer stationary, source moving away from the observer	0	Positive		$\left(\frac{v}{v+V_s}\right)f$

From the above example we see that both when the observe approaches a stationary source or the source approaches a stationary observer there is an increase in frequency (cases 1 and 2). But it is important to note that even when the velocity with which the source or observer approach each other is the same, the observed frequencies are not the same.

Similarly when the observer moves away from a stationary source or the source moves away from a stationary observer (cases 3 and 4) there is a decrease in frequency. Even in this case we find that for the same velocity of recession of the source or observer the observed frequencies are not the same.

But if the velocity with which the source or observer move is small compared to the velocity of sound in the medium, observed frequencies are nearly the same for cases 1 and 2 and for cases 3 and 4

This can be shown as follows:

Let u represent the relative velocity between of source and observer

Let us take case 2: $f' = f\left(\frac{v}{v-u}\right)$

$$= f\left[\frac{1}{1-\frac{u}{v}}\right] = f\left(1-\frac{u}{v}\right)^{-1}$$

$$= f\left(1+\frac{u}{v}+\left(\frac{u}{v}\right)^{+2} \dots \dots\right) \qquad (12.6)$$

If u is small compared to v, $\frac{u}{v}$ is small compared to unity and hence we can neglect $\left(\frac{u}{v}\right)^2$ and higher power in equation 12.6

$$\therefore f' \cong f\left(1+\frac{u}{v}\right) = f\left(\frac{v+u}{v}\right)$$

This is the same as for case 1, when $v_o = u$

Similarly it can be shown for cases 3 and 4

12.5 DOPPLER EFFECT IN LIGHT

Doppler effect is not confined to only sound waves. It applies to waves in general. Since light is propagated as waves the effect can be observed with light waves also. However since the speed of light is so great, that the effect can be detected with astronomical or atomic sources whose velocities are high.

The frequencies of light sources will be the same whether a given atom is on the earth or on a star. The wavelength of light from certain stars is slightly longer than the wavelength of light from same atoms on the earth. In the case of light from some other stars the wave length is shorter. We can thus infer that the stars are moving towards or away from the earth. Their velocities can be determined from the observed differences in wavelengths.

In a gas discharge tube in the laboratory the atoms are in constant random motion with relatively large and varying velocities. Hence light from such sources is a mixture of waves with a spread of wavelength. Thus the broadening of radiation emitted from hot gases is a consequence of Doppler effect.

There is an important difference between Doppler effect in light and in sound.

In sound it is not the relative motion of source and observer that determine the change in frequency. We saw that even when the relative velocity is the same the frequency will depend on whether the source or observer is in motion. This is so because the velocities V_0 and V_s are measured relative to the medium in which the sound is propagated and the medium determines the wave speed (v)

In the case of light for which a material medium is not necessary for propagation the speed of light is the same relative to the source or observer. It does not depend on their motion. This is an important and basic postulate of the special theory of relativity. Hence for light it is only the relative motion of source and observer that will determine the change in frequency.

An interesting example of Doppler effect is the reflection of radar waves from a moving object such as a plane or a car. The wavelength of the reflected waves is increased if the object is moving towards the source and decreased if it is moving away from the source. The velocities of earth satellites are determined from the Doppler shift in the frequency of the radio waves transmitted by them.

Worked Example:

A man standing in front of a large smooth wall holds a vibrating tuning fork of frequency 400 Hz. directly in front of him between him and the wall. He moves the fork toward the wall with a speed of 1.2 metre per sec. How many beats per second will he hear between the sound waves reaching him directly from the fork and those reaching after reflection from the wall? (Velocity of sound in air - 341 metres per sec.

Solution:

Frequency of the fork = F=400 Hz.

In the case of the sound waves reaching directly, the observer is stationary and source moving away from the observer.

Frequency of the note heard by the observer

$$f = \left(\frac{u}{v+v_s}\right)f = \left(\frac{348}{348+1.2}\right)x400 = 398.6\,Hz.$$

In the case of sound waves reflected, the observer is stationary and source approach in the observer.

Frequency of the note heard by the observer

$$f = \left(\frac{v}{v+v_s}\right)f = \left(\frac{348}{348-1.2}\right)x\,400 = 401.4\,Hz$$

Number of beats heard per second = $f_2 - f_1 = 2.8$

12.6 SUMMARY

Achange in observed frequency due to relative motion of observer and the source are in relative motion is called doppler effect. Doppler effect can be applied to waves. The velocity of the earth satellites are determined from the Doppler shift in the frequency of the radio waves transmitted by them.

12.7 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Quetion in About 30 Lines.

Explain the phenomenon of Doppler effect and clearly discuss the various cases.

II. Answer the Following Questions in About 10 Lines.

What is the difference between Doppler effect in light and in sound.

III. Solve the Following Problems

1. A whistle of frequency 540 Hz. rotates in a circle of radius 0.6 metre at an angular speed of 15 radians/sec. What is the lowest and the highest frequency heard by a listener a long distance away at rest with respect to the centre of the circle? velocity of sound in air = 348 metres/sec. [Ans:554 Hz.526 Hz]

2. Two trains travelling in opposite directions at 100 KM/hr cross each while one of them is whistling. If the frequency of the note is 800 Hz find the apparant frequency as heard by an observer in other train before the trains cross each other [942.3Hz]

3. The frequency of the horn of a car is observed to drop from 272 Hz to 256 Hz as the car passes the stationary observer. Find the speed of the car. Velocity of the sound in air is 346.5m/s

   [10.5 M/S]

4. A person is standing on a railway platform. An engine moving away from the person with a speed of 20 M/s blows a whistle of frequency 740 Hz. Calculate the apparant frequency of the whistle as heard by the person. [700 hz]

5. A person is standing on a railway platform. An engine while approaching the platform with a velocity of 20 M/s. blows a whistle of frequency 666Hz. If the velocity of sound is 350M/S calculate the apparant frequency of the whistle as heard by the person

   [700Hz]

BLOCK - VII

INTEREFERENCE

UNIT-13 HUYGEN'S PRINCIPLE YOUNG'S EXPERIMENT AND ITS QUALITIVE TREATMENT

Contents

13.1 AIMS AND OBJECTIVES

This unit explains the dual character of light on the basis of the corpuscular and wave theories, youngs experiment & Interference patterns. After going through this Unit you will be in a position to:

1. Verify the Law's of reflection and refraction using Huygen's principle.
2. Explain the conditions under which total internal reflection occurs.
3. Explain the colours observed in thin films like soap bubble and oil films on the surface of water.

13.2 INTRODUCTION

We know that energy can be propagated from one place to another either by moving matter or by a wave disturbance travelling through a medim which does not itself move as a whole. Water falling from a dam, a glass window getting broken by a stone thrown from a distance, a turbine moving due to the water falling on the blades are some examples where energy is transferred directly from one place to the other of moving matter. Propagation of sound energy from one place to the other takes place via wave motion. Each particle of the medium, by vibrating about its mean position, transfers energy to its neighbour. Two theories namely the corpuscular theory and the wave theory have been proposed to describe the transmission of light energy from one place to the other i.e., from the point of light source to any distant point. The corpuscular theory was proposed by Sir Issac Newton in 1675 and the wave theory was proposed by Huygens in 1678. These two theories have been subjected to experimental tests since then and there have developed two schools of thought. Hooke, Huygens, Descartes and Euler supported the wave theory whereas Newton and Laplace defended the corpuscular theory.

According to the corpuscular theory, light emitted by a luminous body consists of invisible, rapidly moving particles whose size varies with colour. These invisible particles called corpuscules are so small that they can readily travel through the interstices if the particles of matter with the velocity of light. These particles undergo reflection from a polished surface, or transmission through a transparent medium. When these particles fall on the retina of the eye, they produce the sensation of vision.

According to the wave theory, the light source sends out waves in all directions. Since a medium is required the propagation of a wave, a hypothetical medium called ether was assumed to exist pervading all space.

The relative merits of each of the two theories could be judged in terms of the ability of these theories to explain the experimentally observed phenomena. While the corpuscular theory could explain the rectilinear propagation of light, the phenomena of reflection, refraction and dispersion, the theory fails to explain phenomena like interference, defraction and polarization. The wave theory put forward by Huygens could explain the phenomena of reflection, refraction and dispersion. A satisfactory explanation or rectilinear propagation on the basis of wave theory was given by Thomas Young and Fresnel who also demonstrated the phenomenon of interference. Acceptance of wave theory became almost universal when Focault and Michelson showed experimentally that the speed of light is less in water than that in air which is the reverse as per the corpuscular theory and true as per the wave theory. However, the discovery of photoelectric effect led to the conclusion the emission of light is corpuscular in nature. The quantum theory of light postulates the emission of light energy in terms of quanta called photons whose energy can be given by hυ where 'h' is Planck's constant and υ the frequency of the radiation. This theory could satisfactorily explain the phenomenon of photoelectric effect whereas the wave theory had failed to account for the experimentally observed facts.

There are certain phenomena which can be explained only in terms of the particle nature of light and other phenomena which can be explained only on the basis of the wave theory. This dual nature of light is presently accepted beyond doubt. Some times we have to think in terms of photons and some times in terms of waves. In this unit. We shall study in detail Huygen's wave theory and the explanation of the phenomena of reflection and refraction on the basis of wave theory.

A very important characterstic of wave motion is the phenomenon of interference, which occurs when two or more wave motions coincide in space and time. In unit 13 we discussed the super-position of two simple harmonic motions. Interference occurs for example in the region in which reflected and incident waves coincide. Another important example of interefere is found in a wave motion which is confined to a limited region of space such as a string with its two ends fixed or a liquid in a channel.

The phenomenon was first demonstrated by thomas young in 1801. This was the strong experimental evidence in support of Huygen's wave theory of light.

In wave motion we have two types of waves namely longitudinal and transverse waves. In longitudinal wave motion the particles of the medim vibrate about their mean position along the direction of propagation of the wave. In transverse wave motion the particles of the medium vibrate about their mean position in a direction perpendicular to the direction of propagation of the wave. While developing the wave theory of light Huygens assumed light waves as longitudinal. On this basis, Huygens could satisfactorily explain the phenomena of reflection, refraction and double refraction noticed in caystals like quartz and calcite. Fresnel and Young suggested that light waves are transverse and they could explain satisfactorily the rectilinear propagation of light. The phenomenon of interference is characterstic of any wave phenomena, irrespective of whether the waves are longitudinal or transvers. Let us now study the conditions required for interference phenomenon that can be observed with light waves.

13.3 HUYGEN'S PRINCIPLE

According to the wave theory, a point source of light when placed in an isotropic medium emits light waves in all directions. These waves travel in the form of expanding concentric spheres with a uniform velocity of $3x10^8$ ms^{-1}. With the point source as centre, the disturbance will reach all the particles lying on the surface of a sphere simultaneously. Such a sphere is called the wave front. Thus a wave front may be defined as the locus of points having the same phase. The shape of the wave front depends upon the shape of the light source and its distance from the source.

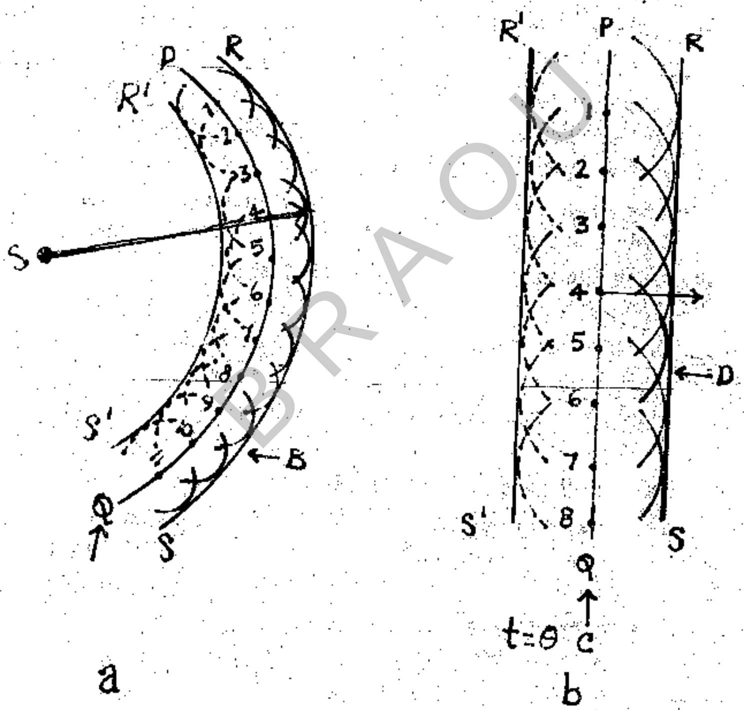


Fig. 13.1 (a) Spherical wave front (b) Cylindrical wave front

i) Spherical Wave front : The spherical wave front is a portion of the spherical surface on which all the particles are in the same phase.

ii) Cylindrical wave front: When the source of light is linear, then in an isotropic medium the wave front takes the cylindrical shape. Every particle on such a cylindrical surface will be in the same phase.

iii) Plane wave front : A plane wave front is a plane surface through which a wave disturbance passes and in which all the vibrating particle are in the same phase. A portion of the spherical wave front or cylindrical wave front which is at an infinite distance from the source of light is a plane wavefront.

Huygens' principle provides a geometrical method of finding the shape and position of the wave front at a certain instant of time 't', if we know its shape and position at any other previous instant. According to Huygens' principle all points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time 't', the new position of the wavefront will be the surface of tangency to the secondary wavelets.

Let us consider a wavefront originating from source 'S' as shown in Fig. 13.1 It reaches the position PQ at t=0. According to Huygens' principle each particle on the wave front behaves as a secondary source from which spherical wavelets spread out. After a time 't', there will be a series of wavelets with centres at 1, 2, 3, 4,......... ..each of radius ct, where 'c, is the velocity of the wave disturbance. The surface which envelops all these wavelets, that is the plane of tangency to the spheres, constitutes a new wave front given by RS. From the examination of the Fig. 13.1a and 13.1b it is evident that the spherical wave front is propagated as a spherical and plane wave fronts are propagated as plane with speed 'c'. It should be noted that Huygens method involves three dimensional construction and Fig. 13.1 is the intersection of this construction with the plane of the page.

Huygens' principle assumes that a wavelet is effective only at the small position which is tangential to the enveloping surface. A ray may thus be defined as the line joining the centre of a wavelet to the small portion as shown in Fig. 13.1 We may also expect a wave radiated backward say $R^1 S^1$ apart from the ones radiated forward as shown in 13.1a. This result may be avoided by assuming that the intensity of spherical wavelets is not uniform in all directions but varies continuously from a maximum in the forward direction to a minimum of zero in the backward direction. According to Voigt and Kirchoff the contribution of a walvelet in any given direction, making an angle θ with OA, is porpotional to $(1+\text{Cos}\theta)/2$. Thus in the forward direction $\theta=0$ and the intensity is the maximum and in the backward direction $\theta = 180°$ and the intensity is zero. Huygen's principle can be applied to all wave phenomena. Let us study of the use 'Huygens' principle to explain the phenomena of reflection and refraction.

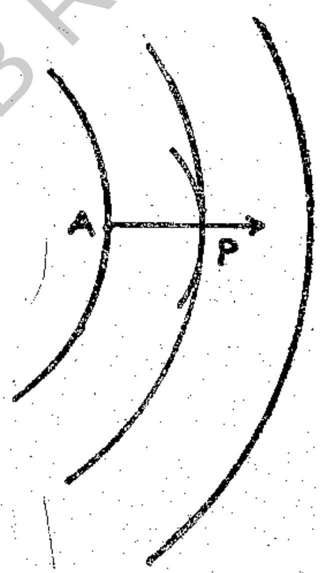


Fig. 13.2

13.3.1 Huygen's Principle - Laws of Reflection

We shall now study the phenomenon of reflection and deduce the laws of reflection by applying Huygens' principle. Consider a plane reflecting surface MM' as shown in Fig.13.3 Let AB be a plane wave front striking the surface MM at A at t = 0. Let c be the velocity of light and t be the time taken for the edge B of the wavefront to reach the point C. In the absence of the reflecting surface MM′ the incident wavefront AB would have reached the position CD. But in the presence of MM′, as the wave front advances, the particles between A and C behave as sources of secondary wavelets. By applying Huygens' principle we can construct the reflected wavefront after a time 't'.

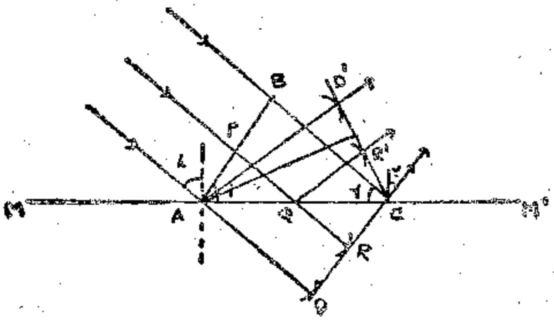


Fig. 13.3

The point of the wave front which touches the surface MM′ first becomes the source of the secondary wavelet. This spreads out in the surrounding medium. By the time the disturbance from the point B reaches C in a time 't' the secondary wavelet from the point A would attain a radius BC=ct. This position can be found by taking the point A as the centre and drawing a sphere of radius ct. Let CD be the the tangent at D to the wavlet from a A then CD^1 represents the reflected wavefront after a time t.

Let us consider a point P on AB. Let PQR be parallel to BC. Let QR and QR′ represent the prependiculars drawn from Q to CD and CD′ respectively. Triangles QCR′ and ACD′ are similar triangles since $\Delta AR'C = \Delta AD'C = 90^o$ and angle QCR′ = Angle ACD′. Hence

$$\frac{QC}{AC} = \frac{QR'}{AD'} \tag{13.1}$$

Also triangles QCR and ACD are similar triangles since angle QRC = angle ADC=90° and QCR = ACD. Therefore

$$\frac{QC}{AC} = \frac{QR}{AD} \tag{13.2}$$

comparing equation 13.1 and 13.2 we get

$$\frac{QR'}{AD'} = \frac{QR}{AD} \tag{13.3}$$

SINCE AD′ = ct = AD = BC we have QR′ = QR (13.4)

Equation 19.4 indicates that the wavelet from Q touches the wavefront CD′ at R′. This is true for all the wavelets emanating from any point between A and C. Hence CD′ represents the reflected wave front.

Let us consider the triangles ABC and AD′C. Here AC is common. BC = AD′ and $\Delta AD'C = \Delta ABC = 90^o$. Therefore triangles ABC and AD^1C are congruent.

Hence angle BAC = andgle ACD′ (13.5)

That angle i = angle R (13.6)

Hence the angle of incidence is equal to the angle of reflection.

13.3.2 Huygen's Principle - Laws of Refrection

Let us consider a plane refracting surface MM' separating the two transprent media (air and glass) as shown in Fig. 19.4. Let c_1 and c_2 represent the speed of light in air and glass respectively. Let APB be the incident plane wavefront. At t = 0 let the wavefront APB meet MM' at A Thus A becomes the source of the secondary wavelet at t = 0. Let t' be the time taken for the edge B of the wavefront to reach the point C. Hence BC = c_1t. In the absence of glass the wavefront APB would have advanced to the position CRD at time t. But in the presence of the refracting surface, as the wave front advances the particles on AC become the sources of secondary wavelets one after the other.

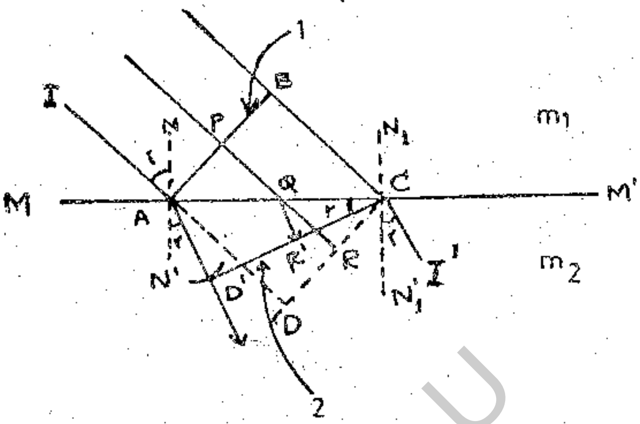


Fig.13.4

At time 't', when the wavelet from C is about to start, the wavelet from A in medium 2 would have developed a radius AD' which will be equal to c_2t. Let the tangent drawn from C to the wavelet developed from A be CD'. Then CD' repesents the refracted wavefront. To prove this let PQR be parallel to BC with the point Q on the refracting surface MM'. Let QR' represent the perpendiculars drawn from Q to CD' and CD respectively. Triangle QCR' and triangle ACD' are similar triangle since

$\Delta QR'C = \Delta AD'C = 90^\circ$. Angle QCR' = angle ACD'. Then

$$\frac{QC}{AC} = \frac{QR'}{AD'} \tag{13.7}$$

Triangle QCR and triangle ACD are also similar triangles since angle QCR=angle ADC=90° and angle QCR = angle ACD. Hence

$$\frac{QC}{AC} = \frac{QR}{AD} \tag{13.8}$$

Comparing equation 13.7 and 13.8 we get

$$\frac{QR'}{AD'} = \frac{QR}{AD} \tag{13.9}$$

$$\therefore QR' = \frac{AD'}{AD}.QR \tag{13.10}$$

$$\text{or } \frac{AD}{AD'} = \frac{QR}{QR'} = \frac{c_1 t}{c_2 t} = \frac{c_1}{c_2} \tag{13.11}$$

Hence AD' is the radius of the secondary wavelet from A then QR' represents the radius of the secondary wavelet from Q. That is the wavelet from Q trouches CD' at R'. This is true for all the wavelets starting from any point on AC. Thus CD' is the commor envelope of all the wavelets from all the points on AC. Hence, by Huygen's principle, CD′ represents the refracted wavefront. From the figure 13.4

Angle IAN + Angle IAB = 90 = Angle NAB + Angle BAC (13.12)

Hence Angle IAN = i =Angle BAC (13.13)

Similarly

Angle N′, CI′ + Angle N′,CR′ = 90=Angle N_1' CR′ + R′CQ (13.14)

gr r = Angle R′CQ = Angle ACD′ (13.15)

From triangles ABC and triangle ACD′

$$\frac{\text{Sin i}}{\text{Sin r}} = \frac{\text{Sin(Angle BAC)}}{\text{Sin (Angle ACD')}} = \frac{BC/AC}{AD'/AC} = BC/AD' \quad (13.15)$$

$$\frac{\text{Sin i}}{\text{Sin r}} = \frac{C_1 t}{c_2 t} = c_1/c_2 = \mu_{21} \quad (13.15)$$

Equation 13.15c represents the law of refraction. Here μ_{21} represents the refractive index of medium 2 with respect to medim 1.

13.4 TOTAL INTERNAL REFLECTION - APPLICATIONS

The phenomenon of total internal reflection can be observed only when light passes from a denser medium to a rarer medium. Consider a source of light situated at O with a denser medium as shown in Fig. 13.5

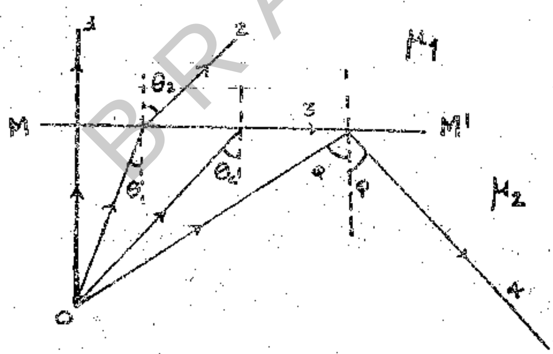


Fig. 13.5

Let μ_1 and μ_2 represent the refractive indices of the denser and rarer media respectively. When a light ray is normally incident to the surface it travels in the same direction. This is shown as ray 1 in the figure. If the incident ray makes an angle θ_1 then the refracted ray (ray 2) moves away from the normal making an angle θ_2 . According to the law of refraction.

$$\frac{\text{Sin}\,\theta_2}{\text{Sin}\,\theta_1} = \frac{\mu_2}{\mu_1} \quad (13.16)$$

Since μ_2 is greater than μ_1 the angle of refraction θ_2 will be great than the angle of incidence θ_1 . As the angle of incidence θ_1 is increased a situation may be obtained at which the refracted ray points along the surface, (ray 3), the angle of refraction being 90°. The angle of incidence for which the refracted ray grazes the surface is called the critical angle θ_c. For angles of incidence greater than θ_c no refracted ray occurs and the rays get reflected back into medium. This phenomenon is called total internal reflection.

Applying Snell's law, we have

$$\frac{\text{Sin }90^{\circ}}{\text{Sin }\theta_c} = \frac{\mu_2}{\mu_1} \quad 1/\text{Sin }\theta_c = \mu_2/\mu_1 \qquad (13.17)$$

For glass and air since $\because \text{Sin }\theta = 1.00/1.50 = 0.667$

$$\theta_c = \text{Sin}^{-1}\, 0.667 = 41.8^{\circ}.$$

Check your Progress 1

What are the necessary conditions for the total internal reflection to occur?

Hence for glass air interface total internal reflection takes place when the light ray is incident only at an angle greater than 41.8°.

The phenomenon of total internal reflection has many applications. Light can be pipied from one point to another with little loss by allowing it to enter at one end of a glass rod or acrylic plastic at an angle greater than the critical angle. The light will undergo total internal reflection at the boundary of the rod and will follow its contour emerging at its far end. Light can be made to emerge even if the rod is bent as shown in Fig. 13.6

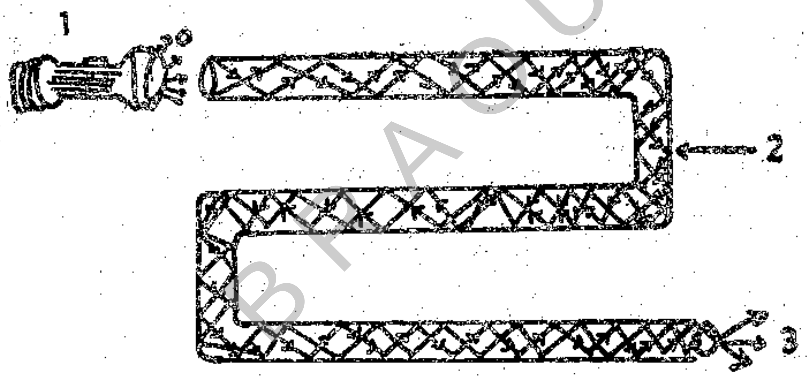


Fig.13.6

Thus the specially made glass rods can be used as light guides. These rods are useful only to transmit light from one point to the other. But image can be transfered from one location to the other using a bundle of fine glass fibles as shown in Fig. 13.7 each fibre transmitting a small fraction of the image.

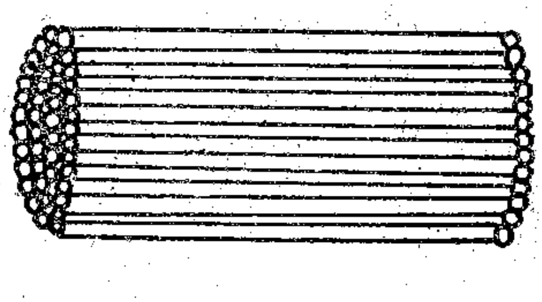


Fig. 13.7

If the radius of the glass fibre is less, then the image transmitted will bé well defined. The instrument designed using optical fibres for transmission of images is called a fibroscope. The gastroscope used to see the minor parts of the stomach, and the hypodermic needle used to examine skin tissues and muscle fibres, belong to the family of fibroscopes. A fibroscope is also employed to examine the inaccessible parts of a machine, to test the turbine blades, boiler tubes and also to examine the various parts of nuclear reactors.

The phenomenon of total internal reflection is utilized in developing total reflection prisms which are useful in optical instruments. A few types are shown in Fig.13.8.

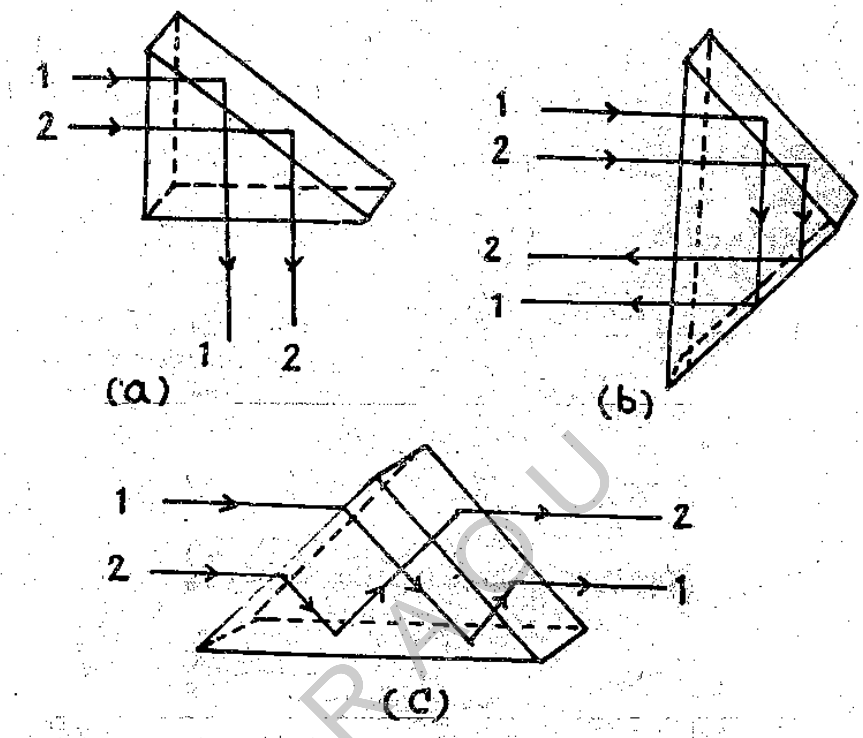


Fig. 13.8

The appearance of the outside world for a fish in water can be found by making use of the phenomenon of total internal reflection. As shown in Fig.13.9 a swimmer inside the water will appear to float on the upper surface of the water, and a tree at the shore looke as if it is floating in the sky (to the fish).

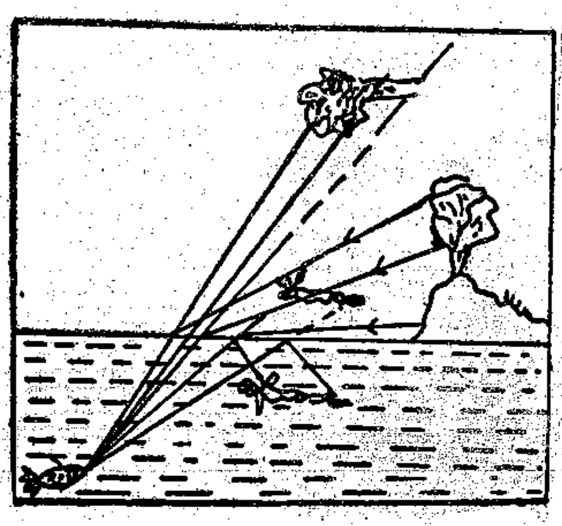


Fig. 13.9

Similarly, the rising and setting sun appear to be little above in the sky. The appearance of mirages in deserts is only due to the phenomenon of total internal reflection. Due to the high temparature the air

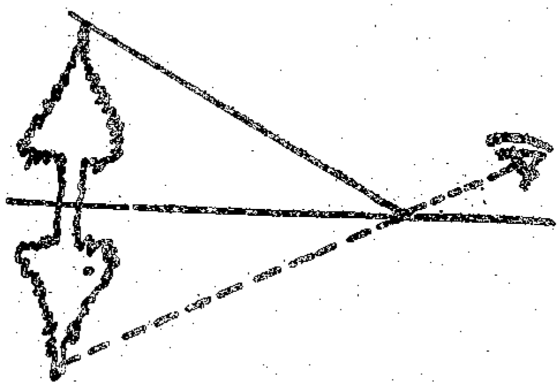


Fig.13.10

near the surface of the earth gets heated up and hence forms a rarer medium. As a result the rays from the trees get totally internally reflected shown in Fig.13.10 resulting in the formation of inverted images if seen by a distant observer. Hence the observer experiences the illusion of water in deserts i.e., the presence of mirages.

13.5 PHENOMENON OF INTERFERENCE

The Light waves travelling along the x-direction may be represented by

$$Y = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \qquad (13.18)$$

or $$Y = A \sin 2\pi\left(\frac{x}{\lambda} - V t\right)\left(\because \upsilon = \frac{1}{T}\right) \qquad (13.19)$$

or $$Y = A \, \mathrm{Sin}(kx - \omega t) \qquad (13.20)$$

Where $k = 2\pi/\lambda$ and $\omega/ = 2\pi\upsilon$

In the above equations $A, \lambda, T, x, \upsilon, \omega$ and k represent the amplitude, wavelength of light, wave period, wave position, frequency, angular frequency and wave vector respectively.

The intensity of light at any point is propotional to the square of the amplitude of the wave at that point. The term $2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$ represents the phase angle.

Before going into the details, let us analyse the effects that take place when two or more waves arrive simultaneously at a particular point.

Let us consider two waves of equal frequency and amplitude travelling with the same speed along x direction. Let the phase difference between the two waves be φ . Then the two waves can be represented by the following equation:

$$y_1 = y_m \text{Sin}(kx - \omega t - \varphi) \tag{13.21}$$

$$y_2 = y_m \text{Sin}(kx - \omega t) \tag{13.22}$$

According to the principle of superposition the resultant displacement at any point, and at any instant of time due to different wave trains is given by the algebric sum of the instantaneous displacements that the individual wave trains would produce independently.

The resultant wave is given by

$$y = y_1 + y_2 = y_m(\text{Sin}(kx - \omega t - \varphi) + \text{Sin}(kx - \omega t) \tag{13.23}$$

using the trignometric relation

$$\text{Sin A} + \text{SinB} = 2\,\text{Sin}\left(\frac{A+B}{2}\right).\text{Cos}\left(\frac{A-B}{2}\right) \tag{13.24}$$

we can rewrite Eq.13.23

$$y = y_m[2\,\text{Sin}(kx - \omega - \varphi/2).\text{Cos}(\varphi/2) \tag{13.25}$$

or $$y = (2y_m \text{Cos}\varphi/2)\,\text{Sin}(kx - \omega t - \phi/2) \tag{13.26}$$

The resultant wave corresponds to a new wave having the same frequency but with an amplitude $2y_m \text{Cos}\,\varphi/2$. If φ =0 the resultant amplitude will be $2y_m$ and the two waves have the same phase everywhere. The crest of one wave coincides with the crest of the other and the trough of one wave coincides with the trough of the other wave. The waves are then said to interfere constructively. If $\varphi = 180^\circ$ the resultant amplitude will be zero and the crest of one coincides with the trough of the other. The waves are then said to be interfering destructively. The superposition of two wave trains are when $\phi = \theta$ (Fig.13.11(a), ϕ =0, (Fig.13.11(b)) $\phi = 2\pi$,(Fig.13.11(c)), $\phi \cong 180$, (Fig.13.11(d)), $\phi = \pi$ shown in Fig.13.11

An examination of the Fig.13.11 clearly shows that a stable interference pattern can be observed if we have two sources of light waves having the same frequency, nearly the same amplitude and possessing constant phase difference at any instant of time. Such sources are called coherent sources.

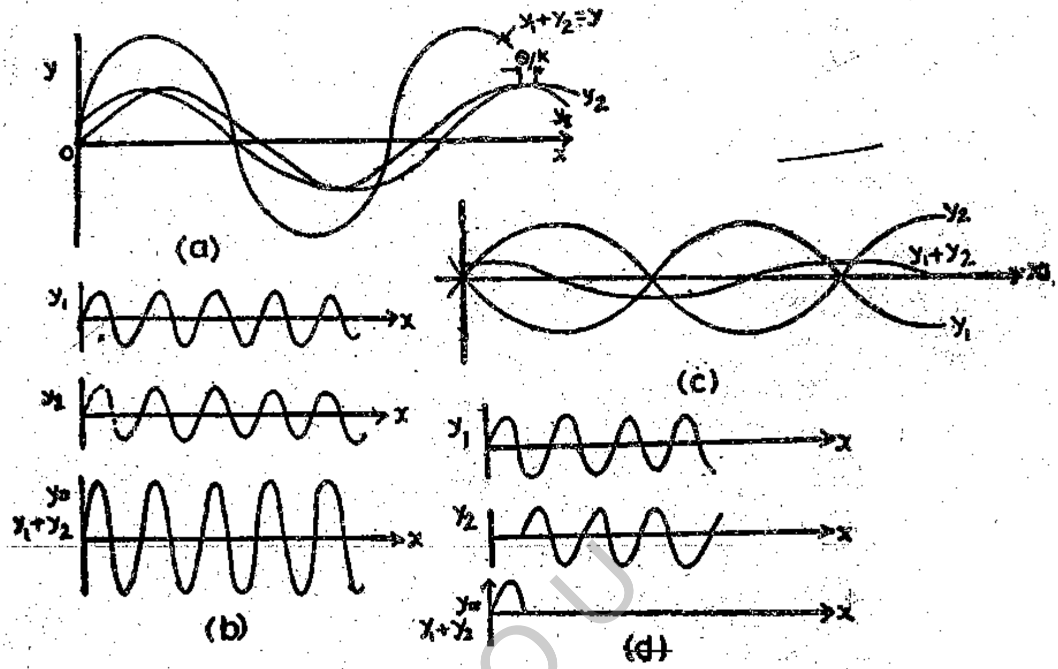


Fig.13.11

Interference effects can be observed on a screen from two identical wave trains which travel in two different directions as shown in Fig.13.12

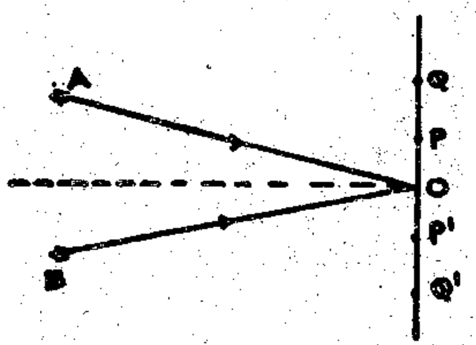


Fig.13.12

Here A and B are two identical light sources emitting light waves. Since OA is equal to OB the path difference between the two waves arriving at O is equal to zero and hence the intensity at O will be maximum. As one moves from O to either direction along the screen there exists a path difference between the two waves and at some point P or P′ the path difference would be $\lambda/2$. Hence the phase difference would be π. Here the two waves interfere destructively and P and P′ would represent the points of complete darkness.

At Q or Q' if the path difference is λ then the phase difference between the waves arriving at Q and Q' would be 2π. Hence the waves interfere constructively and have Q and Q' as bring points. When the path difference is 0, λ, 2λ, 3λ .... the two waves interfere constructively and when the path difference is $\lambda/2$, $3\lambda/2$, $5\lambda/2$ ...... the two waves interfore destructively. Thus on either sides of O on the screen the bright and dark points will form alternately and these bands are termed as interference bands and the phenomenon is called interference.

13.6 YOUNG'S EXPERIMENT AND ITS QUALITATIVE TREATMENT

Interference effects due to light waves was first demonstrated experimentally by Thomas Young in 1801. This experiment was a crucial one at that time since it added evidence to the growing belief in the wave theory of light.

The experimental set up of Young is shown in Fig.13.13

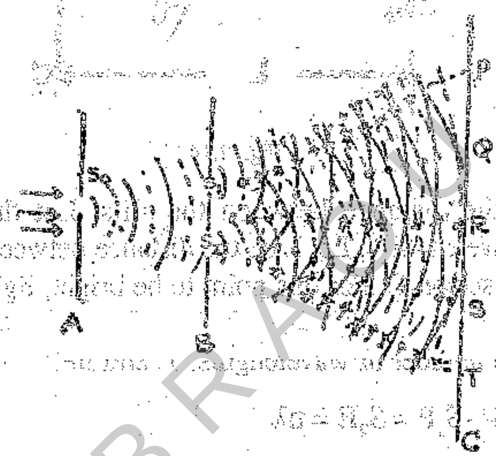


Fig.13.13

Here sunlight was allowed to fall on a pin hole punched in a screen A. The emerging light spreads out and fall on pin holes S_1 and S_2 of screen B. The two waves originating from S_1 and S_2 and the interference pattern was observed on the screen C. Young observed a few coloured bright and dark bands on the screen. By replacing sunlight by monochromatic light and using slits instead of pin holes S_1 and S_2 one can observe the interference pattern of equally spaced bright and dark bands.

In Young's experiment the pin holes do not cast geometrical shadows but act as sources of expading Huygen's wavelets. According to Huygens' principle S_0 acts as a point source so that a spherical wave emerges and falls on S_1 and S_2. These two pinholes act as point sources emitting spherical waves which spread out as shown in Fig.13.13. Since S_1 and S_2 are illuminated by the same spherical wave and S_1 and S_2 are equidistant from S_0 the light wave emerging from S_1 and S_2 are coherent that is they have the same frequency and amplitude and have constant phase difference with each other at any instant of time. The intensity at any point depends on their relative phases at that point. Interference bands are produced on the screen C which are alternatively bright and dark. The points such as T are bright because the crest or trough of one wave coincides respectaively with the crest or trough of the other leading to constructive interference. The points such as S are dark because the crest or trough of one falls respectively on the trough or crest of the other and destructive interference takes place. If any one of the slits say S_1 or S_2 is covered, the dark lines disappear and the screen is illuminated in a broad band. The corpuscular theory cannot account for the fact that a point on the screen is bright only when one slit is exposed and becomes dark when both slits are exposed. The explanation can be offered only in terms of the wave theory of light.

The phenomenon of interference is not limited to light waves alone but is characteristic of all wave phenomena. In the ripple tank the waves are generated by two vibrator that tap the water surface in synchronism, producing two expanding spherical waves.

Let us analyse Young's experiment quantitatively. As shown in Fig.13.14

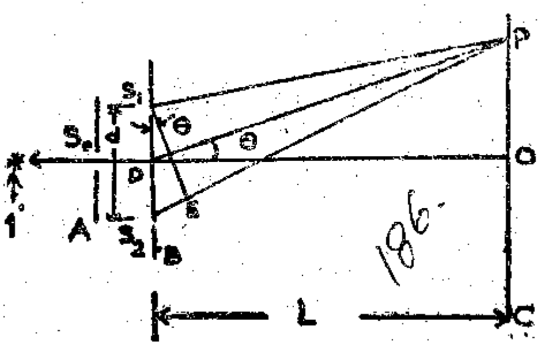


Fig.13.14

Let 'd' represent the distance between the slits. Let the source of light be monochromatic having wavelength λ. Let L be the distance between the screens B and C. Consider a point P on the screen C. For this point to be bright, light reaching it from the two slits must be in phase. Hence, the light path from S_2 to P must exceed the light path from S_1 to P by an integral number of wavelengths. Therefore,

$$S_2P - S_1P = S_2E = n\lambda \qquad (13.27)$$

Where 'n' is an integer. It can take values 0,1,2.....S_2E represents the path difference. If the distance between the slits 'd' is very small as compared to the distance between the screens B and C i.e.,L then S_1E will be almost perpendicular to S_2P. Hence angle S_2S_1E=angle PDO, both marked as θ in the Fig.13.14.From triangle S_1S_2E we have

$$\frac{S_2E}{S_1S_2} = \sin\theta \qquad (13.28)$$

$$S_2E = S_1S_2 \sin\theta = d \sin\theta \qquad (13.29)$$

from the triangle PDO, $\tan\theta = OP/OD = x/L$ (13.30)

If θ is very small, we can take $\tan\theta \cong \sin\theta$

$$\therefore \sin\theta = x/L \qquad (13.31)$$

Here 'x' represents the distance of the point P from the central image. From Equation 13.29 and 13.31 we have

$$S_2E = dx/L \qquad (13.32)$$

Bright Fringes : Bright fringes result whenever the waves interfere constructively. For constructive interference to take place the path difference must be an integral multiple wave length λ. Thus if P were to be bright, then

$$xd/L = n\lambda \tag{13.33}$$

$$\text{or } x = n\lambda\, L/d \tag{13.34}$$

$$\text{If } n=1,\ x\text{-}= \lambda L/d \tag{13.35}$$

$$\text{If } n=2,\ x_2\text{-}= \lambda 2L/d \tag{13.36}$$

$$\text{If } n=3, x_3\text{-}= \lambda 3L/d \tag{13.37}$$

The difference between any two consecutive bright fringes.

$$x_n - x_{n-1} = \frac{n\lambda L}{d} - \frac{(n-1)\lambda L}{d} = \frac{\lambda L}{d} \tag{13.38}$$

Dark Fringes : Dark Fringes results whenever the waves interfere destructively. For destructive interference to take place the path difference must be an odd integral multiple of half wave length.

Hence if the point P were to be dark then

$$x = (2n+1)\frac{\lambda}{2}.\frac{L}{d} \tag{13.39}$$

where n=0,1,2,...... (13.40)

when $$\frac{xd}{L} = (2n+1)\frac{\lambda}{2}$$

$$n=0,\ x_o = \lambda\, L/2d \tag{13.41}$$

$$n=1,\ x_1 = 3\lambda\, L/2d \tag{13.42}$$

$$n=2,\ x_2 = 5\lambda\, L/2d \tag{13.43}$$

The distance between any two consecutive dark fringes is given by

$$x_n - x_{n-1} = \frac{(2n+1)\lambda L}{2d} - \frac{(2n-1)\lambda L}{2d} = \frac{\lambda L}{d}$$

The distance between any two consecutive bright or dark fringes is known as fringe width (δ). It is given by

$$\delta = \lambda L/d \tag{13.45}$$

Since λ is small, L/d must be large so that the bands can be observed distinctively.

Eq.13.45 indicates that the fringe width is directly proportional to (i) the wave length and (ii) the distance between the screen and the two sources, and inversely proportional to the distance between the two sources.

Check Your Progress 1

Define fringe width

Check Your Progress 2

If the path difference between the interfering waves is an odd integral multiple of half wave length do we get a constructive interference pattern.

Worked out Examples

1. Calculate the fringe width, If the source is Sodium light emitting yellow line of wavelength λ=5893 A° the distance between the slit is 0.05cm and the screen is at a distance of 2.50 metres from the slits S_1 and S_2

$$\delta = \lambda L / d = \frac{5893x10^{-8}x250}{0.05} = 2.95mm.$$

2. In Young's double slit experiment the separation of the slits is 2.00 m.m. The fringe spacing is 0.30m.m. at a distance of 1 metre from the slits. Calculate the wavelength of light.

$$\delta = 0.30mm=0.03\ Cm.;\ d=2.00\ mm=0.2\ Cm;\ L=1\ meter=100\ cm.$$

$$\text{since } \delta = \lambda L / d, \text{ then } \lambda = \delta d/L = \frac{0.03x0.2}{100} = 6000A^{\circ}$$

The wavelength of the light source is 6000 A°

Young's experiment indicated that to observe interference effects it is necessary to have two sources emitting light waves of the same frequency and amplitude which, at the start, are alway in phase. This can never be accomplished, for if two sources are in phase at one moment they would shortly get out of phase. The best way of obtaining two coherent sources having the same frequency, amplitude and phase is to start with only one light beam and to split this wave into two as was done by Young. At present laser sources are available which are highly monochromatic and intense. Using laser light one can obtain sharp interference patterns.

Instead of pin holes if slits are used more light passes through and the interference pattern is bright. The general nature of the interference pattern obtainable in Young's experiment is shown in fig.13.15.

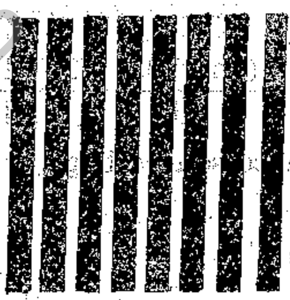


Fig. 13.15

13.6.1. Intensity of Interference Pattern in Young's Experiment

We know light waves are electromagnetic in nature. The electric $\vec{E}$ and magnegic field($\vec{H}$) vectors will be perpendicular to each other and also perpendicular to the direction of propagation. The magnitude of the electric and magnetic field vectors varies with time. The intensity of light waves at any point at any instant of time depends on the instantaneous value of the electric fields vector. At any instant of time to let the electric field components of the two waves at P as shown Fig.13.14 be given by

$$E_1 = E_o \sin \omega t \qquad (13.46)$$

and $$E_2 = E_o \sin (\omega t + \Phi) \qquad (13.47)$$

Hence ω represents the angular frequency and φ represents the phase difference between them : φ depends on P and for fixed geometrical arrangement it depends on the angle. The electric field component of the resultant wave at P can be given, according to the principle of superposition, as

$$E=E_1 + E_2 \qquad (13.48)$$

$$E = E_o \sin \omega t + E_o \sin (\omega t + \Phi) \qquad (13.49)$$

$$E = E_o \sin \omega t + E_o \sin \omega t \cos \phi + E_o \cos \omega t \sin \phi \qquad (13.50)$$

$$E = E_o \sin \omega t (1+\cos \phi) + E_o \cos \omega t \sin \phi \qquad (13.51)$$

Let $E_o \sin \phi = R \sin \beta$ (13.52)

and $E_o(1+\cos\phi) = R \cos \beta$ (13.53)

Then

$$E = R \sin \omega t \cos \beta + R \cos \omega t \sin \beta \qquad (13.54)$$

or $E = R \sin (\omega t + \beta)$ (13.55)

Equation 13.55 represents the equation of simple harmonic vibration of amplitude R. The value of R can be obtained by squaring equation 13.52 and 13.53 and adding the sum.

Hence:

$$R^2 \cos^2 \beta + R^2 \sin^2 \beta = E_o^2 (1+\cos\varnothing)^2 + E_o^2 \sin^2\varnothing \qquad (13.56)$$

$$R^2 = E_o^2(1+\cos^2\varnothing + 2\cos\varnothing + \sin^2\varnothing) \qquad (13.57)$$

$$R^2 = 2E_o^2 (1+\cos\varnothing) \qquad (13.58)$$

or $R^2 = 4E_o^2 \cos^2\varnothing/2$ (13.59)

The intensity at P is given by the square of the amplitude, we have

$$I = R^2 = 4E_o^2 \cos^2\varnothing/2 \qquad (13.60)$$

We also can show that $\beta = \phi/2$ and having and hence $I = R^2 = 4E_o^2 \times \cos^2\beta$

Let us analyse equation 13.60 for different cases,

CASE - 1. When $\phi = 0, 2\pi, 2(2\pi), 3(2\pi)........n(2\pi)$

the path difference is $0, \lambda, 2\lambda, 3\lambda, ... n\lambda$

we have $I = 4E_o^2$ (13.61)

The intensity is the maximum when the path difference is an integral multiple of λ.

CASE - 2 When $\phi = \pi, 3\pi, 5\pi, ... (2n+1)\pi$ or the path difference is $\lambda/2, 3\lambda/2, 5\lambda/2........(2n+1)\lambda/2$

we have $I = 0$ (13.62)

Hence the intensity is the minimum when the path difference is an odd number and a multiple of half wave length

The horizontal solid line I_o, describes the intensity pattern on the screen if one of the slits is coverec up. If the two sources were incoherent, then the intensity would be uniform over the screen and would be $2I_o$ as shown in the Fig. 13.16. For the coherent source, as in Young's experiment the energy is

The intensity pattern for the double slit interference is shown in Fig. 13.16.

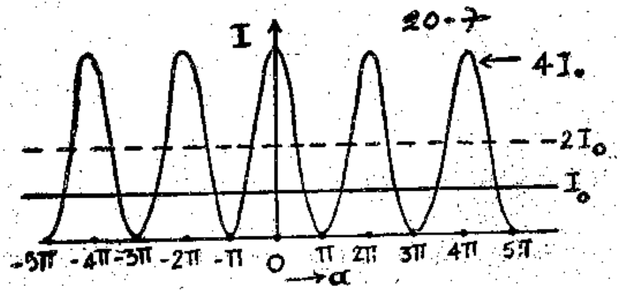


Fig. 13.16

redistributed over the screen. Since energy can neigher be created nor destroyed, the energy is transfered from the points of minimum intensity to the points of maximum intensity. As shown in Fig. 13.16 the intensity varies from 0 to $4E_o^2$ and the average value is still $2E_o^2$ corresponding to the intensity present in the abscence of the interference phenomenon.

13.7 INTERFERENCE DUE TO THIN FILMS

Brilliant colours are often seen when light is reflected from a soap bubler or from a thin layer of oil floating on water. These colours are produced because of interference effects between the two trains of light waves reflected at opposite surface of the thin films. The intensity of the colours depends on the refractive index of the film, thickness of the film and the angle of refraction.

Let us analyse the interference effect in reflected light. Let MM_1 and $M'M_1'$ represent the two surfaces of a transparent film of uniform thickness 't' and refractive index 'μ' as shown in Fig. 13.17.

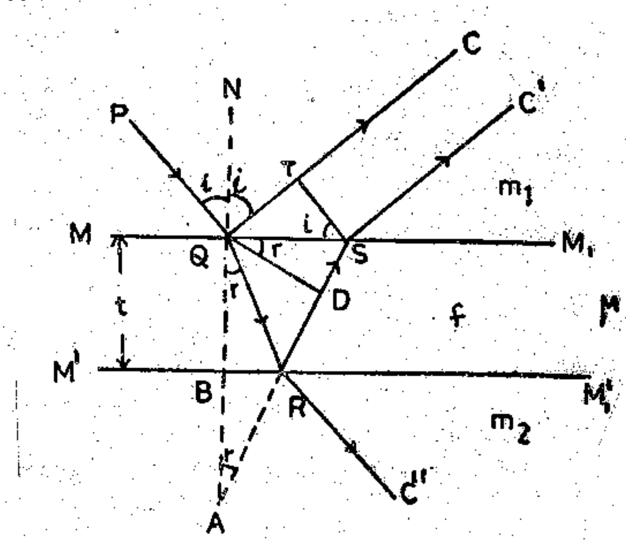


Fig 13.17

Let a ray PQ of monochromatic light of wavelength λ be incident on the upper surface of the film. This ray is party reflected along QC and partly refracted along QR. At R, internal reflection takes place and we have the ray RS. RS gets refracted resulting in the ray SC′ in air. The rays SC′ and QC are paralel to each other. We have to find the effective path difference between the rays QC and SC′ to obtain the conditions required for interference effects to take place. To facilitate this, let us draw a normal ST on QC and another normal QD on RS. Let us extend SR to meet QB at A.

As per the Fig 13.17 the angle of incidence i=angle PQN and the angle of refraction r=angle BQR.

$$\text{angle NQT} + \text{angle TQS} = 90^{\circ} = \text{angle TQS} + \text{angle TSQ} \tag{13.63}$$

$$\therefore \text{angle NQT} = \text{angle PQN} = \text{angle QST} = i \tag{13.64}$$

traingle QBR and ABR are congruent

since BR is common, QR=RS=RA and hence QB=BA. Hence

$$\text{angle BQR} = \text{angle BAR} = r \tag{13.65}$$

The optical path difference between the reflected rays QC and SC′ is given by

$$\Delta = \text{path (QR+RS) in the film - Path QT in air} \tag{13.66}$$

or

$$\Delta = \mu(QR + RS) - QT \tag{13.67}$$

We know that the refractive index μ is given by

$$\mu = \frac{\text{Sin i}}{\text{Sin r}} = \frac{QT}{QS} / \frac{DS}{Qs} \tag{13.68}$$

or

$$\mu = QT/DS \tag{13.69}$$

$$\therefore QT = \mu\, DS \tag{13.70}$$

using equation 13.70 in equation 13.67 we get

$$\Delta = \mu(QR + RS) - \mu\, DS = \mu\,(QR + RS - DS) \tag{13.71}$$

$$\Delta = \mu\,(QR + RD + DS - DS) = \mu(QR + RD) \tag{13.72}$$

since QR = AR, we get

$$\Delta = \mu\,(AR + RD) - \mu\, AD \tag{13.73}$$

From the right angled triangle QAD (13.74)

$$\text{Cos r} = AD/AQ = AD/(AB+BQ) = AD/2t$$

$$\therefore AD = 2t\, \text{Cos r}. \tag{13.75}$$

substituting the value of AD from equation 13.75 in equation 13.73 we get,

$$\Delta = 2\mu\, t\, \text{Cos r} \tag{13.76}$$

Equation 13.76 only represents the apparent path difference. According to the electromagnetic theory when light is reflected from the surface of an optically denser medium a phase change π equivalent to a path difference of $\lambda/2$occurs. Hence the correct **path** difference is given by

$$\Delta = 2\,\mu t \cos\, r = \lambda/2 \qquad (13.77)$$

Constructive interference takes place whenever the path difference is an integral multiple of λ.

Hence $\quad \Delta = 2\mu t \cos r \pm \lambda/2 = n\lambda$ where $n = 0, 1, 2$ $\qquad (13.78)$

or $\quad 2\,\mu t \cos r = (2n \pm 1)\,\lambda/2 \qquad (13.79)$

When the above condition is satisfied the film appears bright.

If the path difference is an odd integral multiple of $\lambda/2$ then destructive interference takes place and the film appears dark.

$$\therefore 2\,\mu t \cos r \pm \lambda/2 = (2n+1)\,\lambda/2 \qquad (13.80)$$

or $\quad 2\,\mu t \cos r = n\,\lambda \qquad (13.81)$

Equation 13.79 represents the condition for destructive interference.

The interference pattern will not generally be perfect because the intensities of the rays QC and SC′ will not be the same and their amplitudes are also different. The amplitudes depend on the amount of light reflected and transmitted through the film. If the film thickness is not uniform, say the film is wedge-shaped, then constructive interference will occur in certain parts of the film and destructive interference occurs in other parts. Lines of maximum and minimum intensity will appear. If the film is illuminated with white light, the light reflected from various parts of the film will be modified by various constructive of destructive interferences that occur and hence we observe the brilliant colours of soap bubbles and oil slicks.

Example 3 :

A water film of refractive index 1.33 is in the air and has a thickness of 3000 A°. It is illuminated by white light at normal incidence. Determine the colour of the reflected light.

For maximum intensity.

$$2\,\mu t \cos r = (2n \pm 1)\,\lambda/2 = (2n+1)\lambda/2 \text{ say.}$$

$$\lambda = \frac{2\mu t \cos r}{\left(n+\frac{1}{2}\right)} = \frac{2\mu t}{\left(n+\frac{1}{2}\right)} \qquad (13.82)$$

for normal incidence since r = 0°

using $\mu = 1.33$, t=3000 A°

$$\lambda = \frac{2(1.33)\text{x}3000}{\left(n+\frac{1}{2}\right)} \qquad (13.83)$$

$$\lambda = 7980/\left(n+\frac{1}{2}\right) \qquad (13.84)$$

For minimum intensity.

$$\lambda = 7980/n \qquad (13.85)$$

* The maximum and minimum occur thus for the following wavelengths.

n	0(max)	1 (min.)	1 (max)	2 (min.)	2 (max)
λ (A°)	15960	7980	5320	3990	3192

The maximum corresponding to n=1 lies with the visible region. This corresponds to the green colour.

Check your Progress 3.

Do you think colours reflected from a soap buble is an example of interference effect?

13.8 SUMMARY

The propagation of light was explained on the basis of corpuscular theory and wave theory. Light has dual character. According to Huygens principle all points on a wave front acts as point sources for the production of secondary wavelets. The phenomenon of total internal reflection can be observed when light travels from a denser medim to a rarer medium.

The phenomenon of interference of light waves was first explained by Young. Two identical light waves having the same frequency, amplitude and phase travelling from different direction, interfere constructively if the path difference between the two waves is an integral multiple of wavelength, and interfere distructively if the path difference is the odd integral multiple of half the wavelength.

In young double slit experiment the fringe width (δ) is given by $\lambda L/d$, where λ is the wavelength of light L is the distance between the screen and the slit and d is the distance between the slits.

(b) The intensity of the interference pattern varies as $\text{Cos}^2\ \phi/2$ where ϕ is the phase angle.

The brillant colours observed in the thin films are due to the interference effects between the light waves reflected on opposite surface of thin films.

13.9 MODEL ANSWERS

Check your Progress 1

The conditions for total internal reflection to occur are

1. light should pass from a denser medium to rarer medium.
2. The angle of incidence in the denser medium should be greater than the critical angle of the denser medim for the two media under consideration.

Check your Progress 2

The distance between two consecutive bright or dark fringe is called fringe width (δ)

Check your Progress 3

No. If the path difference between the interfering waves is an integral multiple of wave length we get constructive interfeence pattern.

Check your Progress 4

Yes - They are due to the interference pattern produced by a thin a film when white light is used.

13.10 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following in About 30 Lines.

1. Explain the conditions under which the phenomenon of total internal reflection can be observed. Describe a few examples of total internal reflection.

2. Explain the term interference. Describe youngs experiment and derive the condition under which interference pattern can be observed.

obtain an expression for the intensity of interference pattern in young's experiment. What conclusions do you draw from these expressions.

II. Answer the Following Questions in About 10 Lines?

1. Show that wave propogation can be in plane. spherical and cylindercal wave forms.
2. Explain the phenomenon of reflection based on the Huygen's wave theory of light.
3. Explain the phenomenon of reflection based on Hygens construction of wave fronts.
4. Discuss the phenomenon of interference due to reflected light from a thin film.

III. Solve the Folowing Problems.

1. Determine the critical angle when light passes from glass to water. μ glass=1.5 μ water = 1.33 [Ans : 62°28']

2. In Young's experiment when the separation between the two slits is 0.2 mm and the screen is at a distance of one meter from slits, the fourth bright fringe is found to be displaced from the central frige by a distance of 10mm. Determine the wavelength of the light source used. (Ans.5000 A°)

3. Yellow sodium light of wavelength 5893 A° passes through two narrow slits 1.5 mm. apart. The interference pattern is seen on a screen at a distance 150 cm away from the slits. Determine the fringe width. (Ans. 0.59mm)

4. A soap flm of 6×10^{-5} cm thickness is viewed at an angle of 45° to the normal Find the wavelengths of light in the visible spectrum which will be absent from the reflected light. The refractive index of the soap film is 1.33. (Ans.5635 A°)

5. A parallel beam of light of λ=5890 A° is incident on a thin glass plate of μ = 1.5 such that the angle of refraction into the plate is 60°. Find the smallest thickness of the plate which will appear dark by reflection [3.93×10^{-5}cm]

6. White light is made to be incident normally on a thin film having a μ=1.5 and thickness 4×10^{-5} cm. what wavelength within the visible region will be intensified in the reflected beam. [4800 A°]

UNIT-14 NEWTONS RINGS

Contents

14.1 AIMS AND OBJECTIVES

This unit explains the formation of Newton's rings. To do so it discusses the conditions required for the formation of such rings.

After studying this Unit you will be in a position to workout the wavelength of the light source used, and also the refractive index of a liquid by forming Newton's rings and using liquid in place of air.

14.2 INTRODUCTION

Newton observed and studied the interference effects produced by the film of air between the convex surface of a lens and the flat surface of a plane glass plate. When the film is viewed in reflected light the centre of the pattern appears black with concentric bright and dark circular fringes. When viewed in transmitted light the centre of the pattern appears bright. When white light is used the colour of the light reflected from the film at any point is complementary to the colour transmitted. These interference rings are known as Newton's rings. Even though Newton could estimate the radius of the ring, he could not offer an explanation for the formation of the rings. Thomas Young was able to offer an explanation for the formation of Newton's rings based on Huygens' wave theory of light. According to Young, the light waves reflected at the upper and lower surfaces of the air film interfere and produce the ring pattern.

14.3 NEWTON'S RINGS BY REFLECTED LIGHT

The experimental arrangement for observing Newton's rings is shown in Fig.14.1 Light from the source 's' is rendered parallel by the convex lens L_2 placed between the source and the glass plate G. The glass plate G which is inclined at an angle of 45° to the horizontal plane reflected the rays downward on to the lensL_1. The rays fall normally on the lens L_1.

These rays are reflected back in the opposite direction by the two surfaces of the air film existing between the lens L_1 and glass plate P. These reflected rays interfere. When the microscope M is focussed on the air film interference fringes will be observed. Dark and bright circular fringes can be seen on the field of view of the microscope.

A black paper is usually placed underneath the plate P in order to absorb the light passing down through the glass plate thereby avoiding any reflection of light from the metalic bed of the instrument.

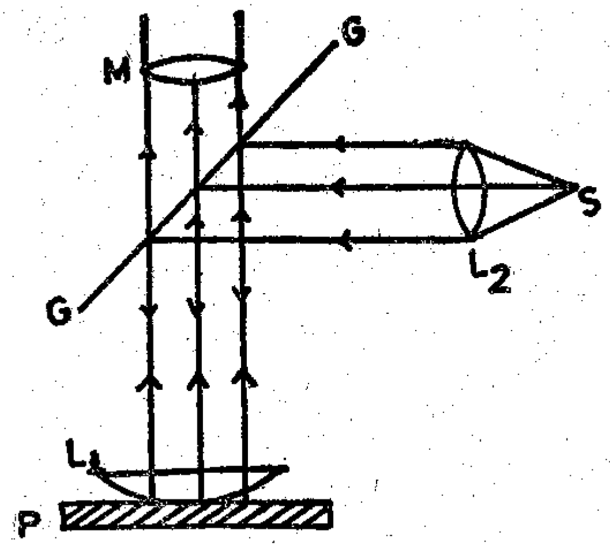


Fig. 14.1

The formation of Newton's rings can be explained with the help of Fig.14.2 Here PQ is a monochromatic light ray which falls on the lens L_1.

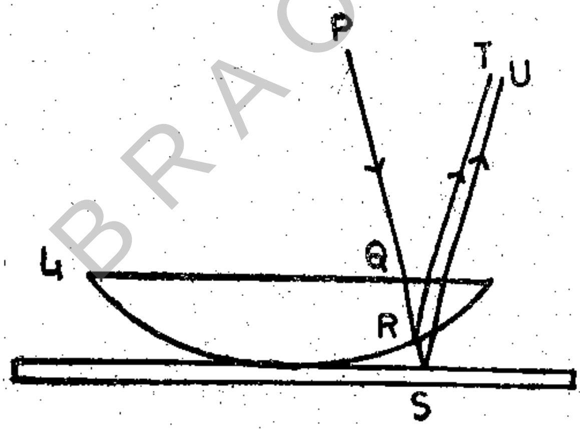


Fig.14.2

PQ undergoes reflection and refraction at R. The reflected ray RT undergoes no phase reversal. The refracted ray RS undergoes reflection at S and takes the path SU with the phase reversal of 180°. The rays RT and SU interfere since they are derived from the same ray PQ leading to Newton's rings. The ray PQS shown to be incident slightly inclimed to the vertical in order to show the two interfering reflected rays seperately. Actually PQ is incident normally.

Since the interference pattern is due to reflected light the path difference Δ is given by

$$\Delta = 2\mu t \cos\gamma + \lambda/2 \qquad (14.1)$$

Where μ is refractive index of the material between the lens and glass plate. γ is the angle of incidence λ is wavelength of incident light and t is the thickness of the air film.

For air film $\mu = 1$ and for normal incidence $\gamma = 0$. Therefore the path difference is given by

$$\Delta = 2t + \lambda/2 \quad (14.2)$$

The condition for the formation of a bright ring is given by

$$2t + \lambda/2 = n\lambda \quad (14.3)$$

or $2t = n\lambda - \lambda/2 = (2n-1)(\lambda/2)$ Where n=0, 1, 2,... ... (14.4)

The condition required for the formation of dark rings is given by

$$2t + \lambda/2 = (2n+1)\lambda/2 \quad (14.5)$$

or $2t = n\lambda$ (14.6)

where n- 0, 1, 2, ...

Let us now calculate the diameter of the bright and dark rings. Let the radius of curvature of the lens be R. Let the thickness of the air film be t which is at a distance of EQ=v as shown in Fig.14.3

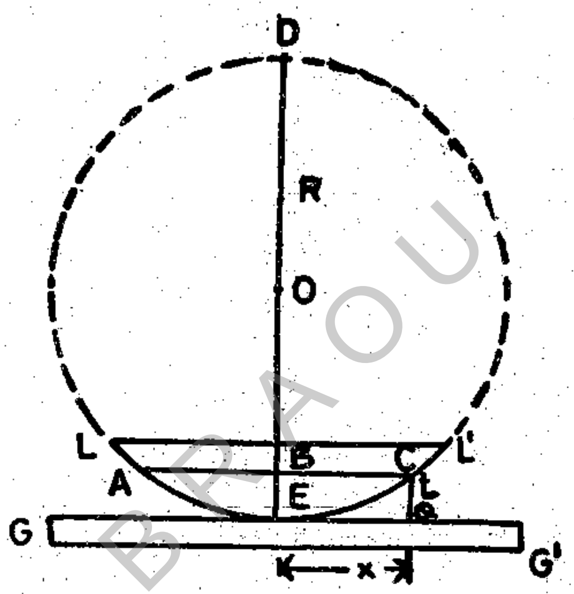


Fig.14.3

Here LEL' represents the lens placed on the glass plate GG' with the centre at E. As per the Fig. 14.3 using the property of the circle we have

$$AB \times BC = EB \times BD \quad (14.7)$$

since BD=ED-EB=2R - T we have AB X BC = X x X=t(2R-t) (14.8)

since t is very small compared to 2R, we can rewrite equation 14.8 as

$$x^2 = 2Rt \quad (14.9)$$

Hence $t = x^2/2R$ (14.10)

For the bright ring, substituting the value of 't' from equation 14.4 we get,

$$\frac{x^2}{2R} = \frac{(2n-1)\lambda}{4} \quad (14.11)$$

Here 'x_b' represents the value of x pertaining to bright ring, therefore

$$x_b = \left[\frac{(2n-1)\lambda R}{2}\right]^{\frac{1}{2}} \quad (14.12)$$

Replacing x_b by $D_b/2$ where D_b represents the diameter of the ring we get

$$D_b = 2\left[\frac{(2n-1)\lambda R}{2}\right]^{\frac{1}{2}} = [2(2n-1)\lambda R]^{\frac{1}{2}} \qquad (14.13)$$

Hence D_b is proportional to $[2\,(2n-1)]^{1/2}$, $\lambda^{1/2}$ and $R^{1/2}$

For dark rings, substituting the value of t from equation 14.1 in equation 14.6 we get

$$\frac{x^2 d}{R} = n\lambda \qquad (14.14)$$

or $$x_d = 2(Rn\lambda)^{\frac{1}{2}} \qquad (14.15)$$

Here x_d represents the value of x pertaining to the dark ring. The diameter of the dark ring

$$D_d = 2\,(Rn\lambda)^{\frac{1}{2}} \qquad (14.16)$$

Hence the diameter of the dark rings is proportional to $R^{\frac{1}{2}}$, $n^{\frac{1}{2}}$ and $\lambda^{\frac{1}{2}}$ for the dark rings when n=0 we get

$$D_d = 2(n\lambda R)^{\frac{1}{2}} = 0 \qquad (14.17)$$

This corresponds to the centre of the Newton's rings. While counting the order of the dark rings 1,2,3........ the central ring is not counted.

From equation 14.17 we have

For the first ring n=1 $$\therefore D_{d1} = 2(\lambda R)^{\frac{1}{2}} \qquad (14.18)$$

for the second ring n = 2 $$\therefore D_{d2} = 2\,(2\lambda R)^{\frac{1}{2}} \qquad (14.19)$$

Similarly for nth ring $$\therefore D_{dn} = 2\,(n\lambda R)^{\frac{1}{2}} \qquad (14.20)$$

Let us consider the first and fourth rings. The difference in the diameter of these two rings is

$$D_{d4} - D_{d1} = 2(4\lambda R)^{\frac{1}{2}} - 2(\lambda R)^{\frac{1}{2}} = 2(\lambda R)^{\frac{1}{2}} \qquad (14.21)$$

Let us consider the ninth and sixteenth rings. The difference in the diameters of these two rings is

$$D_{d_{16}} - D_{d9} = 2\,(16\lambda R)^{\frac{1}{2}} - 2\;(9\lambda R)^{\frac{1}{2}} = 2\,(\lambda R)^{\frac{1}{2}} \qquad (14.22)$$

Equations 14.21 and 14.22 indicate that the fringe width decreases with the order of the fringe and the fringes get closer with the increase in their order. These apsects are illustrated in Fig. 14.4.

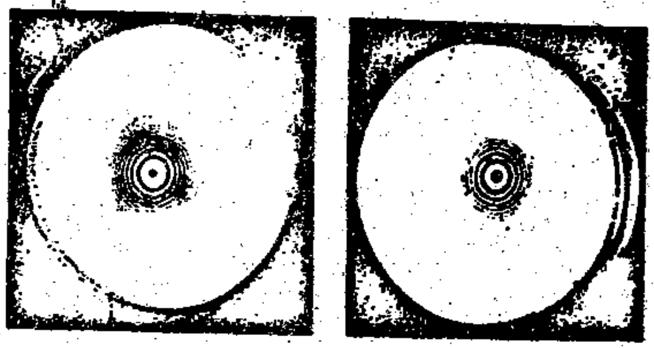


Fig. 14.4

14.4 NEWTON'S RINGS BY TRANSMITTED LIGHT

Let us now discuss the nature of Newton's rings formed due to transmitted light. Consider a monochromatic light ray ABC transmitted through the air film along the direction CD as shown in Fig. 14.5 Part of BC is reflected at C, goes to E and is then reflected at E along the direction EF.

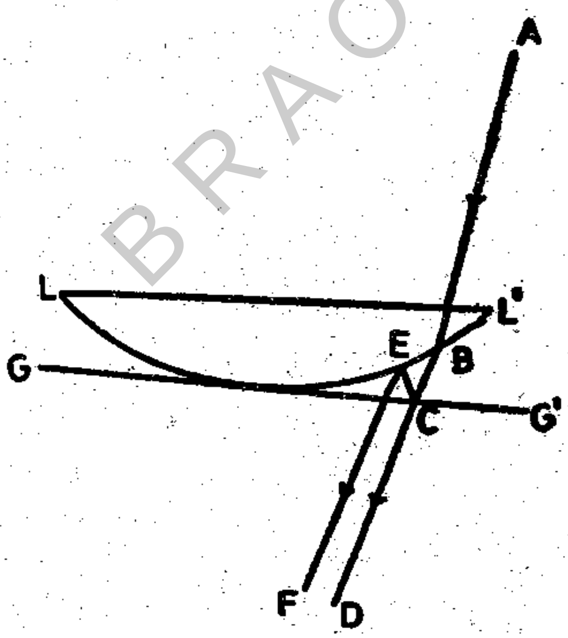


Fig. 14.5

The two rays CD and EF interfere leading to the formation of Newton's rings. A path difference of $2t + \lambda/2 - \lambda/2$ is produced between the two transmitted rays. Here $t = BC$ represents the thickness of the air film at B. The condition required for the formation of bright ring is

$$2t + \lambda/2 - \lambda/2 = n\lambda \tag{14.23}$$

$$\therefore 2t = n\lambda \tag{14.24}$$

since $t = x^2/2R$, we get $2x^2/2R = 2x^2/2R = n\lambda$ (14.25)

$$\text{or } x_b = (n\lambda R)^{1/2} \tag{14.26}$$

The diameter of the ring $D_b = 2(n\lambda R)^{1/2}$ (14.27)

The diameter of the bright ring is proportional to $n^{1/2}$, $R^{1/2}$ and $\lambda^{1/2}$

The condition required for the formation of dark ring is given by

$$2t = (2n-1)\lambda/2 \tag{14.28}$$

Using $t = x_d^2/2R$ and $x_d = D_d/2$, we get the diameter of the dark ring as

$$D_d = [2(2n-1)\lambda R]^{1/2} \tag{14.29}$$

The diameter of the dark ring is proportional to $(2n-1)^{1/2}$, $\lambda^{1/2}$, $R^{1/2}$

The bright rings as per Eq.14.27, when $n = 0$ $D = 0$. Hence in the case of Newton's rings due to transmitted light the central ring is bright. Also the fringe width decreases with the order of the fringe and the fringes get closer with increase in their order. The reflected and transmitted Newton's rings patterns are complementary to each other.

Check your Progress 1

What is the difference between Newtons rings formed by reflected light and transmitted light?

The wavelength of the light source and refractive index of a transparent liquid can be determined by using Newton's rings.

Consider the experimental arrangement shown as in Fig. 14.1. Newton's rings can be viewed through the travelling microscope focussed on the air film. Circular bright and dark rings can be seen when the light used is monochromatic light say Sodium light. The central spot will be dark. The diameter of the n^{th} and m^{th} dark rings are measured using the travelling microscope. Since the diameter of the n^{th} ring is given by

$$D_{dn} = (4n\lambda R)^{1/2} \tag{14.30}$$

Similarly the diameter of the m^{th} ring is given by

$$D_{dm} = (4m\lambda R)^{\frac{1}{2}} \tag{14.31}$$

The wavelength λ is given by

$$\lambda = \frac{D_{dm}^2 - D_{dn}^2}{4(m-n)R} \tag{14.32}$$

In the equation 14.32 R represents the radius of curvature of the lower surface of the lens which can be determined using the spherometer. Knowing D_{dm}, D_{dn}, n and R can determine λ.

The refractive index of a liquid can be determined using the experimental arrangement shown in Fig.14.6. Hence the plano convex lens and the glass plate are firmly kept inside a metal container M.

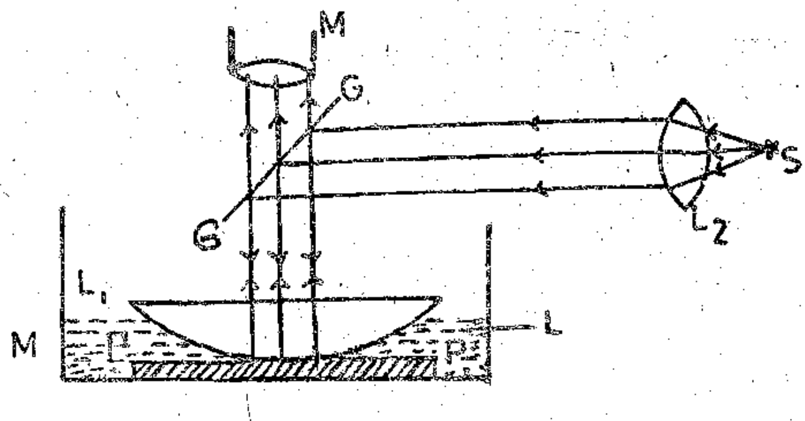


Fig.14.6

Newton's rigns are first observed when there is an air film between the lens and optically plane glass plate. The diameters of the n^{th} and m^{th} dark rigns say D_{dn} and D_{dm} are measured using the travelling microscope. Now, without disturbing the arrangement, the given liquid, say,water, is poured into the container. The air film will be replaced by the liquid. Again the diameters of the n and m rings, say D'_{dn} and D'_{dm} are measured.

For air film $D^2_{dn} = 4(n\lambda R)$ (14.33)

$$D^2_{dm} = 4m\lambda \qquad (14.34)$$

$$\therefore \lambda = \frac{D'^2_{dm} - D'^2_{dn}}{4(m-n)R} \qquad (14.35)$$

For liquid film

$$(D'_{dn})^2 = \frac{4n\lambda R}{\mu} \qquad (14.36)$$

$$(D'_{dm})^2 = \frac{4n\lambda R}{\mu} \qquad (14.37)$$

Where μ represents the refractive index of the liquid.

$$(D'_{dm})^2 - (D'_{dn})^2 = \frac{4(m-n)\lambda R}{\mu} \qquad (14.38)$$

$$\therefore \mu = \frac{D'^2_{dm} - D'^2_{dn}}{4(m-n)\lambda R} \qquad (14.39)$$

Knowing λ,R,m,n, D'_{dn}, D'_{dm} can determine μ can be determined using equation 21.35. Substatituting the value of λ from the equation 14.35 in the equation 14.39 we get,

$$\mu = \frac{(D^2_{dm} - D^2_{dn})4(m-n)R}{4(m-n)R(D'^2_{dm} - D'^2_{dn})} \qquad (14.40)$$

$$\mu = \frac{(D^2_{dm} - D^2_{dn})}{(D'^2_{dm} - D'^2_{dn})} \qquad (14.41)$$

Equation 14.41 indicates that μ can be determined even without the knowledge of λ and R.

Worked Example 1.

In Newton's rings experiment using reflected ligh ne diameters of the 6th and 26th rings were observed to be 0.4 cm. and 0.8 cm respectively. If the radius of curvature of the plano convex lens used is 100 cm, determine the wavelength of the light source used.

we know

$$\lambda = \frac{D'^2_{dm} - D'^2_{dn}}{4(m-n)R}$$

Hence, as per the data given in the problem $D_{dn} = D_{d26} = 0.8$ cm, $D_{dn} = D_{d6} = 0.4$ cm, R=100 cm. m=26 and n=6

$$\lambda = \frac{(0.8)^3 - (0.4)^2}{4(26-6)100} = \frac{0.48}{8000} = 6000 A^\circ$$

Worked Example-2.

Newton's rings are formed by reflected light of wavelength 5895 A° with a liquid between the plane and curved surfaces. If the diameter of the 5th and 20th rings respectively are 0.3 cm and 0.7 cm respectively, determine the refractive index of the liquid. The radius of the curvature of the lens is 100 Cm.

We have as per Eq.14.39

$$\mu = \frac{D'^2_{dm} - D'^2_{dn}}{4(m-n)\lambda R}$$

As per the data given in the problem

$$\mu = \frac{(0.7)^2 - (0.3)^2}{4(20-5)5895 \times 10^{-8} \times 100}$$

$$= \frac{(0.49 - 0.09)}{4(15)5895 \times 10^{-8} \times 100} = \frac{0.4}{4(15)5895 \times 10^{-8}}$$

$=0.4/0.3537 = 1.13$

14.6 SUMMARY

Newtons rings are formed by the reflected light and transmitted light. When light rays are reflected at the upper and lower surface of a thin film of air formed between the convex lens and a glass plate, they give rise to an interference pattern. This interference pattern is known as Newton's rings.

14.7 MODEL ANSWERS

Check your Progress 1

Newtons rings formed by reflected light will have a dark central spot where as rings formed by Transmitted light have bright spot at the center.

14.8 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

Discuss with the necessary theory how you can determine the wavelength of menochromatic source using Newton's rings.

II. Answer the Following Question in About 10 Lines

1. Discuss the basic differences between the Newton's rings obtained by reflected light and transmitted light due to a thin film of air.

2. Show that the fringe width in Newton's rings depends on the order of the ring.

III. Solve the Following Problems.

1. In Newton's rings experiment using reflected light the diameter of the eighth ring is found to be 0.5 cm. When the waelength of the light source used in 5900 A° determine the radius of curvature of the plano convex lens used in the experiment.
(Ans. 1.323 m)

2. A liquid forms the film between the glass plate and the plano convex lens. Sodium light of wavelength $\lambda = 5893$ A° is made incident on the film at normal incidence. In the Newton's rings formed by the light reflected by the film, the fourth and eight rings are found to have the diameter of 2.0 mm and 3.5 mm respectively. Determine the refractive index of the liquid. The radius of curvature of the lens in 1m.
(Ans. $\mu = 1.142$)

3. In a Newton's rings experiment the diameters of 5th and 25th rings are 0.3 cm and 0.8 cm. respectively. If the wavelength of the light source used is 4870 A° find radius of curvature of the convex lens used in the experiment.
(1M)

4. In a Newton's rings experiment the diameters of the 12th amd 20th dark rings are 0.7 cm and 0.906 cm. respectively. Find the diameter of the 4th ring. (0.4 cm)

5. The diameter of the 10th bright ring in the Newton's rings experiment changes from 1.5 cm. to 1.3cms. When a liquid is introduced between the lens and the plate. Find the refractive index of the liquid. (1.33)

6. In a Newton's ring experiment the diameter of the 10th ring changes from 1.4cm to 1.27cm. when a liquid is introduced between the lens and plate. Find the refractive index of the liquid.
(1.215)

BLOCK - VIII

DIFFRACTION

UNIT-15 FRESNAL AND FRAUNHOFER DIFFRACTION

Contents

15.1 AIMS AND OBJECTIVES

This unit introduces the phenomenon of diffraction. In order to explain the phenomenan Fresnel and Fraunhofer types of diffractions are explained in detail.

After going through this unit you will be able to distinguish between Fresnel type of diffraction and Fraunhofer type of diffraction.

15.2 INTRODUCTION

The law of propagation of light in straight lines explains the formation of shadows and, is the basis of geometrical optics. According to this law, light rays proceeding from a very small (point) source should produce sharp shadows, when they pass through apertures (opening in opaque screen) or when passing by obstacles. Euclid in the fourth century B.C.laid the foundation of geometrical optics, using this law. It was only in the 17th century A.D. that certain experiments were done, showing that this law was not absolutely true. The bending of light, into what should be the shadow region according to geometrical optics, is called diffraction of light.

Father GRIMALDI (1618-1663) was a jesuit priest, teaching in a religious institution in Bologna, Italy. He carried out a series of experiments in optics and described them in his book de Lumine which was published two years after his death. In the beginning of this book he states that he prosposes to study the question of the nature of light by developing and interpreting his own experiments without paying any attention to the authority of the masters.

While describing his discovery of diffraction, Grimaldi says "Light propagates in addition to the three modes-direct propagation, refraction and reflection - by a fourth mode namely diffraction."

Grimaldi did his experiments on diffraction in a darkened room, into which sunlight was allowed through a very small hole. On placing various objects in the path of this beam of light, he found that the shadows of these objects on a screen were not sharp. There were complicated dark and bright bands at the edge of the shadow, showing that some light had bent into the geometrical shadow. He studied this phenomenon carefully, by changing the distance between the object and the screen.

He observed this phenomenon, which he called diffraction both with opaque obstacles and apertures in opaque screens.

15.3 NEWTON'S EXPERIMENTS ON DIFFRACTION OF LIGHT

Newton began his famous researchs in optics in 1666. The results were published in his book "Optick" in 1704. He described experiments on diffraction and tried to give their explanation on the basis of his corpuscular (particle) theory of light. So great was Newton's authority in those days of scientific awakening, that for a hundred years nobody even attempted to apply the wave theory to explain diffraction of light. It is interesting to note that Newton himself gave a fairly good wave theory of sound, as shown by his derivation of the formula for the velocity of sound.

15.4 EXPLANATION OF DIFFRACTION ON THE BASIS OF THE WAVE THEORY OF LIGHT.

The wave theory of light had its beginning in Huygens' attempts to explain the double refraction of light in crystals (1678). He put forward the idea, known later as Huygens' Principle, that each point in a wave front becomes a source of secondary wavelets and the new wave front is the envelope of these secondary wavelets. It is not certain whether Huygens knew about Gramnaldi's experiments on diffraction. Huygen was very careful in showing how his principle was in conformity with the law of propagation of light in straight lines. The effect of a portion of wave front was such that only propogation in the forward direction was possible, hence there could be no bending of light beyond the shadow. In other words, Huygen's wave theory did not predict the diffraction of light. For this explanation, one had to wait till Young, a British Physician discovered the interference of light in 1801, which he explained successfully on the basis of the wave theory of light. The first satisfactory explanation of diffraction of light was given by Fresnel, a few years later using the wave theory of light. Fresnel combined the ideas of Huygens with the principle of interference and successfully explained the diffraction pattern obtained by several simple types of objects and aperture. His article on diffraction won him the prize of the French Academy in 1819. In these lessons we shall describe the Huygens-Fresnel theory of diffraction, which is still used to calculate the diffraction pattern in simple cases. Difficulties with these theories arise when one considers the pattern very close to the diffracting object and when the nature of the object (eg. its being a metal or insulator) comes into the picture. In such cases we have to use electromagnetic theory of light put forward by Maxwell in 1873. This is of importance when we consider diffraction of microwaves, in connection with radar etc. Another field in which the study of diffraction has produced important results is diffraction of x-rays when passing through crystals and also of material particles like electrons, which show their wave nature in passing through crystals.

15.5 A SIMPLE MATHEMATICAL APPROACH FOR THE UNDERSTANDING OF DIFFRACTION OF LIGHT-THE HUYGENS-FRESNEL THEORY

The Huygens-Fresnel theory is based on two principles:

1. The Huygens principle
2. The principle of interference (supplemented by the principle of super position)

Huygens principle states that each point in the wave front 1 (ab Fig.15.1) becomes a source of secondary waves and sends out secondary (spherical) wavelets. The wave front at a later time is the envelope 2 of the secondary wavelets as they have moved out in the time t. Each point in 2 then become a source of secondary waves and their envelop gives the next position of the wave front. Thus the propogation of light is explained in terms of the advancing wave fronts. When one such wave front reache a point we say that light has propagated from source to the point. This is a very nice way of describing propagation of light when the wave fronts are infinite in extent and not obstructed and the full wave front is allowed to proceed forward.

Fresnel tried to think about what would happen when the whole wave front was not allowed to proceed further but only a certain part which could get through an aperture (Fig.15.2)

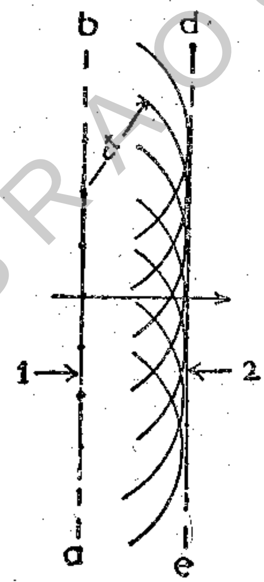


Fig 15.1light waves emerging from a light source.

1. Wave front at t=0
2. New position of wave front.

As experiments (like Grimaldi's) had shown, when the observation point P on the screen is well inside the cone nothing strange happens at P. But near the edge of the shadow diffraction bands (called fringes) appear and the light in this region does not obey the straight line propagation law.

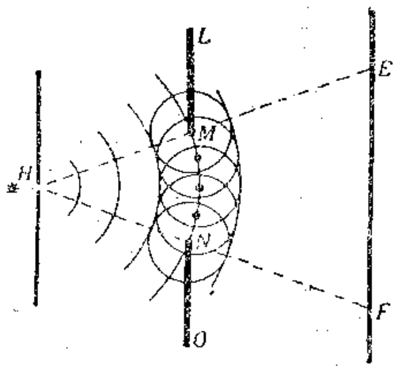


Fig 15.2

Huygen's principle applied to secondary wave light from a narrow opening.

To explain diffraction, Fresnel uses Huygen's idea of secondary waves starting out from every point in that part of the wave front which is not obstructed. To find the effect of the secondary waves he uses the principle of superposition. The resultant effect is then given as the sum of the effects due to all these secondary waves. One can say that the secondary waves produce interference and there may be a bright fringe or a dark fringe dependin on how the secondary waves add up, taking into account their relative phases. Thus in the Huygens-Fresnel theory of diffraction, one calculates the diffraction pattern as the interference pattern due to the very large number of waves coming from the several points in the exposed parts of the wave front. We shall carry out this programme for some simple cases of diffraction in the next three lessons. We shall conclude this lesson by explaining the usual classification of diffraction phenomena into two types, namely, the Fraunhofer and the Fresnel diffraction phenomena.

15.6 FRESNEL AND FRAUNHOFER DIFFRACTION

From our description of Grimaldi's discovery of diffraction it appears that observation of diffraction-phenomena requires special arrangement of point sources, darkened room etc. This is not quite true. Diffraction patterns are also observed in some simple ways. For example if one looks at a small source, (a distant street lamp) through an ordinary piece of cloth (say, a handkerchief) one sees a regular partern of light spots surrounding the source, which is a diffraction pattern. Also, if a distant tube light is seen through half closed eyes so that the light passes through the regular obstacles formed by the eyelashes, a set of diffraction bands are seen. Looking through the narrow gap between two fingers at the tube light, one sees diffraction bands.

Looking at fig. 15.3 one can find a simple way of classifying diffraction patterns. When the source of light S and the point of observation P are at small distance from the diffracting screen (B), the diffraction phenomenon is said to be of the Fresnel type. This is what Grimaldi had studied. When the source and point of observation are very far from the diffractiong screen B (practically at infinite distance), the pattern observed is said to be of the Fraunhofer type. Note that in the Frauhofer case the light falling on the diffracting screen forms a beam of parallel rays and the rays leaving the diffracting screen ("the diffracted rays") are also paralled rays. So one can call the Frauhofer diffraction as diffraction due to parallel rays. But this means that the source and the observation screen must be placed at infinite distances from the diffracting object. A practical way of satisfying this seemingly impossible condition is to place the source at the principal focus of a convex lens and allow the resulting parellel beam to fall on the diffracting screen. To catch the diffracted parallel rays we put a convex lens after the diffracing screen, and place the observing screen at the principle focus of this lens. (Fig.15.3)

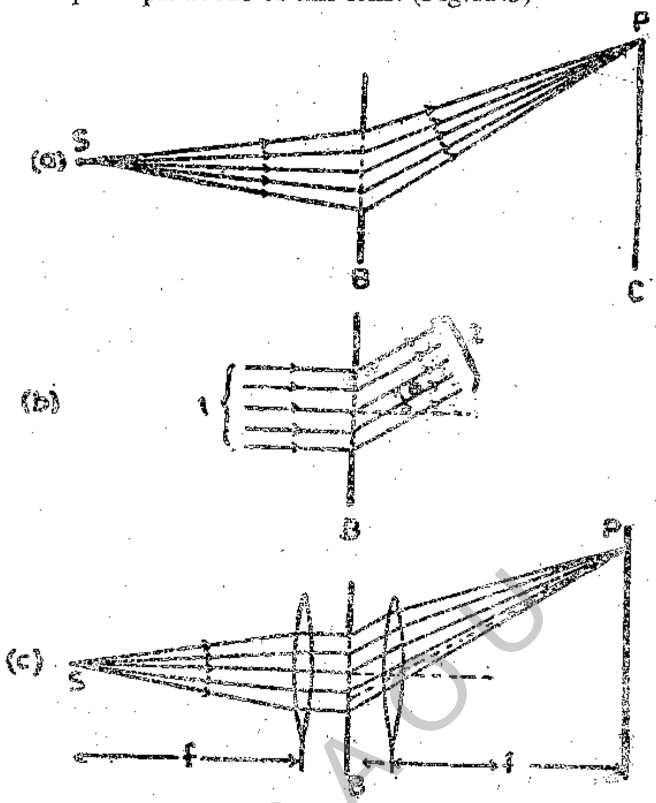


Fig.15.3

(a) Fresnel diffraction. (b) Source S and screen C are moved to a large distance, resulting in Fraunhofer diffraction. (c) Fraunhofer differection conditions produced by lenses, leaving source S and screen C in their original position.

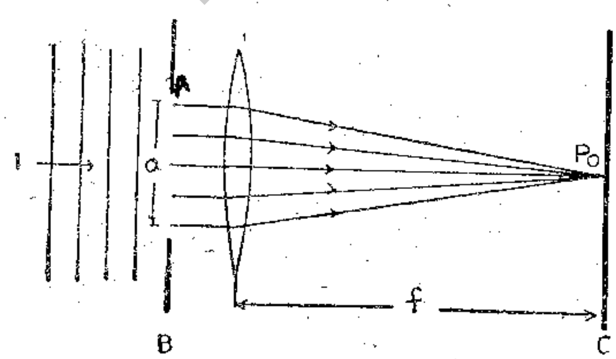


Fig.15.4 Fraunhofer diffractions,
1 Incident wave

One can use an ordinary spectrometer (whose collimator and telescope have been "adjusted for parallel rays") to study the Franuhofer diffraction patterns of various objects by just placing the objects (slits etc.) on the spectrometer table and observing the pattern through the telescope.

Fig 15.4 shows a lens L forming the image of a point source located at infinity on the screen. If an aperture AB is placed near the lens, diffraction fringes appear on the screen surrounding the point where the image should be, according to geometrical optics. This is nothing but the Fraunhofer diffraction due to the aperture AB. To understand this imagine the lens L to be replaced by two lenses L_1 and L_2 on either sides of AB with focal lengths equal to SL and LO respectively. Then L_1 produces parallel rays which are incident on AB and L_2 focusses the diffracted parallel rays in its focal plane. But this is just the way a Fraunhofer diffraction pattern is seen, (Fig 15.4). There is an important conclusion to be drawn from what we have found out just now. In optical instruments, where an image of a certain object is formed by a lens, say, the image of a star in an astronomical telescope, the image is really the Fraunhofer diffraction pattern of the aparture which restricts light coming through the lens.

15.7 FRESNEL AND FRAUNHOFER DIFFRACTION - MATHEMATICAL APPROACH

So far we have talked about Fresnel and Fraunhofer diffraction in terms of certain "very small" or "very large" distances. To make these definitions more exact we shall now use the ideas of Fresnel and put the theory on a mathematical basis, To simplify matters let us consider a plane wave incident on a plance screen containing an opening B. We wish to find the intensity of light at a point P in the observation screen C (fig 15.4) According to Huygens wavelets start out from every point in the exposed portion of the incident wave front, and reach P. We have to find the resultant amplitude due to these waves. According to the wave theory of light, the intensity of light at P will be proportional to the square of the resultant amplitude.

A point Q is the source of secondary spherical waves. If the amplitude at Q is a, the light disturbance at P (distance d away not shown in the fig. due to the spherical wave is given by the formula.

$$\varnothing = \frac{a}{d}\cos(\omega t - kd)$$

Where ω = angular frequency $2\pi f$

f = frequency

t = time

$k = 2\pi/\lambda$

λ = Wavelength

$\omega t - \lambda\, d$ = phase of wave.

Following Fresnel, we now add the light disturbances due to all the points between A and B. So that the resultant wave is given by a sum.

$$\Phi = \Sigma \frac{a}{d}\cos(\omega t - kd) \qquad (15.1)$$

Since point like O are continuosly distributed in the aperture AB the resultant light distrubance may be written as an integral:

$$\psi = \int \varnothing dr = \int \frac{a}{d}\cos(\omega t - kd)dt \qquad (15.2)$$

Where r is the distance of Q from an origin R in the diffractiong aperture.

Expressions 15.1 and 15.2 form the basis of calculating the diffracting patterns in a simple way, and will be used in the next three units.

At present we wish to find a quantitative way of distinguishing between the Fresnel and the Fraunhofer type of diffraction patterns.

Here the size of the aperture, namely, the distance AB = b is important. We shall now find an expression for the phase difference between the waves reaching P from the extreme ends of the opening namely, the point A and B.

path AP=D

path BP=D+ Δ

Δ is the path difference and $(2\pi/\lambda)$ is the phase difference for waves from A and B reaching P. Let us find an expression for Δ using the well known relation between the sides of the triangle ABP.

$$(D+\Delta)^2 = D^2 + b^2 = 2Db\cos\left(\frac{\pi}{2}+\theta\right) \qquad (15.3)$$

When θ is the angle between AP and the normal to the screen AN.

Simplifying 15.3 we get:

$$2D\Delta + \Delta^2 = b^2 + 2Db\sin\theta \qquad (15.4)$$

Here we make an assumption which generally holds in practical situations, namely $\Delta << D$.

Which means that P is sufficiently far away from AB. Thus in eq.15.4 we may ignore Δ^2 in comparison with $2\Delta D$. Thus 15.4 becomes:

$$2\Delta D = b^2 + 2bD\sin\theta \qquad (15.5)$$

This important formula makes the distinction between Fresnel and Fraunhofer diffraction possible: if D becomes very large.

$$D \to \infty, \Delta = \Delta\infty = b\sin\theta$$

But this is just the condition for observing the Frauhofer diffraction. For small values of D the other term $b^2/2D$ becomes important. We can write $\Delta - \Delta\infty = \frac{b}{2D}$. So for smaller values of D the phase will have a part varying as b^2. Since the resultant amptitude dpends on the variation of phase, the resultant in the case of D will be different from that of $\Delta = \Delta\infty = b\ \sin\ \theta$. Which shows a linear relation with b. Thus the diffraction pattern where the path difference is proportional to b^2 is the Fresnel diffraction pattern. But where is the dividing line between the Fresnel and Fraunhofer cases? To find this we use an idea due to Lord Rayleigh, who said that if the path difference infolved in two calculations differs by less than $\lambda/8$ (or phase difference by less than $\pi/2$) than the two intensities calculated will not differ apperciably from each other. In the present case, If $\Delta - \Delta\infty < \lambda/8$ we shall not be able to say in a clear cut manner, whether this is a Fresnel or a Fraunhofer pattern. The dividing line we want is then given by.

$$\Delta - \Delta\infty = \frac{b^2}{2D} = \frac{b^2}{L} = \frac{\lambda}{8}$$

or $\frac{b^2}{L} = \frac{\lambda}{8}$ defines the boundary between Fresnel and Fraunhofer diffraction.

For large value of $Lb^2/L << \lambda$, giving Fraunhofer diffration.

For smaller L, $b^2/L = \lambda$, giving Fresnel diffraction.

For very small values of L, $b^2/L >> \lambda$ and we have fairly sharp shadows, showing practically no fringes. i.e. straigth line propogation is observed. This can be summarised as follows:

Fraunhofer diffraction $\frac{b^2}{L\lambda} << 1$

Fresnel diffraction $\frac{b^2}{L} = 1$

· Straight line propagation $\frac{b^2}{L\lambda} >> 1$

So, the value of the dimensionless constant $b^2/2\lambda$ decides the type of diffraction to be seen. We shall illustrate this with an example. Let a beam of parallel rays of wavelength $\lambda = 6 \times 10^{-5}$ cm fall on an opening of width b=0.1 mm.

Consider three values of $l = 10^{-1}$ cm, 1 cm, 100 cm. Corresponding values of $l = 10^{-1}$ cm, 1 cm, 100 cm. Corresponding values of $b^2/L\lambda$ are 17, 1.7 and 0.0017 respectively. So we expect a sharp shadow obeying the law of straight-line propogation in case 1. Fresnel d. fraction pattern showing fringes near the edge of the shadow in case 11, and finally a Fraunhofer diffraction pattern consisting of a bright spot in the centre, surrounded by fringes in case III. Notice that with a slit of width 0.01 cm., One can produce a Fraunhofer pattern at a screen just a metre or so away. This beautiful experiment can be performed easily with a He-Ne laser.

15.8 SUMMARY

The diffraction of light was explained on the basis of Grimaldis discovery, Newtons work and wave theory of light. The two types of diffraction are Fresnel and Fraunhofer. In Fresnel type of diffraction the source of light, and the point of observation are at a small distance from the diffracting screen. In Fraunhofer type of diffraction source and point of observation are very far from the diffracting screen.

15.9 SAMPLE EXAMINATION QUESTIONS

I Answer the Following Questions in About 30 Lines.

1. Give a critical account of the historical background, discovery and the Fresnel and Fraunhofer's ideas on diffraction.

II Answer the Following Questions in About 10 Lines.

1. Explain the phenomenon of diffraction on the basis of wave theory of light.

2. Give a brief account of the mathematical approach of Fresnel and Fraunhofer diffraction.

UNIT - 16 FRESNEL DIFFRACTION AT A STRAIGHT EDGE

Contents

16.1 AIMS AND OBJECTIVES

In this unit the Fresenl diffraction at a straight edge is explained. In order to make you understand it applies the Huygens-Fresnel theory of diffraction to wave front on a straight edge. After going through this Unit you will be able to find out (1) the amplitude and phase of the waves reaching the screen from elements in the wave front (2) the resultant disturbances at a point on the screen.

16.2 INTRODUCTION

Diffraction phenomenon is the characteristic of wave motion. As we have seen in the unit 15 diffraction is noticeable when a wave is distorted by an obstacle which has dimensions comparable to the wave length of the wave. The obstacle may be a screen with a small opening or slit which allows only a small portion of the incident wave front to pass. In this unit we shall study the Fresnel diffraction at a straight edge.

16.3 MATHEMATICAL TREATMENT OF DIFFRACTION AT A STRAIGHT EDGE

We apply the Huygens-Fresnel theory to Fresnel diffraction at a straight edge. Fig.16.1 shows the straight edge OS is extending to large distance beyond S. A plane wave front is incident from the left. Its portion along OS is completely stopped by the straight edge and the part extending along OQ and beyond is allowed to pass. Elements along OQ become Huygens' secondary sources and send the waves towards observation screen C, where these secondary waves combine and produce the diffraction pattern.

The amplitude of a secondary wave from an element of length dh situated at a distance h from edge O is proportional to dh. The path difference between the waves from O and those from Q is $\Delta = h^2/2s$ and the phase difference is $\delta = (2\pi/\lambda)\Delta$ Where s is the distance between O and the corresponding P at the edge of the geometrical shadow. This formula was derived in unit.15.

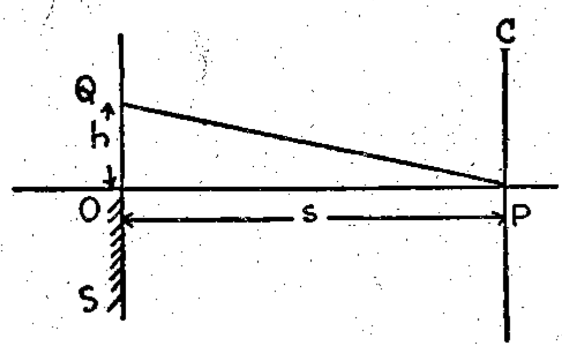


Fig 16.1 Fresnel diffraction at straight edge.

The light disturbance reaching the point P from Q is represented by

$$d\psi = dh\cos(\omega t - \delta) \tag{16.1}$$

Where $\delta = (2\pi/\lambda)h^2/2s$ represents the phase, relative to that of the wave from the element at h=0. There are several ways of mathematically dealing with the addition of such simple harmonic waves. We shall use the vector (or as it is sometimes called, the phase or) method. The wave described by expression 16.1 is represented by a vector whose length is proportional to the amplitude (dh), and whose direction with reference axis (ox) is equal to the phase [$\left(\frac{2\pi}{\lambda}\right)\frac{h^2}{2s}$] Fig. 16.2 shows the vector of length, dh making an angle) $\theta = \left(\frac{2\pi}{\lambda}\right)h^2/2s$ with the X=axis.

The x —— and y —— components of this small vector are

$$dx = dh.\cos\left(\frac{2\pi}{\lambda}\frac{h^2}{2s}\right)$$

$$dy = dh.\sin\left(\frac{2\pi}{\lambda}\frac{h^2}{2s}\right) \tag{16.2}$$

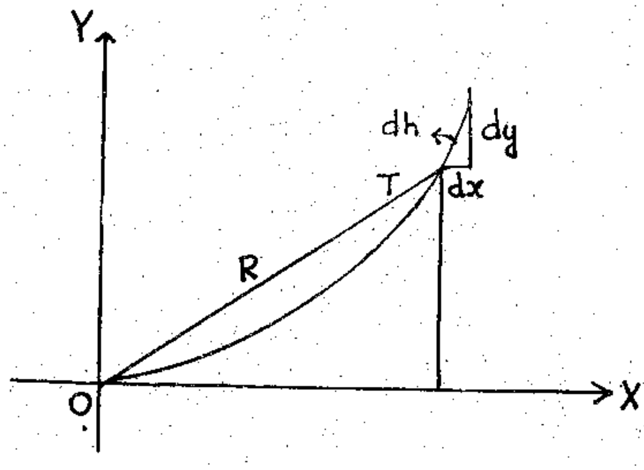


Fig 16.2 Vectorial representation of a wave

We can find the resultant due to the waves from all Huygens secondary sources in the wave front between O and Q by intergrating these expressions between o and h. So.

$$\text{The resultant x — component } A = \int_o^h dh.\cos\left(\frac{2\pi}{\lambda}\frac{h^2}{2s}\right)$$

$$\text{The resultant y — component } B = \int_o^h dh.\sin s\left(\frac{2\pi}{\lambda}\frac{h^2}{2s}\right) \qquad (16.3)$$

The resultant amplitude produced at the point P is equal to the length of the resultant vector (or phasor) R, Now

$$R = \sqrt{A^2 + B^2} = \text{distance OT in Fig 16.2}$$

So the vector OT represents the resultant wave. The phase of the resultant is ϕ. Note that the point T represents the resultant of disturbances from elements along OQ = h reaching the diffracation screen. As the length OQ is increased the point T moves along a curve in Fig 16.2 Equations 16.2, 16.3 are the parametric equations describing this curve, where h is the length OT along the curve. Let us find the shape of the curve as h becomes very large. It is convenient to use a new varible V instead of h, defined by

$$v = h\sqrt{\frac{2}{\lambda s}} \qquad (16.4)$$

So equations become 16.2

$$dx = \sqrt{\frac{\lambda s}{2}}\, d\,v\cos\left(\frac{\pi}{2}v^2\right)$$

$$dy = \sqrt{\frac{\lambda s}{2}}\, d\,v\sin\left(\frac{\pi}{2}v^2\right)$$

Further, let us choose now x and y, so we write

$$dx = d\,v\cos\left(\frac{\pi}{2}v^2\right)$$

$$dy = d\,v\sin\left(\frac{\pi}{2}v^2\right) \qquad (16.5)$$

We wtill write the new variables x and y also as x and y, as no confusion shall arise

The slope of the curve at $v = \frac{dy}{dx} = \tan\left(\frac{\pi}{2}v^2\right)$ the angle of the slope is $\theta = \frac{\pi}{2}v^2$

This tells us that as v increases and the point T moves further on the curve the slope increases as v^2 and the curve turns back on itself several times. In otherwords, the curve is a spiral. The end point (called asymptotic point) Z is reached as $v \to \infty$ its coorcinates are

$$x = \int_o^\infty \cos\left(\frac{\pi}{2}v^2\right)dv, \quad y = \int_o^\infty \sin\left(\frac{\pi}{2}v^2\right)dv \qquad (16.6)$$

$x = 1/2 \qquad y = 1/2$

The integrals X and Y are known as Freesnel's integrals and the curve is called Cornu's spiral. For negative values of v, there is a lower branch of the spiral ending in the asymptotic point Z, at $X = -\frac{1}{2}$, $Y = -\frac{1}{2}$ (Fig 16.3).

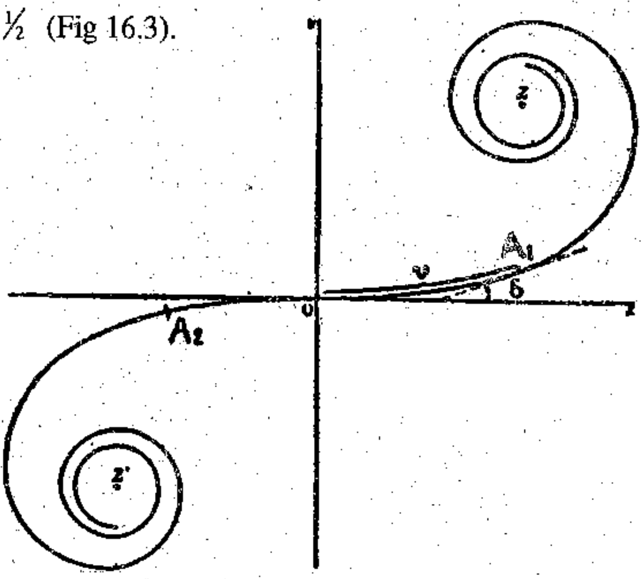


Fig 16.3 Cornu's spiral

The reason for choosing V_{icap} instead of h as the parameter is that V_{cap} is a universal parameter and can be calculated for a given problem. Fig 16.3 represents such a universal curve. Let us now see how the Fresnel integrals and the cornu spiral are used to find intensity at points in the observation screen OS. (Fig 16.1) First let us find the intensity at the point P on OC, just at the edge of the geometricoptical shadow. To find the resultant amplitude at P we must know the limits of values of h corresponding to the open part of the aperture. Here one end is at h=O and the other at $h=\infty$ (beyond Q). So the corresponding resultant vector on the cornu spiral has co- ordinates x=0, y-0 This is the origin O.

$$x = \int_0^\infty \cos^2\left(\frac{\pi}{2}v^2\right)dv, \quad y = \int_0^\infty \sin\left(\frac{\pi}{2}v^2\right)dv$$

This is the asymptotic point Z

The resultant amplitude $\infty O Z \infty \frac{1}{2}$

Hence resultant intensity $\infty\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ To understand the meaning of this fraction let us find the intensity at P when the straight edge is not present and the entire wave front moves unobstructed. So h varies from $h = -\infty$ to $h = +\infty$ and the resultant is the line joining the two ansymptotic points Z & Z_1, Here CD=1. So we see that

Intensity at P with straight edge presents= 1/4

Intensity due to entire wave =1

Thus the intensity at the edge of the shadow on observing screen is one-fourth of the maximum intensity. Now let P move down in the observation screen by a distance h_1 to p_1 (Fig 16.4). The foot of the perpendicular from p' on QOS is O'. To find the intensity at p' we again use the cornu spiral, noting the change in origin from O to O'. So the A_1 representing the upper end of point the straight edge is at

$$v = v_1 = +h_1\sqrt{\frac{2}{\lambda s}} \qquad (16.3)$$

and the second point as before corresponds to $v=\infty$ (point z). So the length is A_1C(Fig16.3). This shows clearly that as the observation point moves into the geometric shadow region, the intensity increases. As h_1 increases so does v_1 and the lower point A_1 moves up the spiral, while z_1 remains constant (corresponding to $v=\infty$). Thus we find light intensity proportional to $(AZ)^2$ in the shadow region-diffraction. For larger values of v_1 A_1 gets close to A and intensity fall to zero deep inside the shadow.

Now let us move p up to p″ (Fig 16.4). We use the cornu spiral with

$$v_2 = -h_2\sqrt{\frac{2}{\lambda s}}$$

So the two extreme values of v in the wavepart are now v_2 in the —ve or lower region of the spiral.

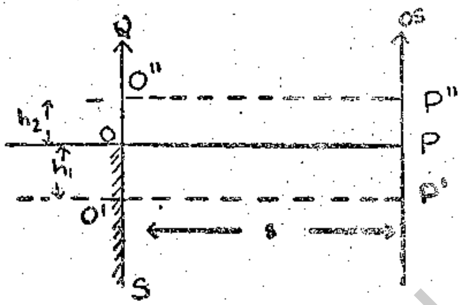


Fig.16.4

The upper limit is still $v=\infty$. (Point Z). As p″ moves up the point A_2 on the spiral moves down towards the lower asymptotic point Z, the length representing the resultant amplitude at p″, increases and decreases as A_2 moves along the turns of the spiral. That is, the resultant intensity just outside the geometrical shadow goes through maxima and minima, and the diffraction pattern shows ring close to p towards p″. Far out, again the intensity reaches the value in the absence of the straight edge.

16.4 SUMMARY

Fresnel diffraction at a straight edge is explained taking plane wave front to incident outer edge. Formulae for the amplitude and phase of waves reaching the screen from elements in the wave front are developed.

16.5 SAMPLE EXAMINATION QUESTIONS

I **Answer the Following Questions in About 30 Lines.**

1. Derive the formula for the amplitude and phase of waves reaching the screen from elements in Fresnel's diffraction at a straight edge.

   Explain the Fresnel diffraction pattern due to an opaque strip qualitatively.

II **Answer the Following Questions in About 10 Lines.**

Discuss Fresnel's theory of diffraction applicable to plane wavefront incident on a straight edge.

UNIT-17 PLANE DIFFRACTION GRATING MEASUREMENT OF WAVELENGTH OF LIGHT

Contents

17.1 AIMS AND OBJECTIVES

This unit explains the construction and use of a plane diffraction grating.

It discusses the preparation of a ruled grating and replica grating.

After going through this unit you will be able to

(1) measure the wavelength of light using a grating

(2) understand how ghosts are formed in a grating spectra

17.2 INTRODUCTION

Already we have studied Fresnel and Fraunhofer type of diffraction let us consider the diffraction patten produced by several parallel slits of equal width b equally spaced the at distance a.

In this unit we shall discuss construction and use of a plane diffraction grating.

17.3 FRAUNHOFFER DIFFERACTION PATTERN OF SINGLE, DOUBLE AND MULTIPLE SILTS

In practice, the Fraunhofer diffraction pattern of single, double and multiple, slits is obtained by an arrangements similar to that shown in Fig. 17.1. The slits are actually rectangular openings (Apertures) with one side much smaller than the other. In multiple slit arrangements, such rectangular apertures are seperated by opaque strips (or "bars"). This set of parallel "slits" and "bars" makes up the "plane diffraction grating". Diffraction patterns obtained by plane diffraction grating in an arrangement are shown in Fig 17.1 photographic plate placed in the focal plane of lens L_2. If the incident light contains several wavelengths the corresponding principal maxima appear as a set of sharp "lines" on the photographic plate. Hence photograph shows a spectrum of the incident light, and we can calculate the wavelength using the formula for the principal maxima (also known as the grating formula)

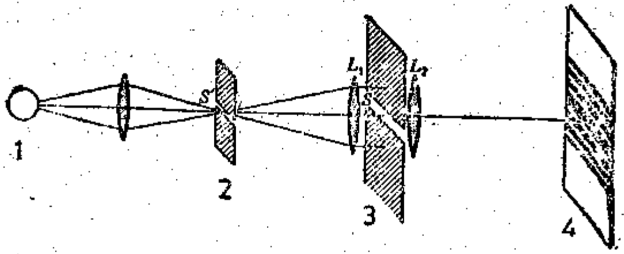


Fig 17.1 Fraunhoffer diffraction pattern of single double and multiple slits
1.Source 2.Source slit 3. Diffracting slit 4.Screen

The slit and bar grating is never used in specutroscopy, where grating with N of the order of thousands of slits per cm, is required. Ingenious methods have been exployed to produce such fine "slits" with small spacing.

The first practical grating was made by Fraunhofer in 1819. Greatly improved gratings were produced by H.A.Rowland in 1882. These are the "ruled grating". A diamond point moves on a glass surface and produces a groove. A screw then moves the diamond point to the next position and an identical groove, parallel to the first one, is now ruled on the glass. This process continues till the required number of lines have been ruled on the glass surface. The grooves, which scatter the incident light, act like opaque "bars", and the portion of glass in between the grooves as "slits". In a perfect grating made by the ruling engine, all the grooves must be of the same shape and size and their seperation must be accurately maintained. This placed a severe demand on the accuracy of the screw and the ruling engine. Rowland succeeded in constructing a ruling engine of high precision. He was able to rule about 14,000 grooves per inch. (i.e.$(b+d) \cong 1.7 \times 10^{-4}$ cm). Each grating was about 15cm. wide.

It was found later that one can rule the groves on a polished metal surface (e.g.Aluminium) and then take a cast of it in a plastic material. The cast is called a replica grating and is usually mounted on a plane glass plate. Such gratings are commonly used in undergraduate laboratories. The art of making gratings of high quality is now very much advanced, thanks to progress in electronics and mechanical devices. As one can imagine, high quality gratings used in sepetroscopy are very expensive. We must mention here a new method of making gratings, developed over the last fifteen years. This uses the principles of "holography", a branch of Modern Optics. Gratings made this way are called holographic gratings and are considerably cheaper than comparable gratings made by the ruling engine.

A simple apparatus for measuring the wavelength of light, using a grating, is the spectrometer. You have learnt about the spectrometer. You have learnt about the spectrometer in experiments done with glass prisms, measuring the angle of minimum deviation and calculating the refractive index. In the spectrometer we have a collimator, which produces a parallel beam of light, using a narrow slit and a convex lens. The parallel beam falls on the grating, which is placed in a holder, mounted on the prism table. The diffracted light is seen through the telescoope, which is previously "focussed" for parallel rays. In the absence of the grating, when the telescope is brought in line with the collimator, one sees a sharp image of the collimator slit. The slit should be adjusted as narrow as possible. Now the grating is placed in position, such that the beam from the collimator is incident normally on the grating. You will learn now to do this accurately, in the physics laboratory. Also, you must make sure that the "lines" in the grating are parallel to the slit of the collimator.

Now, on moving the telescope away from the direct position, one can see the diffracted light. Let us assume that the incident light is monochromatic, with wavelength λ. We shall then find sharp images of the slit for various values of θ such that.

$$(b+d)\sin\theta = m\lambda \tag{17.1}$$

In other words we are looking at the principal maxima, produced by the grating. The direct image of slit, seen through the telescope when it is in line with the collimator is the principal maximum with m=0. On either side of this we have images of the slit, corresponding to the principal maxima with orders $m = \pm 1, \pm 2$. Each principal maximum looks like a sharp image of the collimator slit, because N, the number of slits, in the grating is very large.

If the source of light is mercury discharge tube, the central image looks whitish. But as we move away, a spectrum of mercury lines is seen in the first order (m=1). For higher values of θ another spectrum may be seen with m=2. To calculate the wavelength of a "line" in the spectrum, we must measure θ,the angular seperation between the direct image and the line in order m, using the usual setting of the cross wire in the telescope in the usual way. The value of (b+d), the "grating constant" is supplied by the manufacturer. The number, usually printed on the grating, is the number of "lines" per unit length and (b+d) is equal to the inverse of that number.

In the experiment described above, the rays are incident normally on the grating. In the general case, the incident ray makes an angle i with the normal to the grating (Fig 17.2)

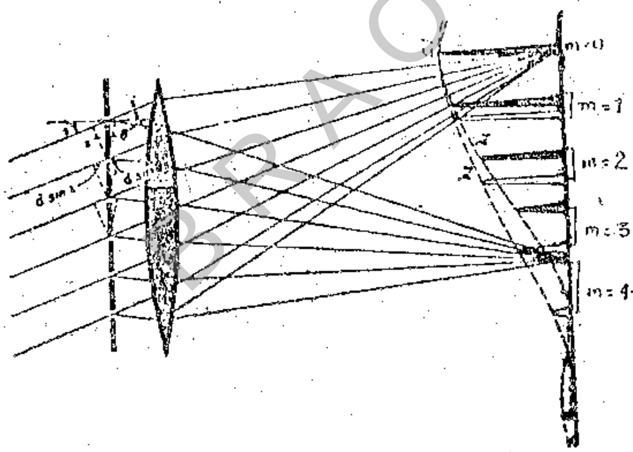


Fig.17.2 Diffraction Spectra in various orders

It is easily seen that the path difference between ray diffracted by adjacent slits is now (b+d) (sin i+sin θ). This path difference is zero which the diffracted rays proceed in the same direction as that of the incident ray. In this case

$$(b+d)(\sin i + \sin\theta) = 0 \tag{17.2}$$

and $\theta = -i$

Note the sign convention followed here. i and θ are taken as positive when they lie "below" the normal to the grating as shown in Fig.17.2. The equation for a principal maximum is

$$(b+d)(\sin i + \sin\theta) = m\lambda$$

Fig.17.2 also shows the diffraction spectra in various orders and their relative intensities.

Check your Progress 1

What is grating constant?

Let us now see what happens when we place two gratings G_1 and G_2 one behind the other on the prism table. Obviously, the light diffracted by the grating G_1 into varous orders, will be diffracted again by G_2. Each principal maximum due to G_1 acts like the incident beam for G_2 and through the telescope one sees a lot more diffraction lines, surrounding the spectral lines produced by G_1. This is the basis of the observation of "ghost" lines in the diffraction produced by imperfectly milled gratings. In a perfect grating the spacing (b+d) remains constant for all lines and each grove is exactly like the other. In practice this is not so. Due to faults in the screw of the ruling engine and the diamond point, the exact periodicity and identy of groves is not maintained. For example, the engine may rule every fifty line a little deeper. This set forms a grating with a (b+d) of its own. It practically acts like the G_2 grating in the imaginary experiment described above and produces its own "lines" in the spectrum. These lines are called "ghosts". In sepetroscopy, special attention has to be paid to detect these "ghosts" as they may cause confusion about the actual spectrum of the incident light.

In spectrographs which measure the wavelength of light with high precision, the " transmission gratings", as we have described above, are hardly used. Reflection gratings are usually employed. Lines are ruled on a polished metal surface. Diffraction of light takes place when light is incident on the grating. One finds light "reflected", in several directions for the same direction of the incident light, corresponding to the principal maxima, which obey the grating equation. Lines may be ruled on a concave reflecting surface (the concave gratings). Which does not need any lenses to focus the diffracted rays. This allows the recording of spectra in the ultraviolet region, where ordinary lenses cannot be used because of their absorption.

Check your Progress 2

How many kinds of gratings are there and what are they ?

17.4 SUMMARY

A two-dimensional slit and bar grating is described. Preparation of a ruled grating and replica grating is discussed. Measurement of wavelength of light using a grating is described. Reflection gratings are mentioned.

17.5 MODEL ANSWERS

Check your Progress 1

The number of lines per unit length is called the grating constant, the inverse of this number is equal to (b+d) in the grating equation.

Check your Progress 2

There are 2 kinds of gratings. They are transmission grating and reflection grating.

17.6 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in Above 10 Lines.

1. Discuss the effect of increasing the slits on the diffraction pattern.

2. What are transmission gratings and reflection gratings?

II. Solve the Following Problems.

1. The angle of incidence on a grating with 5000 lines per cm. is changed from 0° to 90°. Plot the variation in the angle of deviation (D) of the ray diffracted in first order from the direct ray. The wavelengths of light is 4000 A°.

2. In the above example find the relation between the angle of minimum deviation and wavelength.

3. A plane transmission grating having 6000 lines all centimeter is used to obtain the spectrum of light from a radium light in the second order. Find the angular separation between the sodium lines whose wavelengths are 5890 A° and 5816A° [3^1]

4. Light is incident normally on a grating with 250 lines per milli meter. A second order spectral line is observed at 18° with the central zero order image calculate the wavelength of the spectral line [6.18×10^{-8}]

5. Light of wavelength 530×30^{-9} m fals on a transparent grating with a period of 1.5 MM. Find the amgle relative to the grating normal at which make of highest order is observed provided the light falls on the grating (a) at right angles and (b) at an angle of 60° to the normal

   [(a) 0 =$44^6$58'(b)0=1]

6. Light of wavelength 5460A° is normally incident on a grating having 3000 lines per centimeter. What is the angle of diffraction in the 1st order

   [$9^o 26^1$]

UNIT-18 RESOLVING POWER AND DISPERSION OF A GRATING

Contents

18.1 AIMS AND OBJECTIVES

This unit discusses the dispersion and resolving power of a diffraction grating. it determines the limits of the resolving power of an optical instrument by applying Rayleigh's criterian. After going through this unit you will be able to determine the resolving power of a grating.

18.2 INTRODUCTION

When any of the optical instruments like the telescope, the miscroscope, prism spectrometer and a grating spectrometer etc. is in use, information is conveyed by means of light from the object of the detector. In this process, there are mainly three stages, at any or all of which information may be lost leading to the degradation.

(i) Whether the light is capable of conveying the information depends on the relation of the wavelength to the size of detail. This wavelength difficulty is fundamental and has led to the development of electron microscopes and x-ray diffraction etc. In all these cases, the use of wavelength much shorter than those of light allows the information to be obtained at distances which are 10^3 to 10^4 times small than optical wavelengths.

(ii) There will be imperfections in the optical system: geometric aberrations which can be avoided by good design and effects of diffraction within the , instrument are unavoidabel leading to the basic limit of performance. Diffraction must always occur because light on it's way from a point on the object to a point on the image passes through vaiours stops or apertures, so that the image is made up of many superimposed diffraction patterns.

(iii) The final image of any system will be formed on some detecting device such as the retina of the eye, a photographic plate or the surface of a television camera, etc. Such devices will have some sort of grid of detectors which will be of finite spacing and will be unable to detect detail smaller than that.

18.3 RAYLEIGH'S CRITERION

Consider an example of the telescope used to bserve two stars close together. In the focal plane the best possible images (instead of being points) are the two so called Airy diffraction patterns, each being a circular spot surrounded by alternate bright and dark rings. If the angular separtion of the two stars is θ then the centers of the two diffraction patterns are seperated by a distance of θ in the focal plane. This distance can be adjusted to any size simply by increasing f, which may be done by actually altering the focal length of the objective lens. However, this increase in linear separation does not help in resolving the two stars, because the scale of their diffraction patterns increases in its original proporation. The distance to the first minima from the central maximum is given by f θ ≅ 1.22(λ/d)f, where d is the diameter of the objective (lens) so that changing f simply scales the system up or down without altering the relative size of the diffraction patterns and their separation. It is important to appreciate clearly the difference between altering the magnification which simply changes the size of the image and increasing the resolution which allows more details to be seen as illustrated in figure 18.1.

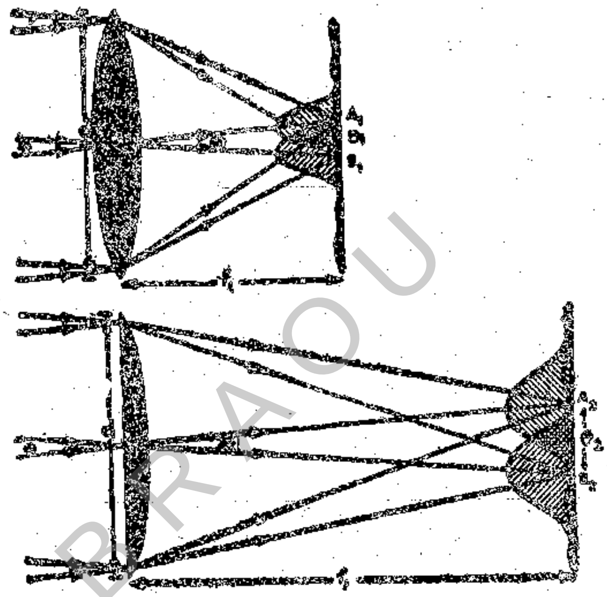


Fig.18.1

To make the diffraction patterns smaller, without altering their separation, it is necessary to increase d, the diameter of the objective lens. It is obviously harder and harder to distinguish two stars as their diffraction patterns get closer together. Therefore, it is useful to have some criterion to measure the resolving power of optical instruments. Rayleigh suggested that the angular resolution of a telescope should be defined as the angle between two stars when the maximum of the diffraction pattern of one falls exactly on the first minimum of the other. That is to say, angular resolving power = $\theta_R = \sin^{-1}\left(\frac{1.22\lambda}{d}\right)$. Since the angle is very samll. we can write $\theta_R = \frac{1.22\lambda}{d}$ is known as Rayleigh's criterion which is illustrated in figure 18.2 Rayleigh's criterion is also usually applied for a slit or rectangle, in which case the maximum if first pattern falling on the first minimum of the second which are now separted by λ/n where 'a' being the slit width.

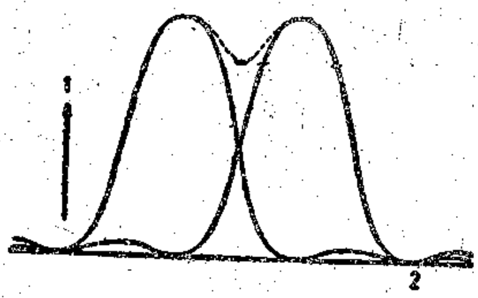


Fig .18.2

Dispersion and Resolving Power of Grating

18.4 DISPERSION OF A GRATING

The dispersion D of a grating, is a measure of the angular separation produced between two incident monochromatic waves whose wavelengths differ by a small wavelength interval.

$$\text{i.e., } D = \frac{d\theta}{d\lambda}$$

From the usual grating formula, we have

$$d \sin \theta = m\lambda; \; m = 0, 1, 2 \ldots \ldots \text{ an integer} \qquad (18.1)$$

Where d is the distance between the rulings and the integer m is the order of principal maximum. Treating λ and θ as variables, the differentiation equation of 18.1 leads to

$$\cos\theta \, d\theta = \frac{m}{d} d\lambda \qquad (18.2)$$

Therefore the dispersion $D = \frac{d\theta}{d\lambda} = \frac{m}{d \cos\theta}$ (18.3)

18.5 RESOLVING POWER OF A GRATING

The resolving power R of a grating determines the smallest difference in wavelength $\Delta\lambda$ that a given grating can resolve in the mth order; the quantity $R=(\lambda/\Delta\lambda)$i.e. the inverse ratio of the difference in wavelength to the average wavelength of two radiations) is called the resolving power. It is usually determined by the same Rayleigh's criterion that is used to determine the resolving power of a lens.

If two principal maxima are to be barely resolved, they must have an angular separation $\Delta\theta$ such that the maxium of one line coincides with the first minimum of the other, i.e., the angular separation between principal maxima whose wavelengths differ by $\theta\lambda$ is given by

$$\Delta\theta = \frac{m\Delta\lambda}{d\cos\theta} \text{ (Using equation 18.3)} \qquad (18.4)$$

The Rayleigh criterion requires that this be equal to the angular separation between a principal maximum and it's adjacent minimum. for m^{th} order, this is equal to

$$\Delta\theta = \frac{\lambda}{d\,N\cos\theta} \qquad (18.5)$$

Where N is the total number of slits. Using equations 18.4 and 18.5 we get

If θ is the angular position of the maximum of the order m and $\theta + \Delta\theta$ that of the adjoining minimum, we have

$$\sin\theta = \frac{m\lambda}{d}, \sin(\theta + \Delta\theta) = (m + \frac{1}{N})\frac{\lambda}{d} \qquad (18.5a)$$

$$\sin\theta\cos\Delta\theta + \cos\theta\sin\Delta\theta = \left(m + \frac{1}{N}\right)\frac{\lambda}{d}$$

Since $\Delta\theta$ is very small, cos $\Delta\theta = 1$ and sin $\Delta\theta = \Delta\theta$

$$\text{or } \Delta\theta = \frac{m\lambda}{dN\cos\theta}$$

$$R = \frac{\lambda}{\Delta\lambda} = Nm \text{ just as 18.5a} \qquad (18.6)$$

Check your Progress 1

What the dispersive power of a grating measures?

As expected the resolving power is zero for the central principal maximum (m=0); (all) wavelengths being undeflected in this order.

To distinguish between dispersion and resolving power, some characterstics of three gratings (each illuminated with same light of $\lambda = 5890\,A°$, the diffracted light being viewed in the first order) are tabulated and illustrated by figures 18.3

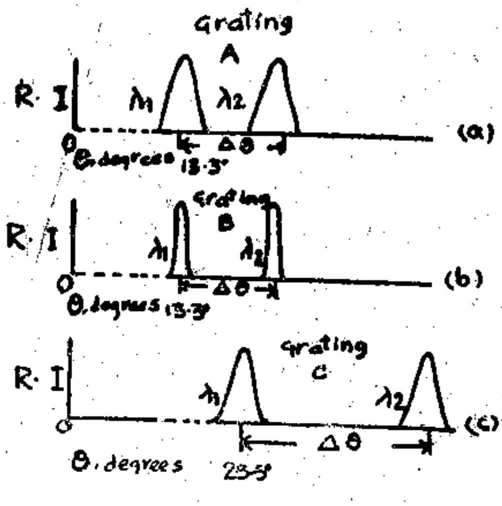


Fig 18.3

SOME CHARACTERSTICS OF THREE GRATINGS

(λ = 5820 A°, m = 1)

Grating	N	d(A)	θ	R	D, 10^{-3}degrees/A
A	10,000	25,400	13.3	10,000	2.32
B	20,000	25,400	13.3	20,000	2.32
C	10,000	13,700	25.5	10,000	4.64

It may be noted that gratings A and B have the same dispersion and A and C have the same resolving power.

18.6 SUMMARY

The dispersive power of a grating is given by the formulae $D = (d\theta / d\lambda)$ Where $d\theta$ is the angular seperation between two incident monochromatic waves is the wavelength. The Resolving power of a grating is

$$R = \frac{\lambda}{\Delta\lambda}$$

18.7 MODEL ANSWERS

Check your Progress 1

The dispersion of grating measures the angular seperation produced between two incident monochromatic waves whose wave length differ by a small wave length interval.

18.8 SAMPLE EXAMINATION QUESTIONS

I Answer the Following Questions in About 30 Lines.

1. Suppose a large adjustable rectangular aperture is placed just behind a grating on the spectrometer table. The width of this aperture is now gradually reduced so that a smaller and smaller portion of the grating passes the light. what changes in the diffraction spectrum will you observe?

2. The relation R=Nm suggests that the resolving power of a given grating can be made as large as desired by choosing an arbitrarily high order of diffraction. Discuss.

3. Explain the phenomenon of the dispersion of a grating.

II. Solve the Following Problems

1. How many rulings must a grating have if it barely to resolve the sodium doublet in the fourth order ($\lambda_1 = 5895.9 A°$ and $\lambda_2 = 5890$ A°)

2. A grating has 6000 rulings/cm and is 6.0 cm wide

   What is the smallest wave length interval that can be resolved in the third order at $\lambda = 5000$ A° ?

3. What must be the minimum number of lines per centimeter in a half inch width grating to resolve the D_1, D_2 lines of wavelengths 5896 A° and 5890 A° [775]

4. A diffraction grating having 4000 lines in a centimeter is used for normal incidence. Calculate the dispersive power of the grating in the third order spectum in the wavelength region of 5000 A° [10,000]

5. Using a plane transmission grating having 12,700 lines perinch in normal incidence position the angular seperation two lines is found to be 2.5 minutes. Find the difference in the wavelength of the two lines in the sodium light. [5.873A°]

6. A grating contains 40,000 lines for 8 cm. What is the smallest wavelength interval that can be resolved in the third order in the region 54000 A°. [0.02A°]

UNIT-19 : X-RAY DIFFRACTION

Contents

19.1 AIMS AND OBJECTIVES

This unit explains the production of X ray and their use for the determination of cyrstal structure.

The cyrstal structure is determined with the help of X-ray diffraction pattern.

After going through this unit you will be able to determine the wavelength of the X-ray beam.

19.2 INTRODUCTION

In the year 1895, a german physicist named Wilhelm Roentgen discovered an unusual type of radiation which he called X-rays because their nature was unknown at that time. These rays are known to be of electromagnetic nature. Laue, in 1912, was able to show that the X-rays can be utilized for the diffraction by crystals. The work of Bragg has clearly demonstrated that the X-rays can be used not only for the crystal structure determination but also for the X-ray wavelength determinations.

19.3 PRODUCTION AND NATURE OF X-RAYS

Unlike ordinary light x-rays are invisible. But they travel in straight lines and effect the photographic film in the same way as light. Now, we know that X-rays are a part of electromagnetic radiation of exactly same nature as light and the wavelength range is about 0.1 to 10 A. The X-rays used in the crystal diffraction studies have wavelengths in the range of 0.5 to 2.5 A.

X-rays are produced when electrically charged particle of sufficient kinetic energy is rapidly decclerated. In practice the high energy electrons are allowed to strike the anode (target) in an X-ray tube for producing X-rays. For this purpose, the voltage required is of the order of tens of thousands of volts (generally 10 to 40 KV) with a current of the order of a few milli amperes (mA).

Hard X-rays are of short wavelength where as the long wavelength radiation is known as soft X-rays. Characteristics of radiation, including it's intensity and wavelength depends on the chemical composition of the target material and the voltage applied to the tube. The X-ray spectrum from the target consists continuous radiation superimposed by characterstic radiation. While continuous spectrum is caused by the rapid decelaration of electrons in the target, the origin of the charactersistic radiation lies in the atoms of the target material itself. Figure 19.1 shows how X-rays are generated when electrons from heated filament F, accelerated by potential difference V, are brought to rest on striking metallic target T.

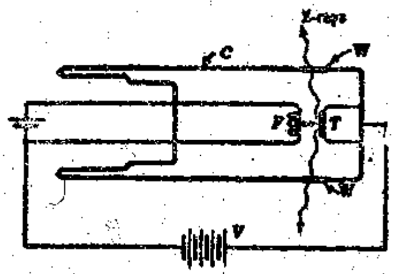


Fig 19.1 X-Ray Tube

Check your Progress 1.

What are hard and soft X-rays?

Check your Progress 2.

What are characterestic X-rays

19.4 THE GEOMETRY OF CRYSTALS

We must now consider the geometry and structure of crystals in order to discover what there is about crystals in general that enables them to diffract X-rays. A crystal may be defined as a solid composed of atoms arranged in a periodic pattern in three dimensions. In thinking about crystals, it is often convenient to imagine a three dimensional array of points repeating themselves at regular intervals and assume that the crystal is obtained by adding an identical group of atoms to each point such that each point has an identical environment. The array of points is called the crystal lattice and the points are called lattice points. The lattice points divide the space into identical cells which are in the form of parallel pipeds. The smallest possible cell by the repititions of which the crystal can be built up is called a primitive cell. We can call a primitive cell as a unit cell also.

Since all the cells of the lattice shown in Fig 19.2 (a) are identical, we may choose any one, for example the heavily outlined one, as a unit cell. The size and shape of the unit cell can in turn be described by the three vectors a, b and c. These vectors define the unit cell and are releated to crystallographic axes.

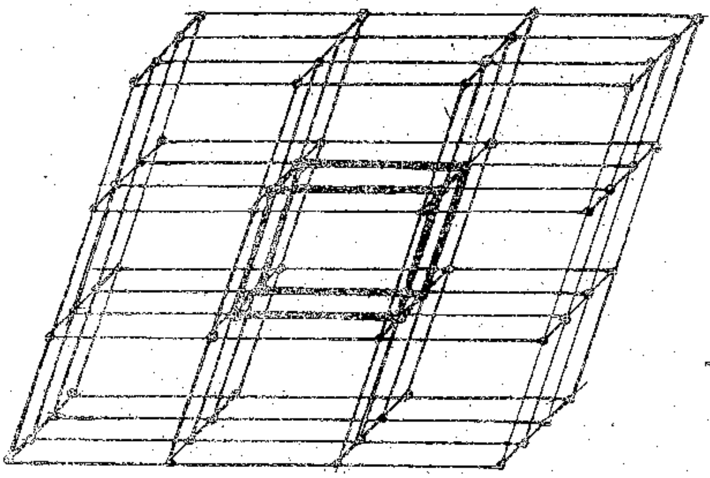


Fig.19.2 (a) A Point Latice

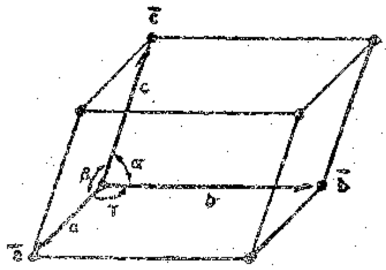


Fig.19.2 (b) A Unit cell

They may also be described in terms of their lengths (a, b & c) and the angles between them (α, β, γ). These lengths and angles are the lattice parameters of the unit cell (Fig19.2(b).

By deviding space by three sets of planes, we can produce unit cells of various shapes, depending on how we arrange the planes. For example, if the planes in the three sets are equally spaced and mutually perpendicular the unit cell is cubic (a=b=c and $\alpha = \beta = \gamma = 90°$) From geometry, it turns out that only seven different systems of cells are necessary to include all the possible point lattices. These correspond to seven crystal systems into which all crystals can be classified. These systems are listed.

System	Axial lengths	angles
Cubic	$a = b = c$	$\alpha = \beta = \gamma = 90°$
Tetragonal	$a = b \neq c$;	$\alpha = \beta = \gamma = 90°$
Orthorombic	$a \neq b \neq c$	$\alpha = \beta = \gamma = 90°$
Rhombohedral (or) Trigonal	$a = b = c$	$\alpha = \beta = \gamma \neq 90°$
Hexagonal	$a = b \neq c$	$\alpha = \beta = 90°\ \gamma = 120°$
Monoclinic	$a \neq b \neq c$	$\alpha = \gamma = 90° \neq \beta$
Triclinic	$a \neq b \neq c$	$\alpha \neq \beta \neq \gamma$

Example - 1:M

Determine the interatomic spacing of a NaCl crystal if the density of NaCl is 2.16×10^3 kg/m^3 and the atomic weight of sodium and chlorine are 23.00 and 35.46 respectively.

The molecular weight of NaCl is 23.00 + 35.46 = 58.46. The number of molecules per 58.46 kg of NaCl is

$$= \frac{1\,\text{kmol}}{58.46\,\text{kg}} \text{x} 6.025\text{x}10^{26} \frac{\text{molecules}}{\text{kmol}} = \frac{6.025\text{x}10^{26}}{58.46} \frac{\text{molecules}}{\text{kg}}$$

Since there are two atoms per molecule, we have

$$\frac{\text{Number of atoms}}{\text{volume}} = \frac{\text{Numberofatoms}}{\text{mass}}, \frac{\text{mass}}{\text{volume}}$$

$$= \frac{2 \times 6.025 \times 10^{26}}{58.46} \frac{\text{atoms}}{\text{kg}} \times 2.16 \times 10^{3} \frac{\text{kg}}{\text{m}^3}$$

$$= 4.45\text{x}10^{28} \frac{\text{atoms}}{\text{m}^3} \tag{19.1}$$

To relate this to the interatomic spacing a, consider the NaC1 unit cell shown in figure 19.3 The vollume of the cube is $(2a)^3$. As to the number of ions assigned to the cube, there are:

8 corner ions, each shared by 8 of the cubes ;

12 edge ions, each shared by 4 of the cubes ;

6 face ions, each shared by 2 of the cubes ;

and 1 unshared centre ion.

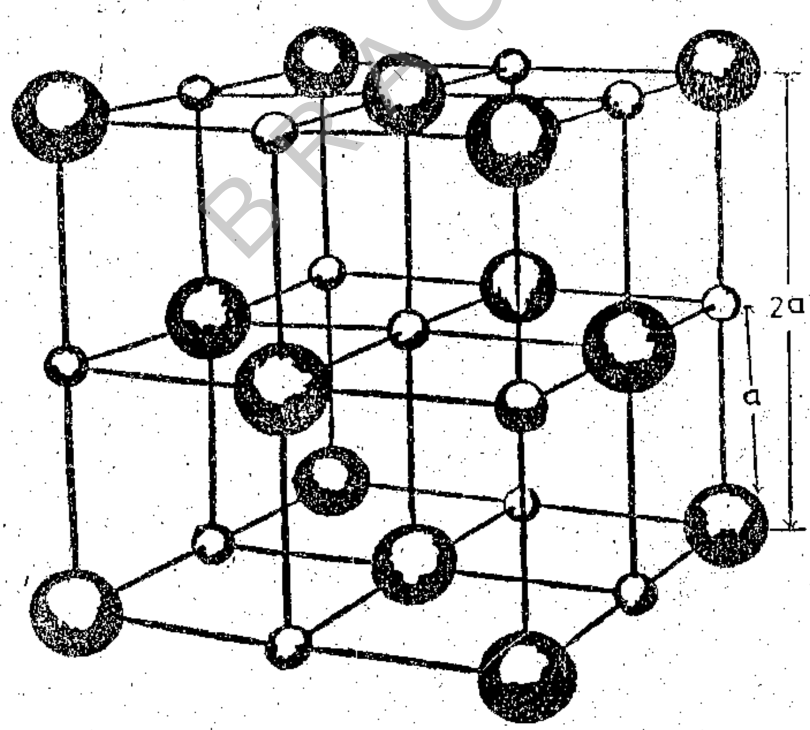


Fig 19.3 Unit Cell of Nacl Crystal

Thus the number of ions $1/8\,(8) + \frac{1}{4}(12) + \frac{1}{2}(6) + 1 = 8$

$$\text{and} \quad \frac{\text{Number of ions}}{\text{volume}} = \frac{8}{(2a)^3} = \frac{1}{a^3} \tag{19.2}$$

Equating 19.1 and 19.2, we get

$$\frac{1}{a^3} = 4.45 \times 10^{28} m^{-3} \text{ or } a = 2.82 \times 10^{-10} m$$

$= 2.82$ A.

19.5 X-RAY DIFFRACTION

Having known something about X-rays and crystals, we wish to understand whether an incident X-ray beam will be diffracted by the crystal, and if so, under what conditions. A diffracted beam may be difined as a beam composed of large number of scattered rays by atoms in the crystal reinforcing one another. Therefore, diffraction is essentially a scattering of electromagnetic radiation.

Laue conceived the idea of using ordered arrangement of atoms or molecules of a crystal as a grating for the investigation of X-rays, because it is impossible to construct a grating with appropriate spacings.

He showed that if a pencil of X-rays was made to traverse a crystal, diffracted pencil would be formed, arranged about the primary beam in a regular pattern according to laws which he formulated (beyond the scope of present discussion) A photographic palte placed perpendicular to the primary rays and behind the crystal would show a strong central spot, where the primary rays struck it, and other spots arranged in regular fashion round the central spot in the places struck by the diffracted pencils. The experiment was carried out by Messrs. Friedrich and Knipping in the spring of 1912, and was a brilliant succes. The arrangement is shown in Fig 19.4. The beautiful geometrical arrangement of the pattern is in reality a manifestation of the regularity of crystal structure.

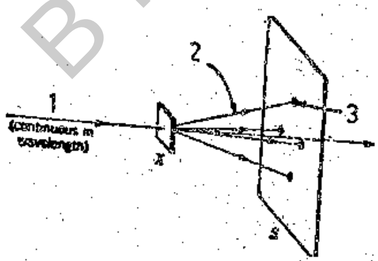


Fig. 19.4 Laue Diffraction

19.6 BRAGG LAW

The interference of X-rays is caused by the crystal lattice in the following manner. The rays from the anode of a X-ray tube impinge on a crystal. The electrons of the atoms begin to oscillate and thus scatter the X-rays. The scattered rays are called secondary and their presence can be detected only if they constructively interfere. W.H.Bragg and W.L.Bragg (Father & Son) worked out a formula for this phenomenon.

Let us imagine a section through a crystal lattice as shown in Fig.19.5

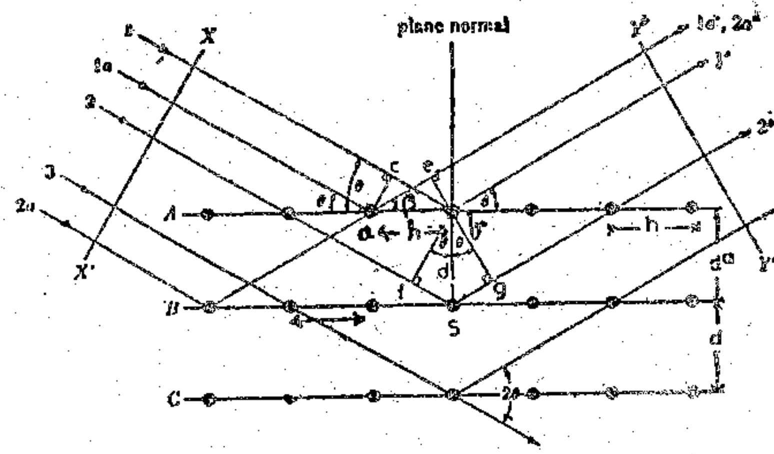


Fig 19.5 Diffraction of X-rays by a crystal

A plane wave that lies in the place of the figure falling on a member of the family of places of atoms having inter planar spacing d. A bundle of parallel X-rays (primary rays) strike this set of planes at an angle θ called the glancing angle, which is the angle between the direction of the rays and the plane of atoms (and not the normal to this plane as in optics). Consider a family of diffracted rays from atoms lying in the plane. A of Fig 19.5 making an angle θ with the plane. The diffracted rays will combine to produce maximum intensity if the path difference between adjacent ray is an integral multiple of wavelengths i.e. the path difference after scattering is

$$ae - bc = m\lambda \quad \text{where } m = 0, 1, 2, \ldots \text{ an integer}$$

$$\text{or } h(\cos\beta - \cos\theta) = m\lambda$$

for $m = 0$, $\beta = \theta$ and the plane of atoms acts like a mirror for the incident rays, no matter what the value of θ. That is, the rays scattered by all the atoms in the first plane in a direction parallel to 1, are in phase and add their contributions to the diffracted beam. This will be true for all the planes separately and it remains to find the conditions for constructive interference of rays scattered by atoms in different planes. For example, rays 1 and 2 are scattered by atoms b and s, and the path difference between the rays 1 b 1′ and 2s 2′ is

$$fs + sg = d\sin\theta + d\sin\theta = 2d\sin\theta$$

These rays (1′ and 2′) willbe completely in phase if this path difference is equal to a whole number of wavelengths, i.e.

$$2d\sin\theta = n\lambda \qquad (19.3)$$

Where n is an integer. This relation was first formulated by W.L. Bragg and is known as the Bragg law; n is called the order of reflection; it may take on any integral value consistant with $\sin\theta$ not exceeding unity. Therefore for fixed values of λ and d, there may be several angles of incidence $\theta_1, \theta_2, \theta_3$ at which diffraction may occur corresponding to n= 1, 2, 3...

19.7 DETERMINATION d OR θ

Experimentally the Bragg law can be used in two ways. By using X-rays of known wavelength and measuring θ we can determine the spacing of various planes in a crystal. This is known as structure analysis. Alternatively, we can use a crystal with planes of known spacing d, measure θ and thus determine the wavelength λ_2 of the radiation used.

$$\lambda = \frac{2d \sin\theta}{n} \qquad (19.4)$$

As shown in figure 19.6, X-rays from the tube T are incident on crystal C which may be set at any desired angle θ to the beam by rotation about an axis through 0, the centre of the spectometer circle.

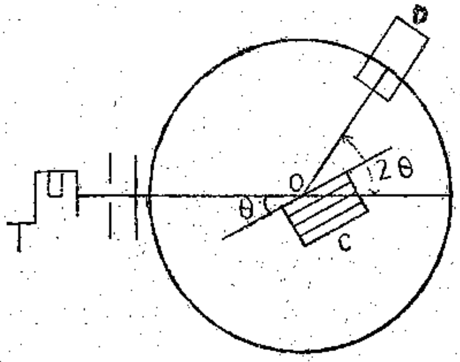


Fig.19.6

D is the detector which measure the intensity of the diffracted X-rays. The crystal is usually cut (or cleaved) so that particular set of reflecting plane a of known spacing is parallel to its surface. In use the crystal is positioned such that its reflecting planes make some particular angle θ with the incident beam, and D is set at the corresponding angle 2 θ. The intensity of the diffracted beam is measured and its' wavelength is calculated from equation (2). Figure 19.7 shows a typical spectrum of Mo.

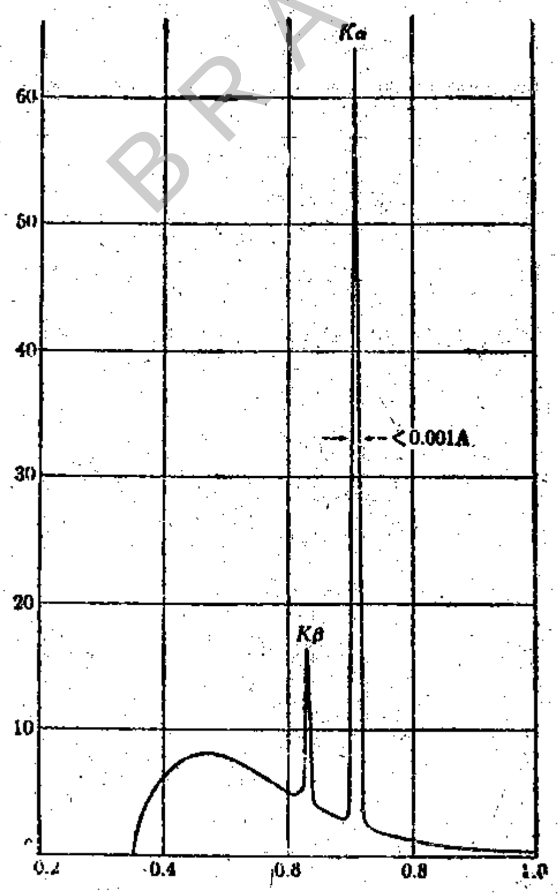


Fig.19.7 Spectrum of Mo (Schematic) line widths not to scale

19.8 SUMMARY

The principles involved in the production of X rays are discussed. Elementary notions of the crystal classification in three dimensions are presented. Use of X-rays for the determination of crystal structure is derived. Application of diffraction principles for the wavelength determination of incident X ray beam is discussed and illustrated with an example.

19.9 MODEL ANSWERS

Check your Progress 1

Hard X-rays are more penetrating and have shorter wave length than the soft X-rays

Check your Progress 2

These are due to jumping of L shell electron to the K shell and M shell electrons to the k shell etc. These are characteristic of the target material.

19.10 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. How are X-rays used in the determination of crystal structures.

2. If parallel beam of X-rays of wavelength λ is allowed to fall on a randomely oriented crystal of any material, generally no intense diffracted beams will occur. Such beams appear if (a) the X-ray beam consists of continuous distribution of wavelengths or (b) the specimen is powder instead of a single crystal. Explain?

II. Answer the Following Question in About 10 Lines.

1. Discuss the nature of X-rays.

2. Mention the different crystal systems into which crystals are classified.

3. Derive Bragg's equation.

III. Solve the Following Problems.

1. At what angles must an X-ray beam with λ=1.10A° fall on the family of planes of spacing 2.52 A° if a diffracted beam to occur.

2. Calculate the wavelengths of X-rays in a band from 0.80 to 1.40A° to obtain diffracted beams associated with a set of planes of spacing 2.75 A°. Assume glancing angle 0.30°.

3. Calculate the glancing angle on the face of a cube of rock salt crystal 2.82A° Corresponding to second order diffraction maximum for x rays of wavelength 0.8442 A°. [17° 28']

4. The spacing between the principal planes Nacl crystal is 2.82 A°. It is observed the

first order Brag reflection of monochromatic X-rays occur at an angle of 10° Calculate the wavelength of X-rays [x=0.98A°]

5. An x ray beam of wavelength 0.0124A° is reflected from sylvine crystal at glancing angle $10^{\circ}.31'$. Calculate the value of d [3.14A°]

19.11 RECOMMENDED BOOKS

1. Physics Part-I and Part II by Resnick and Halliday
2. Optics by P.G. Smith and J.H.Thomson
3. Fundamentals of Optics by F.A. Jenkins and H.E. White
4. Elements of X-ray diffraction by B.D.Cullity
5. X-rays and Crystal structure by W.H.Bragg
6. University Physics by F.W.Sears and M.W.Zemansky
7. Modern College Physics by H.E.White

UNIT-20 : HOLOGRAPHY

Contents

20.1 AIMS AND OBJECTIVES

In this unit we are going to study the production of a hologram and reproduction of the image.

After going through this unit you will be able to

(1) Explain the principle involved in the preparation of a hologram and reproduction of the image.

(2) Apply holography principles in various fields.

20.2 INTRODUCTION

Holography is the technique by which images are produced in a three diamensional form. The images are as true as the objects themselves. This method was originally introduced by Dennis Gabor in the year 1947. We know that the ordinary photograph gives us only a two dimensional image of the object. In ordinary photography we make use of lenses to focus the image on photographic plate. This focusing takes place only in a single plane and all the other planes are out of focus. We record only the intensity distribution prevailing at the plane of the photograph while the light from the object is focused on it. The photographic plate thus records only the intensity variations. But the phase distribution prevailing at the plane of the photographic plate is completely lost.

Gabor suggested a way in which we can record both amplitude of the light wave and phase as well. This can be achieved by combining the two phenomena of interference and diffraction. Interference pattern produced by two or more coherent beams of light can be recorded on a light sensitive medium and is called a hologram. In a hologram intensity and phase of the light wave will be recorded. Hence we can get a three dimension picture. We can have different perspectives of the image when viewed it at different angles. A perfect coherent beam of light is essential to get a perfect hologram. With the discovery of laser, production of holograms has became possible. Holography was received and fully developed by Leith and Uptanics.

20.3 PREPARATION OF HOLOGRAM AND REPRODUCTION OF THE IMAGE

Now we shall try to understand the main principle in preparing a hologram and then reproducing the image from such a hologram.

20.3.1 Preparing a Hologram.

The output from a laser is first seperated into two different beams. The first beam consisting of rays numbered i, i is spread out by means of a lens or a curved mirror. The light reflected by the curved surface will be directed on to the photographic plate PP. This beam is called the reference beam. The second beam from the laser with rays numbered i^1, i^1, i^1 falls on the three dimensional object and illuminates the object. The light scattered by the object falls on the same photographic plate PP. This beam (0-0-0) is called the object beam (O). As the two beams are dirived from the same laser and hence are highly coherent and form a specific interference pattern. The resulting interference pattern is recovered on the photographic plate PP. Now this plate is called the hologram and Contains all the information of intensity and phase required to reproduce the wave front from the object.

The exposed and developed photographic plate will be like a diffraction grating. The grating element will be equal to the wavelength λ of the incedent light.

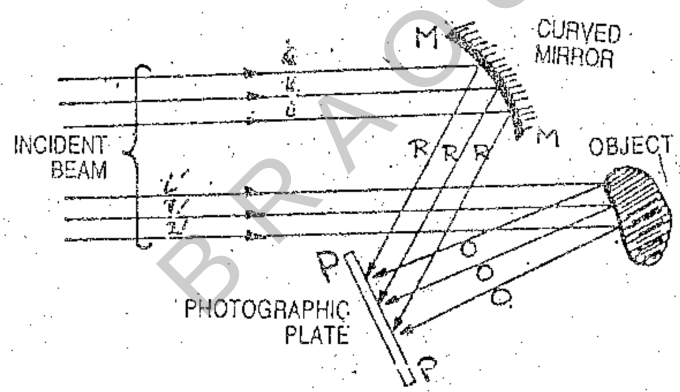


Fig.20.1 Preparing a Hologram

20.3.2 Reproduction of the Image.

To reproduce the image the hologram is now illuminated from a single beam (the reference beam R). from the laser. The hologram now acts as a complex grating and diffracts the light.

The diffraction pattern recreates precisely the wave front emanating from the original three dimensional object. Thus we get a precisely true image of the original wavefront reflected by the object. Looking at the hologram we can see the image in depth.

By moving the position of our eye we can see different perspectives of the object in the image. Two images will be formed. One is real and another is virtual. The real image can be photographed. No lenses will be used. This can be obtained by just placing a photo graphic plate at the position where the real image is formed.

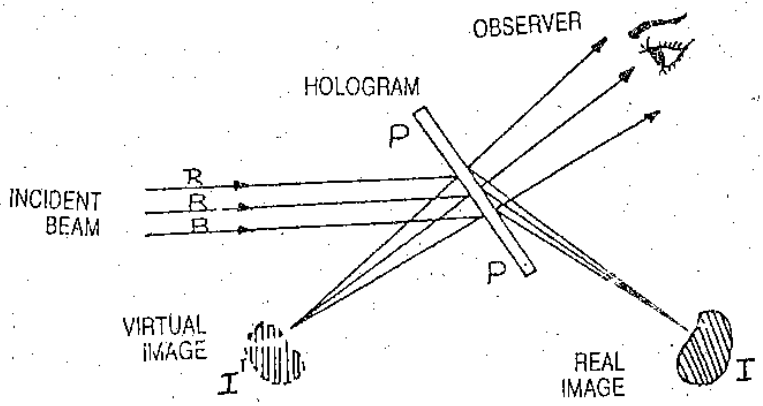


Fig.20.2 Reproduction of the Image from a hologram

20.4 APPLICATIONS OF HOLOGRAPHY.

The principle of holography has many applications in different fields.

20.4.1 Holographic Interferometry

One of the most important applications of holography is in the field of interferometry. Surfaces that undergo deformations due to stress can be studied under optical technique. This special technique is known as "the double exposure technique". In this technique the distribution of strain can be studied in detail in a three dimensional perspective. In the double exposure technique two seperate exposures are made on a single hologram. This hologram reveals deformations, vibrations, strain of surfaces.

20.4.2 Holographic Microscopy

We know that from a hologram we get a three dimensional picture. From this picture we get the information about the depth also. This ability is useful in the study of time varying phenomena that occur in a certain region. In ordinary microscope we can have only a still photograph. If a hologram is recorded of the scene as and when it occurs, the event gets preserved in the hologram. We can focus through the depth of reconstructed image & study the phenomena. Hence with the help of a hologram we can study the transient microscopic events.

20.4.3 Accoustic Holography

We can use holography techniques to study the images formed by sound waves. Ultra sound waves are used in place of sound waves. These ultra sound waves generate the reference beam (R) and the object beam (O). The ultra sound generator is kept inside the water. It emits the reference beam as well as the beam that is directed towards an underwater object. We can observe ripples on the still surface of the water. These ripples are formed due to interference of incident beam and reflected beam. This ripple pattern itself forms hologram.

Accoustic holography can be used to get a three dimensional image of our internal organs. Cohesent ultrasonic beam is split into two parts, the first serves the purpose of a reference beam R and the second part is scattered by our internal organs and forms the object beam (O). These two beams interfere and form a hologram. From the reconstructed three dimensional image we can study in depth the details of the concerned organ.

20.4.4 Other Applications

Spatial filtrations character recognation, long distance holography using microwaves, rain hologram, integral stereogram, focussed image hologram holographic optical elements, phase conjugation etc.

20.5 SUMMARY

Holography is the technique by which images are produced in a three dimensional form. The images are as true as the object themselves. This method was originally introduced by Dennis Gabor in the year 1947. Gabour suggested a way in which we can record both amplitude of the light wave and phase as well. A perfect cohesent beam of light is essential to get a perfect hologram. Laser will be used as Coherent beam.

The principle of holography has many application in different fields like (1) holographic interferometry (2) holographic microscopy (3) accoustic holography etc.

20.6 SAMPLE EXAMINATION QUESTIONS.

I. Answer the Following Questions in About 30 Lines.

(1) Explain how do we prepare a hologram and reproduce the image.

II. Answer the Following Question in About 10 Lines

1. Holographic Interferomentry.
2. Accoustic Holography.
3. Holographic Microscopy.

BLOCK - IX

POLARISATION

UNIT-21 PLANE POLARISATION,POLAROID, POLARIZATION BY REFLECTION

Contents

21.1 AIMS AND OBJECTIVES

This unit explains the phenomenon of polarization on the basis of reflection and double refraction on the basis of Huygen's wave theory.

After going through this unit,

(1) You will learn that the light waves have transverse nature;

(2) You will be able to convert unpolarized light into plane polarized light by using a polaroid;

(3) You can estimate the intensity of the polarized light with the help of law of Malus.

(4) You can find out the refractive indeces of ordinary and extra ordinary rays

(5) Describe the construction and working of a nicol prism.

21.2 INTRODUCTION

The phenomenon of polarization exhibited by light waves established the fact that they are transverse in nature. There are three types of polarized light which can be produced from an unpolarized light. These are linearly polarized, circularly polarized and elliptically polarized. Polarized light has extensive applications in understanding the nature of substances.

When a ray of light is incident on a glass plate, a part of the light energy is reflected, and a part refracted. For an incident ray, there will be one refracted ray. The reflected and refracted rays obey certain laws. An interesting situation arises when light is incident on certain Crystals like Calcite (Ca Co_3). This Crystal produces two refracted rays for a single incident ray. This phenomenon is called double refraction.

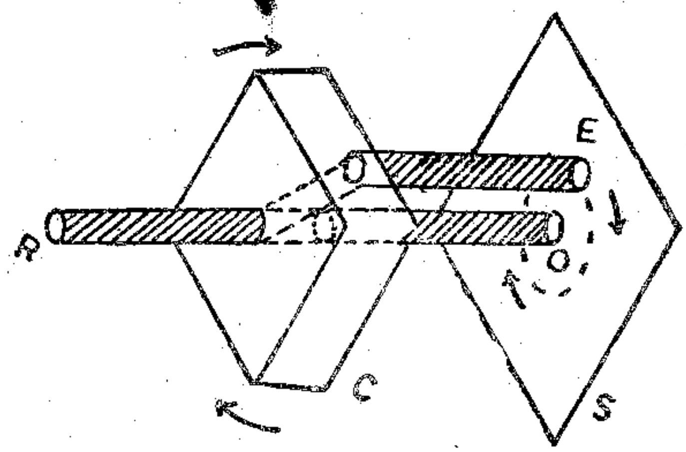


Fig 21.1

Fig.21.2 depicts double refraction. A narrow beam of light is incident normally on any of the faces of a Calcite Crystal. Two spots (O and E) will be observed on a screen placed in the path of the emergent light. If the Crystal is now rotated about an axis parallel to the beam the spot O remains stationary. The other spot E rotates about the axis. Thus, a single incident beam gives rise to two refracted beams. One of the beams (O) correspond to the single beam which would have been transmitted had a substance like glass been used. This beam, which obeys the ordinary laws of refraction, is called the ordinary beam, The other extraordinary (E).

As mentioned earlier, the ordinary beam obeys the Snell's law i.e.

$$\frac{\sin i}{\sin r} \text{ is constant (i=angle of incidence, r=angle of refraction)}$$

For the E beam the $\frac{\sin i}{\sin r}$ value depends on the angle of incidence.

What does this mean? It may be remembered that

$$\frac{\sin i}{\sin r} = \frac{\text{velocity of light in air}}{\text{velocity of light in the medium}} = \mu$$

The constancy of ratio for O beam indicates that velocity of light in the medium is same in all directions. Conversely, the lack of constancy of this ratio for the E beam indicates that the velocity of this Component (E) is different in different directions in this Crystals.

In calcite, double refraction can be observed in all directions excepting one. This particular direction is called the optic axis. It may be concluded that velocities of the two beams are equal in this direction.

Double refraction may be observed in crystalline substances like calcite, quartz, ice, tourmaline, apatite, nitrate of soda, Sorax, mica, selenite, topaz and agaonite. These crystals may be divided into two groups: Uniaxial and biaxial. In biaxial crystals there are two directions along which O and E beams travel with the same velocity whereas in uniaxial crystals there is only one such direction. Of the crystals mentioned above the first six are uniaxial and the rest biaxial.

Check your Progress 1

Distinguish between ordinary and extra ordinary rays.

Check your Progress 2

What is an optical axis of a crystal?

Check your Progress 3

Distinguish between uniaxial and biaxial crystals.

21.3 POLARIZATION

Interference and diffraction phenomena can be explained on the assumption that light energy travels from place to place in the form of waves. Waves could be of different types: Longitudinal, transverse, and surface. What is the type of wave phenomenon involved in propagation of light? All the types of waves mentioned above produce interference and diffraction effects. Polarization gives a clue to the exact nature of the light waves.

According to the electromagnetic theory, light waves like all other electromagnetic radiation, are transverse in nature. The electric and magnetic vectors in a light wave vibrate in mutually perpendicular directions. In addition, these two vectors are perpendicular to the direction of wave motion. Of the two vectors, the electric vector excites our visions. As such it is customary to consider only the electric vector for purposes of discussion.

The end-on-view of an ordinary light beam shown in fig 21.2a illustrates schematically all probable planes of vibration of the electric vector which are perpendicular to the direction of propagation. It is possible to resolve these vibrations along two mutually perpendicular (X, Y) directions as show in Fig. 21.2b.

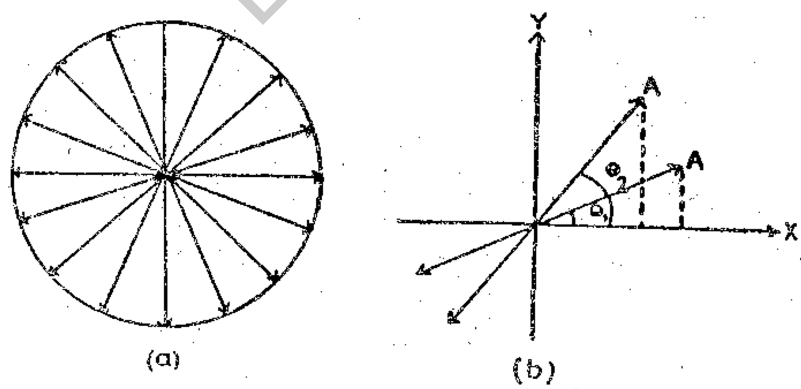


Fig. 21.2

If by some means or other, all the waves in a beam of light are to vibrate in planes parallel to each other, the light is said to be plane polarized. Fig 21.3 illustrates such light. The top diagram(a) represents Plane-polarized light waves travelling to the right and vibrating in a vertical plane. The second diagram(b) represents a ray of plane-polarized light vibrating in a horizontal plane, the plane of vibration. A plane perpendicular to the plane of vibration is called of polarization. These planes are illustrated in Fig 21.4.

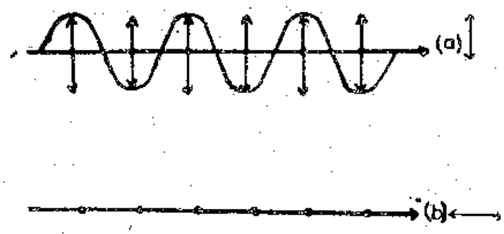


Fig 21.3

In a Plane-polarized light, the vibrations are in one plane.

The electric field vector (and hence the magnetic fiels vector also) moves such that its tip traces a circle with constant angular velocity about the direction of propagation, we have circularly polarized light. If the tip of the electric field vector (and hence the magnetic field vector also) traces an ellipse as it rotates with constant angular velocity about the direction of propagation. We have elliptically polarized light.

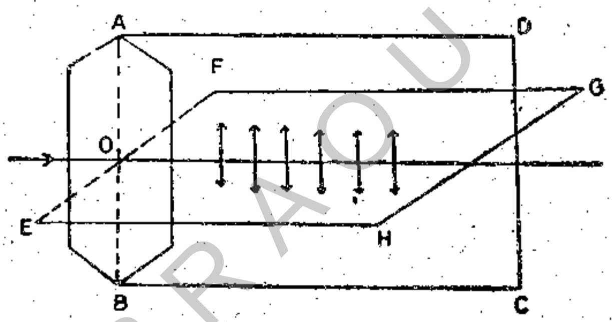


Fig 21.4

21.4 PRODUCTION OF PLANE-POLARIZED LIGHT: POLAROID.

To convert unpolarized light into plane-polarized light, we should think of a device or material that allows light vibrations in particular direction to pass through and absorb all other vibrations. There are many materials that achieve this purpose. Calcite Crystal splits the light beam incident on it into two components. They are called Ordinary and Extraordinary beams. This phenomenon is called double refraction. Both the ordinary and extraordinary beams are plane polarized. The two beams may be separated and used.

Some crystals like tourmaline not only exhibit doubt refraction but also absorb the two rays to different extents. This phenomenon is called dichroism. By using such crystals of proper thickness, it is possible to completely suppress one of the rays and transmit only the plane-polarized wave. Polaroid is a commercially available polarizing material. It transmits vibrations only in one direction and absorbs those that vibrate at right angles to this direction. Consequently, the emerging light will be plane-polarized.

The Polaroid is fabricated by embedding certain long-chain molecules in a flexible plastic-sheet. The sheet is stretched so that the molecules are aligned parallel to each other.

Light vibrations parallel to the long chain molecules are absorbed and vibrations perpendicular to their length are transmitted.

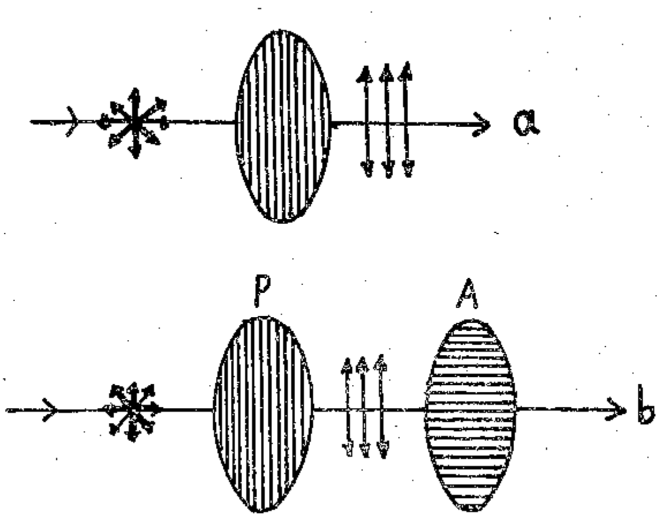


Fig 21.5

Fig 21.5 shows propagation of light throught a pair of polariods the light that emerges out of the first polaroid is plane-polarized. The polarizing direction is shown in the figure. Let us now place a second polaroid in the path of the light beam. If the second one has the same orientation (Fig.21.5a), the polarized light is all allowed to pass through the second polaroid. If, on the other hand, the second polaroid is rotated by 90° (21.5b) the plane-polarized light that comes out of the first polaroid is not allowed to pass throuth the second one. It is completely absorbed by the second polaroid. The first polaroid is called the Polarizer and the second one the analyser. A pair of polaroids arranged with mutually perpendicular directions is called 'Crossed Polaroids'. A pair of polaroids arranged with mutually perpendicular directions is called 'Crossed Polaroids'. In this position no light emerges out of the system.

Check your Progress 1

How can we change unpolarized light to plane polarized light?

A light beam is said to be polarized in a particular plane if there are no vibrations of electric vector in that plane. Therefore the vibrations of electric vector are always perpendicular to the plane of polarization.

21.5 LAWS OF MALUS

Let us now examine what happened to the intensity of the light emerging out of the analyser (Fig 21.5b) as the analyser is rotated about the direction of propagation of the light beams. As we have seen earlier, the intensity is Zero when the two polaroids are 'Crossed" i.e. when the angle between the two polarizing

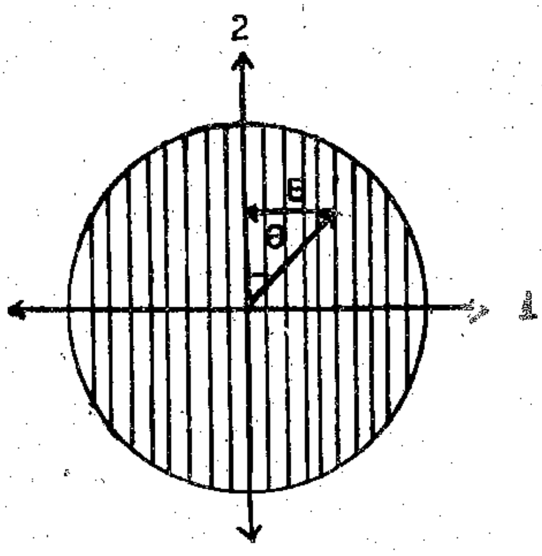


Fig. 21.6

directions is 90°. The intensity is maximum when the angle is Zero. For any other angle θ if the amplitude of the plane-polarized light falling on the analyser is E, the amplitude of the light that emerges is E Cos θ where θ is the angle between the polarizing directions of the analyser and the polariser (Fig 21.6). Since the intensity of the light is proportional to its amplitude.

$$I = I_m \cos^2 \theta \qquad (21.1)$$

Where I_m is the maximum value of the intensity of light transmitted. Equation 21.1 is called the Law of Malus.

Worked Example-1

The intensity of light emerging out of a system consisting of two polaroids is I_m when the angle between the polarizing directions of the analyser and polariser is zero. Calculate the intensity of the light emerging out of the analyser when the angle is 135°.

$$\text{Since } I = I_m \cos^2 \theta$$

$$= I_m \cos^2 (135^\circ)$$

$$= I_m/2 \qquad (21.2)$$

21.6 POLARIZATION BY REFLECTION

The French Physicist Malus discovered that when ordinary unpolarized light is incident at angle of about 57° on the polished surface of a glass plate, the reflected light is Plane Polarized. The experimental set-up shown in Fig. 21.7 demonstrates the discovery of Malus.

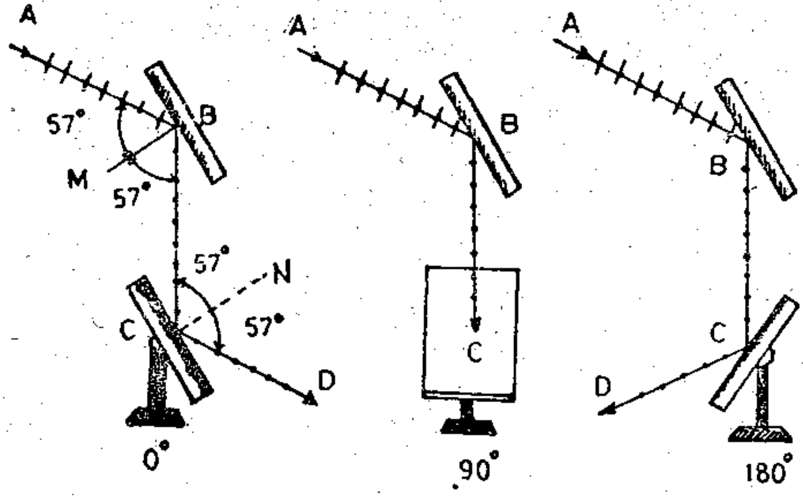


Fig. 21.7 (a) (b) (c)

The experimental set-up consists of two glass plates B and C. The second glass plate C is mounted on a pedestal such that it can be rotated about BC. In Fig 21.7a an unpolarized beam is incident on the first glass surface at B. The light is again reflected at the same angle by the second glass plate C. B and C are parallel to each other. Let us now rotate C about the axis BC by slowly, turning the pedestal. The intensity of light reflected by C decreases and vanishes completely at an angle 90° (Fig. 21.7b). With further rotation the reflected beam CD appears again, reaching a maximum at an angle 180° as shown in 21.7c. Further rotation causes the intensity to decrease to zero again at 270° and maximm at 360°. During this one complete rotation, the angle of incidence on the both the plates remains at 57°.

If the angle of incidence is diffeent from 57°, the light intensity goes through maxima and minima every 90° as before, but the minima will not go to zero. In other words, there will always be a reflected beam CD.

The experiment described above can be made clear by examining the change that take place in reflection. Fig 21.8 shows reflection of an unpolarized beam at a glass surface. As described earlier, the electric vector of the incident light beam can be resolved into two components; dots representing vibrations perpendicular to the plane of the paper and arrows representing vibrations lying in the plane of the diagram. In general, both the components will be of equal amplitude. It was found experimentally that at a particular angle of incidence θ_p only the vibrations represented by dots are reflected. The other component is entirely refracted. The reflected beam, naturally, is of low intensity. It is plane polarized. This angle of incidence θ_p is called the polarizing angle.

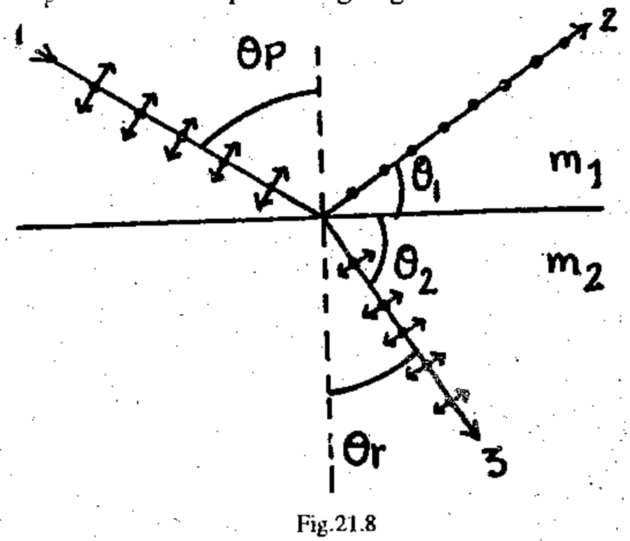


Fig.21.8

The transmitted beam is partially polarized. The intensity of the reflected plane-polarized beam can be increased by using a stack of glass plates as shown in Fig.21.9. Reflections from successivesurfaces add to the intensity of the beam. The stack of plates also make the transmitted beam plane polarized.

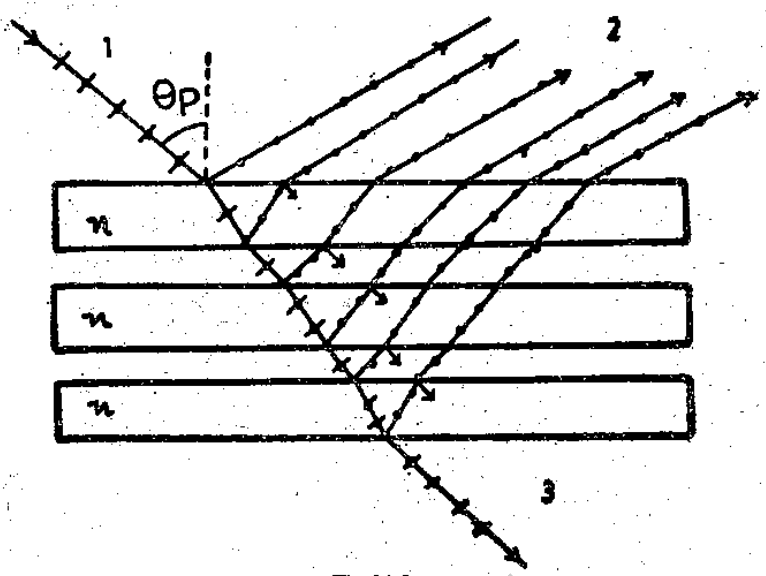


Fig.21.9

At the polarizing angle θ_p Brewster found that the reflected and transmitted beams (Fig.21.8) are perpendicular to each other i.e. $\theta_1 + \theta_2 = 90^\circ$

$$\therefore \theta_p + \theta_r = 90^\circ \tag{21.3}$$

From Snell's Law, $\mu_1 \sin \theta p = \mu_2 \sin \theta r$ (21.4)

Combination of the above equations leads to

$$\mu_1 \sin \theta_p = \mu_2 \sin(90^\circ - \theta_p) = \mu_2 \cos \theta_p \tag{21.5}$$

$$\text{or } \mu = \frac{\mu_2}{\mu_1} = \tan \theta_p \tag{21.6}$$

Where μ $(=\mu_2/\mu_1)$ is the index of refraction of medium 2 with respect to medium 1. Equation 21.6 is called Brewster's law.

Worked Example

Calculate the polarizing angle for a glass plate of refractive index 1.55

$$\theta_p = \tan^{-1} 1.55 = 57.2^\circ$$

21.7 DOUBLE REFRACTION - HUYGENS EXPLANATION

Let us now examine Huygens explanation of the phenomenon of double refraction. We shall confine our discussion to uniaxial crystals. Huygen's explanation is based on the following facts.

a) In a doubly refracting medium, there are two beams travelling the crystals. Hence every particle in the medium sends out two secondary wavelets.

b) The velocity of the O beam is the same in all directions. Therefore, the corresponding wavelet must be shperical.

c) The velocity of the E beam depends on the direction in the crystal. As mentioned earlier, the O and E beams travel with the same velocity along the optic axis.

Based on the above facts. Huygens assumed E wavelet to be an ellipsoid (formed by the revolution of an ellipse about an axis coinciding with the optic axis. Since the velocities of the O and E rays are equal along the optic axis, the O and E wavelets touch each other in this direction. A cross section of the two wavelets is shown in Fig.21.10

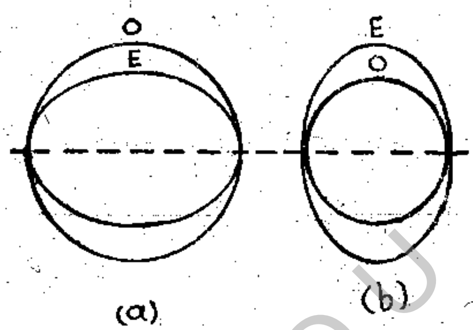


Fig.21.10

Fig.21.10a depicts the situation in a crystal like quartz while that in Fig.21.10b gives the situation in crystal like calcite. In Calcite, the velocity of light is maximum in a direction perpendicular to the optic axis, hence the shperical wavelets is inside the ellipsoidal wavelet. Such crystals are called 'negative' crystal. In quartaz, the velocity of light is maximum along the optic axis; hence the ellipsoid is inside the sphere. Such crystals are called 'positive' crystals.

21.8 CONSTRUCTION OF WAVEFRONTS IN CALCITE-SIMPLE CASES

Let us now utilise the ideas put forward by Huygens to construct refracted wavefronts in some special cases.

Case-1

Fig.21.11 shows an unpolarized plane wave incident normally on plane AB of a negation uniaxial crystal like Calcite. The figure shows a section through the crystal which is normal to the crystal surface and parallel to the optic axis. The dotted lines indicate the direction of the optic axis.

The moment light is incident on the crystal surface, every point on the surface acts as a secondary source of spherical and ellipsoidal wavelets. Fig.21.11 shows only certain points like A,B&C. Circles of radius $v_o t$(v_o - velocity of the ordinary wave) drawn with these points as centres represent ordinary wavelets after t seconds. If v_e is the maximum velocity of the extraordinary wave, the major axis of the ellipse at time t will be $2 v_e t$. The length of the minor axis is equal to $2v_o t$. Since the sphere and the ellipsoid touch each other along the optic axis, a line of the length $v_e t$

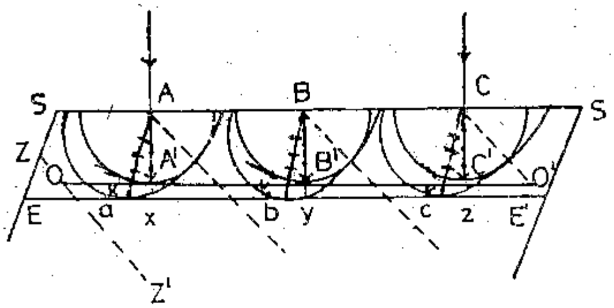


Fig.21.11

drawn perpendicularly to the optic axis represents half of the major axis. One can draw the ellipse using $v_o t$ and $v_e t$ as shown in the figure. The line OO′ tangential to the circles represents the ordinary wave front while EE′ represents the extraordinary wavefront. The lines Aa, Bb, Cc indicate the direction of the refracted E beam.

Case-2

Two special cases of normal incidence are of importance. They are illustrated in Figs. 21.12 and 21.13 In Fig 21.12 light is incident normally on a crystal face cut perpendicular to the optic axis i.e.

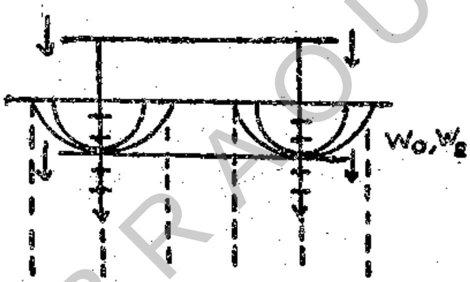
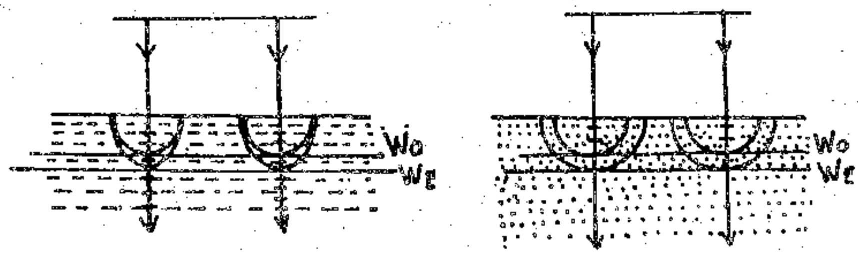


Fig.21.12

the light is travelling along the axis. Since both the waves travel with the same velocity in this direction, there is no double refraction.



(a) (b)

Fig. 21.13

Fig. 21.13 a and b show light incident normally on a crystal face cut parallel to the optic axis. (a) shows the principal section (section containing the optic axis and (b) a section at right angles optic axis there. Will be no double refraction. But the waves advance with different speeds.

21.9 REFRACTIVE INDICES

It is clear from the discussion presented above, the doubly refracting materials have two indices of refraction, corresponding to the ordinary ray (μ_o) and the other corresponding to the extraordinary ray (μ_o). There is no difficulty in defining μ_o as sin i/sin r is constant. The index of refraction for the extraordinary ray is a function of direction. It is customary to state the index for the direction at right angles to the optic axis, in which the velocity is maximum or minimum. Thus,

$$\mu_o = \frac{\text{velocity of light in vacuum}}{\text{velocity of the ordinary wave}}$$

and $$\mu_o = \frac{\text{velocity of light in vacuum}}{\text{maximum velocity of the E wave for negative Crystals}}$$

For calcite $\mu_o = 1.65836$ and $\mu_e = 1.48641$; at 18°e

For quartz $\mu_o = 1.54425$ and $\mu_e = 1.55336$; at 18°e

The values of μ_o and μ_e may be determined experimentally using prisms made out of the doubly refracting material and determining the angle of minimum deviation using a spectrometer.

21.10 NICOL PRISM

Fig. 21.14 is a sketch of the Calcite Crystal, sometimes called iceeland spar. At each of the corners A and B, The angles formed by the three inteersecting faces are the same, each angle is 101° 55'. A line at A or B drawn so that it makes equal angles with the three intersecting edges in the direction of the optic axis. The directions of vibration of the ordinary and extraordinary rays can now be specified with respect to the optic axis.

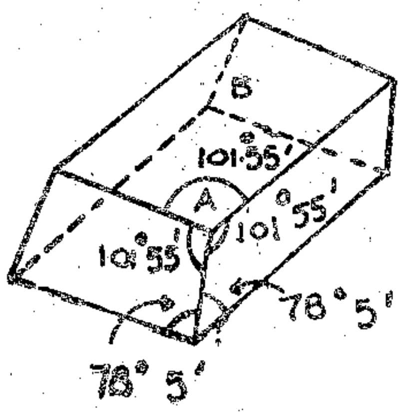


Fig. 21.14

Fig.21.15a shows a plane section through the calcite crystal. It contains the optic axis and is perpendicular to opposite faces of the material crystal. A narrow beam of

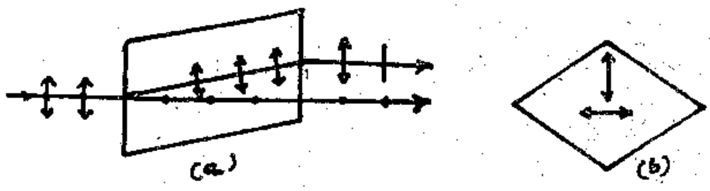


Fig. 21.15

unpolarized light passing through such a plane section (Principal section) gets split into two beams which emerge as two parallel beams. Each of these beams is linearly polarized. The direction of vibration in the ordinary ray is perpendicular to the plane of the principal section. The direction of vibration in the extraordinary ray is parallel to or in the plane of the principal section.

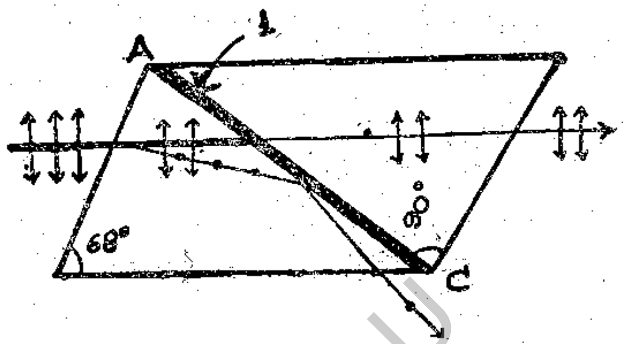


Fig. 21.16

The calcite crystal may be reshaped and used as a polarizer or analyzer. This is achieved by removing one of the polarized rays from the transmitted beam. The method is illustrated in Fig. 21.16. We have to select a fairly long and clear crystal to start with. The end faces are cut so that the angle in the principal section is reduced from 71° (Fig.21.15a) to 68° (Fig.21.16). The crystal is then cut into halves along the diagonal AC perpendicular to the plane of the principal section. The two halves are then cemented together with clear Canada Balsam (cement). The optical device so constructed is called a Nicol Prism.

The index of refraction of the orcinary ray for calcite is 1.66 and for Canada balsam it is 1.55. The angle of incidence of the Ordinary ray at the Canada balsam surface is greater than the critical angle. As such it is totally reflected. The index of refraction of the extraordinary ray in calcite depends on the direction of propagation. For the possible directions in the Nicol prism the index of refraction of the E ray is approximates 1.49; hence it will be transmitted through the Canada balsam and emerge from the prism as shown in the figure.

Nicol prisms are used in optical devices as analyzer and polarizers. The light that is transmitted through this prism vibrates in a direction parallel to the short diagonal of its end face.

Thus when unpolarized light is incident on the Nicol prism, it produces plane polarized light whose direction is parallel to the short diagonal of its end face. This property is utilised in producing linearly polarized light. It allows plane polarized light with its direction of vibration parallel to its short diagonal. It completely cuts off linearly polarized light with its direction of vibration perpendicular to the shorter diagonal of its end face. This property is used for analysis of polarized light.

21.11 SUMMARY

Ordinary light is unpolarized. The electric vector of the light wave vibrates in all possible directions perpendicular to the direction of wave propagation. By using polaroid unpolarized light can be made plane polarized. Polarized light can be plane polarized, circularly polarized and elliptically polarized.

The Law of Malus state that the light emerging from an analyser is given by $I=I_m \cos^2\theta_m$

When ordinary light is incident on the surface of calcite crystal it splits into two rays which emerge out from the crystal. Out of these beams one is known as ordinary (o) ray and the other ray is called extraordinary ray. "O" ray obeys snell's law. Where as E ray does not. This phenomenon of separation into two beams of an ordinary light by crystals like calacite is known as double refraction.

The velocity of E ray depends on the direction of propagation, in the crystal where as O ray travels with the same velocity in all directions inside the cryatal. The velocities of O & E ray are equal along optics axis.

In uniaxial crystal the O & E rays have equal velocities along only one direction. Where as in biaxial crystals the number of directions are two.

21.12 MODEL ANSWERS

Check your Progress 1

Unpolarized light can be changed to plane polarized light by using a polaroid. A polaroid transmits vibrations only in one direction and absorbs those vibrations which vibrate at right angles to this direction.

Check your Progress 2

The beam which obeys snell's law of refraction is called the ordinary ray and the other ray which does not obey the snell's law is called extraordinary ray.

Check your Progress 3

The direction along which the velocities of O and E rays are equal in a crystal is called an optic axis.

Check your Progress 4

In Uniaxial crystals there is only one direction along which the velocity of O and E rays are same where as in a biaxial crystal there are two directions along which O & E rays have same velocity.

21.13 SAMPLE QUESTIONS

I Answer the Following Questions in About 30 Lines.

1) Describe the experimental method to produce plane polarized light from unpolarized

light by reflection.

2) Explain how a Nicol prism is constructed. What are its applications?

II Answer the Following Questions in About 10 Lines.

1. Discuss the fabircation and function of a polaroid.
2. State and explain the law of Malus.
3. Discuss Huygen's explanation of double refraction.

III Solve the Following Problems

1. Determine the angle of refraction when a glass plate of refractive index 1.55 is used as a polarizer. [Ans 32°]

2. Find the polarising angle for a flint glass of index of refraction 1.768 [60° 30']

3. A beam of light travelling through water strikes the surface of the glass plate. When the angle of is adjusted to be 50.82° the reflected beam found to be plane polarised. Find the refractive index of the glass. (given refractive index of water 1.33) [1.637]

4. What is the angle of incidence for the complete polarisation to occur on reflection at a boundary between water of refractive index 1.3 and flint glass of refractive index 1.963 of the light is made to incident from the side of water. [55.84°]

5. The critical angle of incidence of water for total reflection is 45°. Calculate the polarising angle [53°22']

UNIT - 22 PRODUCTION AND ANALYSIS OF DIFFERENT TYPES OF POLARIZED LIGHT

Contents

22.1 AIMS AND OBJECTIVES

This unit discusses the production and properties of circularly and clliptically polarized light. It discribes the experimental setup for the production of circularly and elliptically polarized light.

After going through this unit you will be able to define plane polarized light, circularly polarized light and elliptically polarized light.

22.2 INTRODUCTION

Plane Polarization is a simple type of polarization. We have learnt in the last unit that a doubly refracting crystals like quartz or calcite splits an ordinary unpolarized beam into two components i.e. ordinary and extraordinary rays. Both these rays are plane or linearly polarized. But vibrational directions are mutually perpendicular.

The material presented in the previous unit clearly shows that under certain conditions the ordinary and extra ordinary rays travel along the same path but with differing speeds. For example, consider a calcite crystal with its faces cut parallel to the optic axis. Light incident normally on one of its faces traverses the Crystal in a direction perpendicular to the optic axis. The extraordinary and ordinary rays take the same path but with different speeds. As they emerge out of the Crystal, the two rays develop a phase difference between them. This phase difference depends on the frequency of the incident light, thickness of the crystal and the speeds of the two rays.

Superposition of the two rays after emergence from the crystal leads to interesting results. The problem reduces to one of combination of two simple harmonic motions in mutually perpendicular directions. We know that superposition of two SHMs at right angles leads to different shapes as shown in Fig. 22.1. When the phase difference 0, 2π or any other even multiple of π the result is a linear vibration at 45° to both the original vibraions. When the phase difference is an odd multiple of π the resultant is again a linear vibration but at right angles to the previous one. A phase difference of n $\pi/2$(n=1, 3, 5, 7.) leads to a circle. At all other phase differences, the resultant is an ellipse.

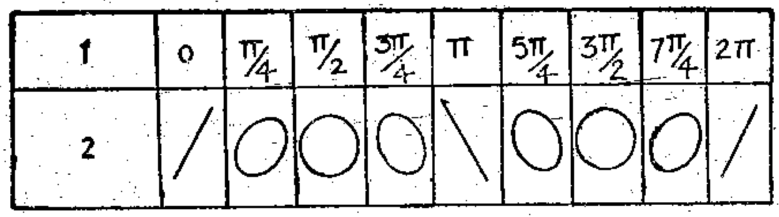


Fig. 22.1 Combination of SHMs at right angle

What do we mean by circularly polarized light? Elliptically polarized light? By the light vibration, we mean the periodic variation of the electric field in space. Hence in a circularly polarized light beam the tip of the electric vector moves in a circle in a plane perpendicular to the direction of propagation. Similarly the tip of the electric vector moves in an ellipse in elliptically polarized light.

Check your Progress 1

What is a circularly polarized light

Check your Progress 2

What is an elliptically polarised light.

22.3 QUARTER WAVE PLATE

When the ordinary and extraordinary rays traverse the same path in crystal with different speeds, the distance travelled by them during a particular in traval of time difference they develop a path/phase difference. If the thickness of the plate is so chosen as to introduce a path difference of $\lambda/4$ (λ wavelength of the incident light) the crystal plate is called Quarter Wave Plate. We have also half wave plates which introduce a path difference of $\lambda/2$ and full wave plates which introduce a path difference of λ.

In wave propagation a path difference of λ corresponds to a phase difference of 2π. Hence, a quarter wave plate introduces a phase difference of $\pi/2$.

Let us now calculate the thickness of the quarter wave plate. As mentioned earlier, The thickness depends on λ. if μ_o and μ_e are ordinary and extraordinary refractive indices of the material chosen and 't' the thickness of the Crystal plate, the optical path difference Δ is given by

$$\Delta = (\mu_0 \approx \mu_e)t \qquad (22.1)$$

For a quarter wave plate made out of a negative crystal like Calcite

$$(\mu_0 \approx \mu_e)t_q = \lambda/4 \qquad (22.2)$$

$$\text{or } t_q = \frac{\lambda}{4(\mu_o - \mu_e)} \qquad (22.2a)$$

For positive Crystals like quarz $t_q = \frac{\lambda}{4(\mu_o - \mu_e)}$ (22.2b)

and $t_h = \frac{\lambda}{2(\mu_o - \mu_e)}$ for negative Crystal (22.3)

$t_h = \frac{\lambda}{2(\mu_e - \mu_o)}$ for positive Crystal (22.3b)

Worked Example

The principle refractive indices of quartz for light of λ(5000A°) are 1.5533 amd 1.5422. Calculate the thickness of the Quarter Wave Plate.

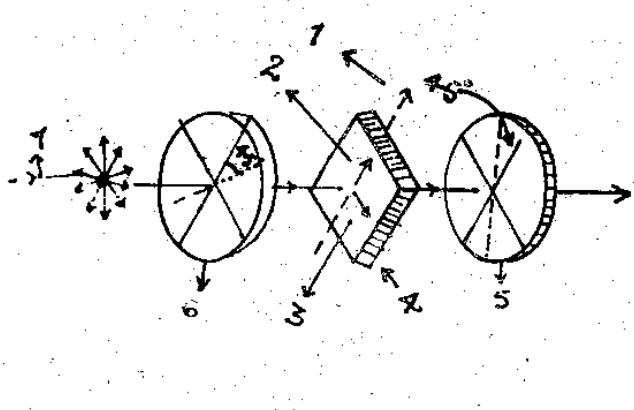
$$t = \frac{5000 \times 10^{-8}}{4 \times 0.0011} = 1.126 \times 10^{-3} \text{cm}$$

This is a very thin plate. We may use more robust plates of thickness which is an odd multiple of 't' calculated above as quarter wave plates.

Quarter wave Plates are usually made of quartz or mica. Mica splits easily into thin sheets. Hence it is easy to prepare a plate of the required thickness.

22.4 PRODUCTION OF CIRCULARLY AND ELLIPTICALLY POLARIZED LIGHT

The experimental set up for production of circularly and elliptically polarized light is shown in Fig.22.2. It makes use of a nicol polaraid to produce plane polarized light form the incident unpolarized light beam. It is called polarizer. Unpolarized, monochromatic light enters the polarizer and emerges as, a linearly polarized beam. The direction of vibration of the linearly polarized beam is indicated by the dotted line drawn on the polarizer. This enters the doubly refracting Crystal (say Calcite) plate. It is cut such that light travels in a direction perpendicular to the optic axis. The Crystal plate has to be Oriented



1. Unpolarized light
2. E-vibration
3. O-Vibration
4. Crystal Plate
5. Analyser
6. Polarizer.

Fig.22.2

Such that the optic axis makes 45′ with the direction of vibration of the incident light. The incident light gets split up into E and O components. In this case the E vibrations are parallel to the optic axis and the O vibrations are perpendicular to the optic axis as shown in the figure. Since the incident beam makes 45 with the optic axis, the amplitudes of E and O vibrations are equal. The E and O waves move along the same paths in the crystal but with different speeds. Hence they develop a phase difference. Superposition of these two waves of equal amplitudes with their vibrations in mutually perpendicular directions leads to circular or elliptical vibrations depending upon their phase difference.

The phase difference depends on the crystal thickness chosen and the wavelength of the light used. If we use a quarter wave plate for the particular wavelength, it introduces a phase difference of $\pi/2$. The resultant, vibrations are circular. Such a light beam is said to be circularly polarized.

If the crystal thickness is so chosen as to introduce a phase difference of π, the emerging beam is linearly polarized.

If the crystal thickness is such that it introduces a phase difference other than n π (n any integer) and R$\pi/2$ (R-Odd integer), the emerging light is elliptically polarized.

2? 5 DETECTION OF POLARIZED LIGHT

22.5.1 Plane Polarized light

Allow the light beam to be incident on an analyser (Nicol or Polaroid) that is capable of being rotated. If on rotation of the analyser, the light is completely extinguished twice in a complete rotation, the light is linearly polarized.

22.5.2 Circularly Polarized Light

An examination of intensity of the light transmitted by the analyser when the incident beam is circularly polarized, shows that the intensity does not vary with the orientation of the analyser. Unpolarized light also exhibits the same behaviour.

To distinguish between these two, the beam is allowed to pass through a quarter wave plate. A circularly polarized beam can be thought of as a combination of two plane polarized beams with a path difference of $\lambda/4$. Such a beam after passing through the quarter wave plate develops a further path difference of $\lambda/4$ between its components. Hence the path difference between the two components is now $\lambda/2$. Thus the beam after passing through the quarter wave plate becomes plane polarized. Consequently when the analyser is rotated, the intensity becomes zero twice in one complete rotation.

If the incident beam is unpolarized, introduction of the quarter wave plate does not alter the intensity variation. Thus if the incident beam after passing through the quarter-wave plate is extinguished when observed through analyser, it is circularly polarized.

22.5.3 Elliptically Polarized Light

When examined, using a rotating analyser, the intensity goes through maxima and minima. The minimum does not go to zero. Same behaviour may be observed when the incident beam is a mixture of unpolarized and plane polarized light.

To distinguish between those two, allow the beam to pass through a quarter-wave plate and an analyser. The plate and the analyser have to be rotated independently. There are two possibilities (a) the light is extinguished for one setting and (b) the intensity varies but never fails to zero. Case (a) indicates that the light is elliptically polarized. Case (b) indicates a mixture of unpolarised and polarized light.

22.6 SUMMARY

When a plane polarized light emerging from Nicol prism falls on a plate of calcite cut with faces parallel to opposite axis two light waves emerge out with difference in phase. Depending on the phase difference the emerging light may be circulary polarized, plane polarizied, or elliptically polarized.

Circularly polarized light is resultant of two waves of equal amplitude vibrating at right angles to each other and having a phase difference of $\pi/2$. Elliptically polarized light is the resultant of two waves of amplitude vibrating at right angles to each other and having a phase difference of $\pi/2$

22.7 MODEL ANSWERS

Check your Progress 1

Circularly polarized light. It is the resultant of two waves of equal amplitude vibrating at right angles to each other and having a phase difference of $\pi/2$

Check your Progress 2

An elliptically polarized light is the resultant of two waves of unequal amplitude vibrating at right angles to each other and having a phase difference of $\pi/2$

22.8 SAMPLE EXAMINATION QUESTIONS

I. Answer the Following Questions in About 30 Lines.

1. Discuss the construction and working of a quarterwave plate.

2. Discuss the method of production of circularly and ellipticaly polarized light waves.

II. Answer the Following Question in About 10 Lines.

Discuss the experimental method to detect plane polarized, circularly polarized and elliptically polarized light waves.

III. Solve the Following Problems

1. For sodium light of wave length=5893A° the refraction indices of quartz are 1.5442 and 1.5533. Calculate the thickness of half wave plate formed by quartz crystal.

(Ans: 3.24 X 10^3cm)

2. A half wave plates is constructed for a wavelength of 6000A°. For what wavelength

does it work as a quarter wave plate. [12000 A°]

3. Calculate the thickness of a mica sheet required for making a quarter wave plate using a light of wavelength 5460A° given that μ_e=1.592 and μ_o = 1.586.

4. For a light of wavelength 5000 A calculate the thickness of a quarter waveplate and half wave plate given that μ_e=1.533 and μ_o=1.544.

[1.14×10^{-3} cm and 2.27×10^{-3} cm]

UNIT-23 ROTARY POLARIZATION

Contents

23.1 AIMS AND OBJECTIVES

This unit introduces the phenomenon of optical activity. It is explained on the basis of Fresnel's theory.

After going through this unit you will be able to:

(1) makeout which subtance is dextrorotatary or laevorotary

(2) define the specific rotation of a solution

(3) measure the angle of rotation produced by an optically active substance with the help of a polarimeter.

(4) improve the angle of rotation produced by optically active substance with help of a Biquartz crystal;

23.2 INTRODUCTION

In this unit we shall study the phenomenon of optical activity and what is meant by specific rotation. Fresnel explained the Rotary polarisation based on a simple concept in mechanics.

23.3 OPTICAL ACTIVITY

Certain substances like quartz are found to rotate the plane of vibration of a linearly polarized light passing through them. This phenomenon is called rotation of the plane of polarization. Substances which exhibit this effect are said to be optically active. Optical activity may be observed using a pair of nicols or polaroids arranged in the crossed position. In the crossed position no light emerges out of the analyser. If an optically active substance is now introduced in between the polarized and the analyser, some light emerges out of the analyser. The light can get extinguished by rotating the analyser through certain angle θ. It follows, therefore, that the vibration plane of the light leaving the substance must make an angle θ with that of the light incident on it.

Some of the substances that exhibit optical activity are : (a) quartz crystals along the optic axis and (b) liquids such as turpentine, sugar solution, tartaric acid solution etc. A substance may rotate the plane of vibration either to the left or to the right. Consequently they are classified and laevorotatatory (also called left-hand) and dextrorotatory (also called right-handed) substances respectively. Some substances may occur in nature in both the varieties. One such substance is quartz.

The actual amount of rotation depends on temperature, the wavelength of the light used thickness of the sample and concentration in the case of solutions.

To facilitate comparison of the optical activity of different substances, one usually defines specific rotation (α) It is the rotation produced by a 10cm. column of liquid containing 1 gm. of active substance per c.c. of solution. Thus, for a solution containing m.gm/c.c of the active substance and a path length of 1.

$$\theta = \frac{\alpha m l}{10} \tag{23.1}$$

$$\text{or } \alpha = \frac{10\theta}{ml} \tag{23.2}$$

For a pure liquid m is simply the density

Check your Progress 1

Define optical activity

Check your Progress 2

Distinguish between dextro rotatory and laevorotation.

Worked Example -1

A 20cm. tube containing cane sugar solution having specific rotation of 60° shows optical rotation of 6°. Calculate the strength of the solution.

we know $\alpha = \frac{10\,\theta}{m x 1}$

$$\therefore m = \frac{10\theta}{\alpha 1}$$

$$= \frac{10x6^\circ}{60^\circ x20} = 0.05gm/cc.$$

23.4 FRESNEL'S EXPLANATION OF ROTARY POLARIZATION

Fresnel's explanation of the rotary polarization is based on a simple concept in mechanics. It visualises simple harmonic motion along a straight line as superposition of two circular motions in opposite direction. This may be illustrated with the help of Fig. 23.1

In this figure, P_1 and P_2 are two points executing circular motion with equal speeds but in opposite directions as indicated in the figure. These points meet each other at Y and Y'. The vectors and OP_1 and OP_2 can be resolved into two components; one along the X - axis and the other along the Y - axis. For example the components of OP_1 along X and Y directions are

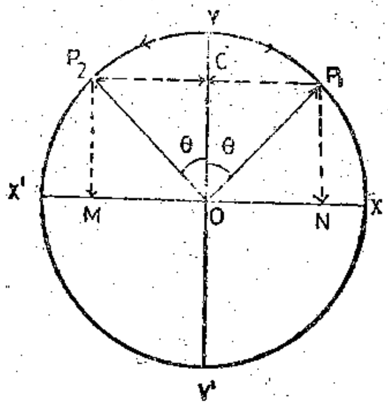


Fig. 23.1

$$ON = a \sin \theta \quad OC = a \cos \theta \tag{23.3}$$

Similarly the components of OP_2 are

$$OM = a \sin \theta \quad OC = a \cos \theta \tag{23.4}$$

Since at any point ON = -OM the resultant component along the X-axis is always zero. The resultant along the Y - axis is 2a Cos θ. Thus the results of the two circular motions is a simple harmonic motion along Y - axis with amplitude 2a.

From the above discussion it may be concluded that the simple harmonic motion along a straight line can be throught of a combination of two circular motions in the opposite direction. From this it is clear that a linearly polarized light may be thought of as two circularly polarized lights.

When a plane polarized light enters a substance, we may infer that it gets split up into two circularly polarized components. they travel with equal velocity in the material and recombine before leaving the substance to produce again the linearly polarized light. However, Fresnel assumed that in optically active substance, the circularly polarized light can travel with different speeds. Consequently, when they emerge from the substance there is a phase difference between them.

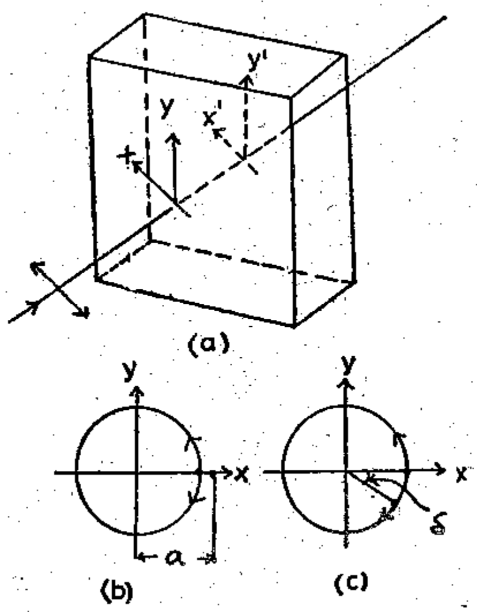


Fig. 23.2

Analytically, let the clock-wise and anti clock wise rotations be represented by the following equations when they enter the crystal.

Clockwise rotation

$$x = \frac{a}{2}\cos\omega t,\ y = -\frac{a}{2}\sin\omega t \tag{23.5}$$

Anti clock wise rotation

$$x = \frac{a}{2}\cos\omega t,\ y = \frac{a}{2}\sin\omega t \tag{23.6}$$

When they reach the other Face of the Crystal they develop a phase difference as shown in Fig. 23.2

Hence for clock wise rotation

$$x' = \frac{a}{2}\cos(\omega t + \delta),\ y' = \frac{-a}{2}\sin(\omega t + \delta) \tag{23.7}$$

Anti clock wise rotation

$$x' = \frac{a}{2}\cos\omega t$$

$$y' = \frac{a}{2}\sin\omega t \tag{23.8}$$

Super position gives

$$x' = \frac{a}{2}[\cos\omega t + \cos(\omega t + \delta)] \tag{23.9}$$

$$y' = \frac{a}{2}[\sin\omega t - \sin(\omega t + \delta)] \tag{23.10}$$

$$\therefore x' = a\cos(\delta/2)\cos(\omega t + \delta/2) \tag{23.11}$$

$$\text{and } y' = -a\sin(\delta/2)\cos(\omega t + \delta/2) \tag{23.12}$$

Since these two perpendicular linear components are in phase the resultant is a line inclined to x' axis at an angle of $-\delta/2$.

According to Fresnel's theory, in the right handed (dextro rotatory) crystal, the clock wise circularly polarized component travelsmore quickly than the other. In the left handed substance the clock wise component is slow.

Fresnel proved his assumption using the system shown in Fig. 23.3. It consists of 3 right handed and 2 left handed Quartz prisms cemented together.

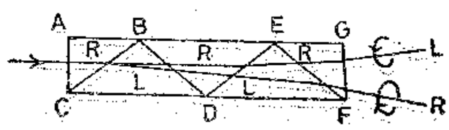


Fig 23.3

A beam of plane polarized light is incident on face AC. It gets split up into two circularly polarized components. On striking the inclined surface CB and entering the second prism, the disturbance which faster in the first prism now becomes the slower. However, according to the law of refracton, one beam is bent away from the normal and the other towards it. Since at other inclined faces similar effects take place, the two beams get separated as they emerge from the crystal. This was actually observed by Fresnel.

23.5 POLARIMETER

Instruments that measure the rotation of the plane of vibration are called polarimeters. The instruments intended for measurements on sugar soltions in particular are called saccharimeters.

A polarimeter in its simplest from consists of a polarizer and an analyser as shown in Fig.23.4. The analyser is mounted on a graduated scale.

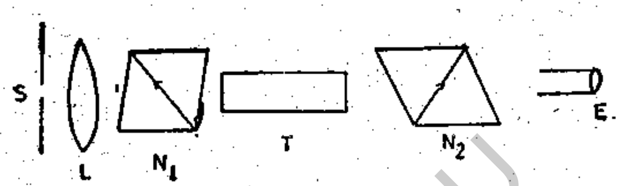


Fig.23.4

Intially, the analyser and polarizer are adjusted to be in the crossed position. The reading of the analyser is noted. The active substance is now introduced in between the polarizer and the analyser. The "Crossed Position" is disturbed. The analyser is now rotated to extinguish the light coming out of it. This reading is noted. The difference between the two readings of the analyser gives the rotaion.

With this simple system, the intensity of the field of the view is very insensitive to small rotations of the analyser from the position of extinction. Consequently, the analyser cannot be set with accuracy. The performance of this simple sytem may be improved by using half-shadow end point devices immediately after the polarizer. Such a device enables accurate determination of the extinction position as it compares intensities of two halfs of the field of view.

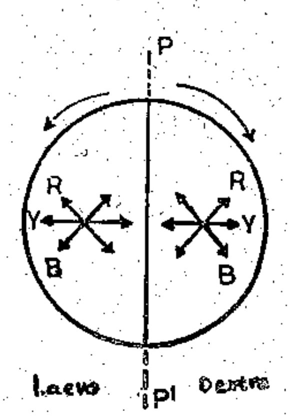


Fig. 23.5

Biquartz is one such half shadow device. It consists of two semicircular quartz devices Cemented together. One half is cut from a right handed quartz and the other from the left handed quartz. The thickness of the disc ($\approx$3.75mm.) is such that for yellow light, the plane of vibration under goes a rotation of $\pi/2$ - clockwise in one half of the disc and anticlockwise in the other. Thus for yellow light the biquartz consists of two quarter wave plates. With white light the rotatory dispersion of the quartz causes the vibration directions of red yellow and blue light to be as indicated in Fig.23.5. If the principal section of the analyser is parallel to that of the polarizer, the yellow light is exinguished. The two halfs of the field are equally illuminated with a reddish violet tint. This is called the tint of passage. A smaller rotation of the analyser causes one half of the field to become distincly red and the other to become blue. the setting of the analyser for a colour match can be done very accurately.

Check your Progress 3

What is a saccharimeter

23.6 SUMMARY

The phenomenon of rotation of vibration of plane polarized light about the direction of propagation exhibited by certain substances like quartz, sugar solution is called optical activity, the amount of rotation is directly proportional to the length of the path in the optically active substance and inversely proportional to the square of the wavelength. If the rotation is clockwise the substance is said to be dextrorotatary. If the rotation is anti-clock wise the substance is said to be laevorotatory. The specific rotation of a solution is defined as the amount of rotation produced by one decimeter length of the solution when the concentration is one gram per C.C. Polarimeter is used to determine the angle of rotation produced by optically active substance. The precision of measurement of angle of rotation produced by optically active substance is improved by using a Biquartz.

23.7 MODEL ANSWERS

Check your Progress 1

The phenomenon of rotation of plane of vibration of a plane polarized light about the direction of propagation exhibited by certain substances is called optical activity.

Check your Progress 2

In optically active substances if the plane of vibration is rotated is in the clock-wise direction when viewed opposite to the direction of propagation in the substance is said to be dextro-rotatory and anti clock wise direction the substance is said to be laevorotatory.

Check your Progress 3

Saccharimeter is an instrument used to measure optical activity of sugar solutions.

23.8 SAMPLE EXAMINATION QUESTIONS

I Answer the Following Questions in About 30 Lines.

1. Discuss Fresnel's explanation of the phenomenon of optical activity exhibited by

certain substances.

2. Describe a simple experimental setup to determine the specific rotation of a substance.

3. Discuss the use of biquartz in a polarimeter.

II Solve the Following Problems.

1. The plane of polarization of plane polarized light is turned through 24° when it traverses through 20 sugar solution contained in a 30 cm length glass tube. Calculate the Specific rotation of the solution. [Ans: 40°]

2. A glass tube of 20 cm. length contains sugar solution. The plane of polarization of plane polarized light passing through this liquid, is turned through an angle of 11°. If the specific rotation of the sugar solution is 66 calculate the strength of the solution. [Ans : 0.083 gm/cc]

3. Calculate the specific rotation if the plane of polarization is turned through 26.4° traversing a length of 20cm. in a 20% sugar solution. [66°]

4. A tube of 20cm. long is placed between two crossed Nichols and illuminated with light of wavelength $6x16^{-5}$ cm. If the optical rotation produced is 13° and specific rotation is 65° determine the strength of the solutions. [10%]

5. The tube in a polarimeter is 20cm long filled with a solution of sugar formed by dissolving 10 gm of sugar in 40cc of water. If the rotation of the plane of polarization is 34° find the specific rotation. [68°]

6. The solution of cane sugar contained in a tube of length 10cm. rotates plane of polarization through 17°30'. If the specific rotatory power is 66°30' find the amount of sugar in 1lt. of solution [263gm/lit]

BLOCK -X

RELATIVITY

UNIT-24 GALILEAN TRANSFORMATION

Contents

24.1 AIMS AND OBJECTIVES

In this unit we will study the Galilean transformation and the concept of invariance equations.

24.2 INTRODUCTION

Relativity of physical quantities like, accleration velocity force are studied using Galilean transformation equations. When an event is observed in two frames, one moving with a constant velocity with respect to the other the physical quantities can be expressed in terms of Galilean transformation equations.

24.3 GALILEAN TRANSFORMATION

Relativity of physical quantities in classical physics are explained using Galilean Transformation equations. Here we know the events in one reference frame should appear to an observer situated in another reference frame which moves with a constant velocity with respect to the first.

Let a reference frame S with a origin 'O' be at rest and let another reference frame S^1 with origin 'O^1' move with a constant velocity v relative to the first frame in the positive x-direction. Let the two sets of axes be parallel. We measure the time from the instant when "O" & "O^1" coinside. Consider an event happening at P at any particular instant of time. Let the coordinate of P with respect to S be (x, y, z, t) and S^1 be (x^1, y^1, z^1, t^1). let x & x^1 be parallel to v and let Y and Z be paralell to y^1 and z^1 respectively. As shown in Fig. 24.1

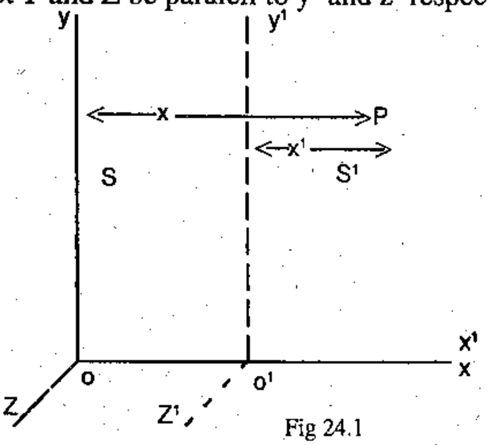


Fig 24.1

Let the observer be present in the reference frame S^1 in a time t the frame S^1 covers a distance Vt.

$$\therefore x^1 = x - vt \qquad (24.1)$$

$$y^1 = y$$

$$z^1 = z$$

According to classical relativity, time is taken as independent of any frame of referance.

$$\therefore t^1 = t$$

Similarly x, y, z, t can be written in terms of (x^1, y^1, z^1, t^1)

as
$$x = x^1 + vt$$

$$y = y^1$$

$$z = z^1$$

$$t = t^1 \qquad (24.2)$$

these equations are known as Galilean transformation.

24.4 VARIANCE AND INVARIANCE (CONCEPTS)

Further let two events A and B occur at times t_A and t_B with reference to S and t^1_A and t^1_B with reference to S^1.

But
$$t^1_A = t_A$$

$$t^1_B = t_B$$

$$\therefore t^1_B - t^1_A = t_B - t_A \qquad (24.3)$$

Let the two events occur at distances x_A and x_B at t_A and t_B with respect to S; and x^1_A and x^1_B at t^1_A and t^1_B with respect S^1. According to equation 24.1

$$x^1_A = x_A - vt_A$$

$$x^1_B = x_B - vt_B$$

Subtracing

$$x^1_B - x^1_A = (x_B - x_A) - v(t_B - t_A)$$

$$\because t_A = t_B$$

$$\therefore x^1_B - x^1_A = x_B - x_A \qquad (24.4)$$

From Equation 24.3 and 24.4 it can be concluded that the time interval between the occurance of two events and the space interval between the two points are invariant and as the same for all inertial frames.

using these transformations, velocity and accelerations of a body as measured by an server with frame S^1 in terms of those measured in S can be calculated.

We know that $x^1 = x - vt$

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \qquad (24.5)$$

$$\because t = t', \frac{d}{dt} = \frac{d}{dt'}$$

$\therefore$ equation 24.5 can be rewritten as

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

similarly $\frac{dy'}{dt'} = \frac{dy}{dt}$ (24.6)

$$\frac{dz'}{dt'} = \frac{dz}{dt}$$

But $\frac{dx}{dt} = u_x$ and $\frac{dx'}{dt'} = u'_x$)

$\therefore$ Equations 24.6 can be written as

$$u'_x = u_x - v$$

$$u_y^1 = u_y, u_z^1 = u_z \qquad (24.7)$$

That is the velocity is variant under Galeliau transformation

Again differentiating equation 24.7

$$\frac{du'_x}{dt'} = \frac{du_x}{dt} \quad \because \text{ v is constant}$$

$$\frac{du'_y}{dt'} = \frac{du_y}{dt}$$

$$\frac{du'_z}{dt'} = \frac{du_z}{dt}$$

(if) $a'_x = a_x$

$$a_y^1 = a_y \qquad (24.8)$$

$$a_y^1 = a_z$$

$$\because \frac{du}{dt} = a$$

That is the acceleration is invariant under Galelian transformations. Therefore force is also same for each inertial frame. Thus Newton's laws of motion and the laws of mechanics which follow from Newton's laws are also invariant.

24.5 SUMMARY

Under the Galelian transformation, the transformation equation from one frame at rest to the other frmae moving with constant velocity are derived. The priniciple of variance and invariance of physical quantities like velocity, accelaration, force are dealt in detail.

24.6 MODEL EXAMINATION QUESTIONS.

I. Answer the Following Question in 30 Lines

Explain Galelian transformation and derive the transformation equations.

II. Answer the Following Questions in 10 Lines

1. Show that the velocity is variant under Galelian transformations.
2. Show that the Newton's law of motion is invariant under Galelian transformation.

UNIT-25 MICHELSON-MORLEY EXPERIMENT, POSTULATES OF SPECIAL THEORY OF RELATIVITY

Contents

25.1 AIMS AND OBJECTIVES

In this unit we shall study the Michelson Morely experiment in detail and the postulates of special theory of relativity

25.2 INTRODUCTION

In this unit we study Michelson - Morley experiment and its consequences. According to classical mechanics the velocity of any body has different values for observers moving relative to each other. Michelson-Morley experiment indicated that the apparent velocity of light relative to the observer is not affected by the motion through the ether. This result leads to the conclusion that (i) either the earth is not moving or (ii) there is something very wrong with Newtonian concept. The first conclusion is nonsense. To accout for this observation Einstein proposed the view that motion through an ether medium is a meaningless concept and one has to consider only motion relative to material bodies. Einstein also observed that the conflict between the experimental results of Michelson Morely experiment and the prediction based on the mechanical experience must be due to an imperfection of the classical ideas about measuring space and time.

Einstein had put forth the Special theory of relativity in the year 1905 by proposing two postulates which are discussed below. This theory has completely revolutionized our ideas about matter, space and time.

25.3 MICHELSON AND MORELY EXPERIMENT

The experiment was first devised by Michelson to determine the velocity of light. But in order to explain the postulates of the theory of relatively (special or general) a modified experimental set up was found to be convinient.

The appparatus is as shown in fig.25.1. This essentially consists of a light source A, a partially silvered glass plate B and two perpendicular mirrors C and E, all mounted on a rigid base. The mirrors are placed at equal distances L from B. The beam of light from the source A is split by the partially silvered mirror into two beams, beam 1 being transmitted through B and beam 2 being reflected off B, Beam 1 reflected back to B by the mirror E and beam 2 by mirror C will interfere at T of the telescope. The interference is constructive or distructive depending upon the path difference between the two beams. The fringes consisting of nearly parallel lines. If the apparatus is at rest in the ether the times should be equal. If it is moving towards right with velocity u. there should be a difference in the times.

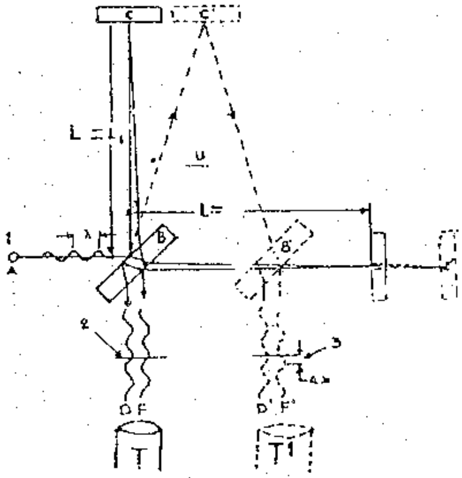


Fig:25.1 Experimental set up of Michelson and Morely

First let us calculate the time required for the light to go from B to E and back. Let us suppose that the time taken by the light to go from plate B to mirror E is t_1 and the time for the return is t_2. When the light travelling from B to E the apparatus moves through a distance of ut_1. Hence the distance traversed by the light is $L_1 + ut_1$. Where L_1 is the distance between the mirror B and the mirror E and that of B and C is L_2. We can express this distance as ct_1 where c is the velocity of light. Hence

$$ct_1 = L_1 + ut_1$$

$$\text{or} \quad t_1 = \frac{L_1}{c-u}$$

In a similar way the time t_2 can be calculated. During this time plate B advances a distance ut_2, so the return distance of light $L_1 - ut_2$, then we have

$$ct_2 = L_1 - ut_2$$

$$t_2 = \frac{L_1}{c+u}$$

Then the total time is

$$t_1 + t_2 = \frac{2L_1c}{c^2 - u^2}$$

$$\text{or} \quad t_1 + t_2 = \frac{\frac{2L_1}{c}}{1-\frac{u^2}{c^2}} = \frac{2L_1}{c}\left(1-\frac{u^2}{c^2}\right)^{-1} \approx \frac{2L_1}{c}\left(1+\frac{u^2}{c^2}\right) \qquad (25.1)$$

Now let us calculate the time t_3 required for the light to go from B to the mirror C. As before during time t_3 the mirror C moves to the right a distance ut_3 to the position C, in the same time the light travels ct_3 along the hypotenuse of a triangle which is BC. For this right angle triangle have

$$(ct_3)^2 = L_2^2 + (ut_3)^2$$

$$L_2^2 = c^2t_3^2 - u^2t_3^2$$

$$= (c^2 - u^2)\, t_3^2$$

or $$t_3 = \frac{L_2}{\sqrt{c^2 - u^2}}$$

For the return trip from 'C' the distance is the same, as can be seen from the symmetry of the figure, therefore the return time is also the same, and total time is $2t_3$ we can write

$$2t_3 = \frac{2L_2}{\sqrt{c^2 - u^2}} = \frac{\frac{2L_2}{c}}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{2L_2}{c}\left(1 - \frac{u^2}{c^2}\right)^{\frac{-1}{2}} \qquad (25.2)$$

$$\frac{2L_2}{c}\left(1 + \frac{u^2}{2c^2}\right)$$

The difference in transit time

$$\Delta t = 2t_3 - (t_1 + t_2) = \frac{2}{c}\left[L_2\left(1 + \frac{u^2}{2c^2}\right) - L_1\left(1 + \frac{u^2}{c^2}\right)\right]$$

Suppose the instrument is rotated through 90° thereby by making L_1 equal to L_2 and L_2 equal to L_1. if the corresponding transit time is

$$\Delta t^1 \approx 2t_3^1 - (t_1^1 + t_2^1)$$

$$= \frac{2L}{c}\left[\left(L + \frac{u^2}{2c^2}\right) - \left(1 + \frac{u^2}{c^2}\right)\right]$$

$$= + \frac{2L}{c}\frac{u^2}{Lc^2} = 4\frac{Lu^2}{c^3}$$

So there will be a shift in the fringe pattern.

$$\text{If } L_1 = L_2 = 11 \text{ meters and } \frac{u}{c} = 10^{-4}$$

Michelson Morely theoretically estimated a fringe shift of about 0.4.

Michelson Morely could not observe any shift in the fringe width. They repeated the experiment at different points of the earth surface and at different seasons of the year. They could not detect any measurable shift and hence a negative result.

The negative result suggests that it is impossible to measure speed of earth relative to the ether or concept of fixed frame of reference (like ether filling space). This suggests the speed of light in vaccum is the same in all frame of references which are in uniform relative motion.

25.4 POSTULATES OF SPECIAL THEORY OF RELATIVITY

The two fundamental postulates of the special theory of relativity can be stated as follows.

Postulate-1.

The laws of physics have the same mathematical form in all inertial systems.

The above postulate is also called as the principle of relativity. The inertial system referred in the postulates are unaccelarated systems that move at constant velocity relative to one another. The physical phenomena in a given reference system are independent of the translational motion of the systems as a whole. There exists a triply infinite set of reference systems moving rectilinearly and uniformly relative to one another in which the physical phenomena occur in an identical manner since the Galilean law of inertia holds in them. The concept of ether as a substance of is removed from the physical theories with the statement of the principle of relativity. The principle of relativity states that it is impossible by means of any physical measurements to designate an intertial system as intrinsically stationary or moving. One can only speak of the relative motion of two systems. No theory contains any reference what so ever to an absolute speed of translational motion of the reference system. No physical experiment of any kind made entirely within an inertial system can tell the observer about the motion of the system with respect to any other inertial system.

Postulate-2

The speed of light in vaccum is constant and the same for all observers, independent of the motion of the source and of the observer.

The second postulate contradicts the Galilean Velocity transformation and is consistent with the results of Michelson - Morely experiments. The postulate implies that Newtonian mechanical phenomena since the Galilean transformation representing the relativity principle natural to mechanics is incompatible with the second postulate.

The special theory of relativity is based on the two postulates discussed above. The transformation of coordinates obtained by Einstein which is compatible with the second postulates is called Lorentz transformation. The Lorentz transformation is detailed in the next unit.

25.5 SUMMARY

In Michelson Morley experiment the velocity of light is found in two perpendicular directions. Postulates of special theory of relativity are explained qualitatively.

25.6 MODEL EXAMINATION QUESTIONS

I. Answer the Following Questions in 30 Lines.

Discuss Michelson Morley expariment bringing out the significance of results.

II. Answer the Following in 10 Lines.

State the laws of special theory of relativity.

UNIT 26 : LORENTZ TRANSFORMATION

Contents

26.1 AIMS AND OBJECTIVES

In this unit Lorentz transformations and the consequences of Lorentz transformations namely space quantization and time dilation are explained.

After going through this must you will be able to deduce the mass energy relation from Lorentz transformation.

26.2 INTRODUCTION

In 1905 Einstein published his special theory of relativity which mainly deals with what two observers moving relative to each other observe and measure. The treatment mainly involves certain mathematical equations known, as Lorentz transformations. These transformations deal with two basic concepts mainly the variation of mass and time with the velocity of the body if it is comperable to the velocity of light. And these concepts explain the mass energy relation also successfully in this theory.

26.3 LORENTZ TRANSFORMATION

To make deductions from relativity, it is necessary to compare the descriptions of some phenomenon in terms of two inertial frames which are moving relative to each other. As shown in Fig.26.1 let us consider two inertial reference frames S and S′ which are in relative motion. Let S′ be moving with a constant velocity v relative to S. Let the two observers O and O′ be located at the origin of the coordinate system of the two inertial systems of reference and are at rest. Since the inertial reference systems are in relative motion, the two observers are also in relative motion, one moving with uniform velocity v relative to the other along x-axis.

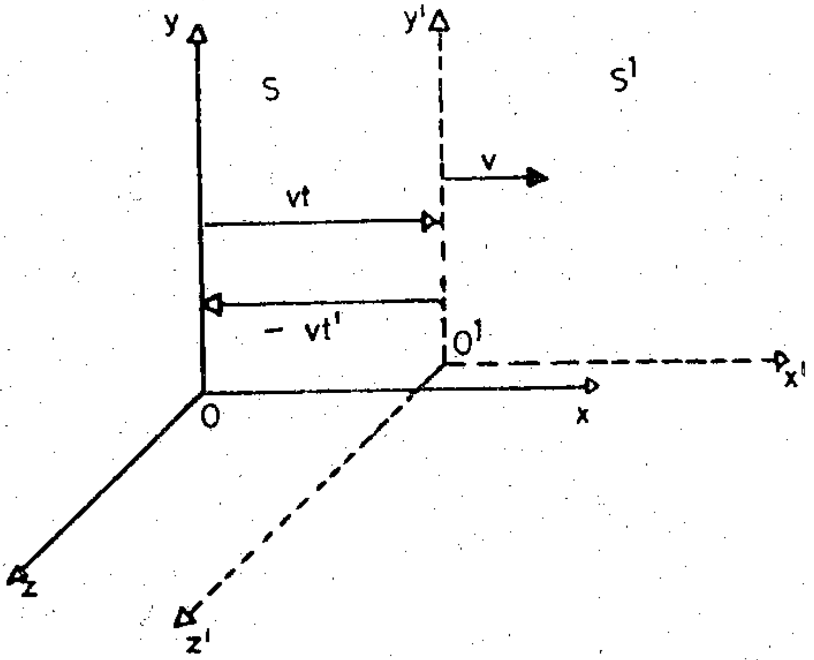


Fig.26.1 Lorentz transformation of co-ordinates.

Any point events is defined by the first observer O with the spatial coordinates and time (x, y, z, t). The same event is defined by the second observer O′ by (x′, y′, z′, t′). The transformation equations relate (x′y′z′) and time t′ of the event if the first observer assigns the values (x,y,z) and t to the same event.

Let us assure that at the instant the two origins of the inertial reference systems coincide, a small pulse of light be produced at the origin. The second postulate of the special theory of relativity implies that both the observers should see the light propagating outward with the same speed c in all directions. Hence, with respect to his own reference frame each observer must see the wave front as a sphere of radius equal to c times the time t or t′ as the case may be.

The equations of motion of the spherical wave fronts as seen by the observers O and o' are

and $X^2 + Y^2 + Z^2 - c^2 t^2 = 0$ (26.1)

$$X'^2 + Y'^2 + Z'^2 - c^2 t^2 = 0 \quad (26.2)$$

$$X^2 + Y^2 + Z^2 - C^2 t^2 = X'^2 + Y'^2 + Z'^2 - C^2 t'^2 \quad (26.3)$$

The quantity on either side of Eq.26.3 must be invariant with respect to the transformation leading from one system to another.

Since the relative motion of the inertial reference systems is in the x direction, let us assume that the transverse Co-ordinates remain unchanged. That is

$$\therefore y = y', \; z = z' \quad (26.4)$$

Substitution of Eq. 26.4 in Eq. 26.3 gives

$$x^2 - c^2 t^2 = x'^2 - c^2 t'^2 \quad (26.5)$$

Since there is only one relative velocity V, the transformation formulae should fulfill the condition that the coordinates of the origin $0'$ in S be vt and the coordinates of the origin O in S′ be - vt′. Since the two origins coincided at $t = t' = 0$ we have the following requirements to be satisfied.

$$x = vt \quad \text{if} \quad x' = 0$$

$$x' = -vt' \text{ if } x = 0 \tag{26.6}$$

x' and t' should be linear functions of both x and t. Similarly x and t also should be linear functions of x^1 and t^1. This linearity is a fundamental property of the transformation equations. The simples relations which satisfy Eq. 26.6 are given by

$$x' = r(x-vt)$$

$$x = r'(x' + vt') \tag{26.7}$$

Where r and r^1 are constants to be determined t^1 can be expressed in terms of unprimed quantities by eliminating x^1 from Eq. 26.7 Thus

$$x = r'[r(x - vt)+vt'] \tag{26.8}$$

$$\frac{x}{r'} = r(x - vt) + vt' \tag{26.9}$$

$$t' = \frac{x}{r'v} - \frac{r(x - vt)}{v} \tag{26.10}$$

$$t' = \frac{x}{r'v} - \frac{xr}{v} + rt \tag{26.11}$$

$$t' = r\left(t - \frac{x}{v} + \frac{x}{v}\frac{1}{rr'}\right) \tag{26.12}$$

$$t' = r\left[t - \frac{x}{v}\left(1 - \frac{1}{rr'}\right)\right] \tag{26.13}$$

substituting for x^1 and t^1 from Eqns 26.7 and 26.13 in Eq. 26.5 we get

$$x^2 - c^2t^2=r^2(x-vt)^2-c^2r^2[t-x/v(1-1/rr')]^2 \tag{26.14}$$

$$x^2-c^2t^2=r^2(x^2+v^2t^2-2xvt)-c^2r^2[t^2-2xt/v(1-1/rr')+x^2/v^2(1-1/rr')^2] \tag{26.15}$$

$$x^2\left[1-r^2+\frac{r^2c^2}{v^2}(1-1/rr')^2+2xt[r^2v-\frac{r^2c^2}{v^2}(1-1/rr')]+t^2[c^2(r^2-1)-r^2v^2]=0 \tag{26.16}$$

In order that Eq.26.16 will always be zero for all possible values of x and t, it is necessary that the coefficients of $x^2 xt$ and t^2vanish independently. Equating the coefficient of t^2 to zero we get

$$c^2(r^2-1)-r^2v^2=0 \tag{26.17}$$

$$r^2(c^2-v^2)=c^2 \tag{26.18}$$

$$r = \left[\frac{c^2}{c^2 - v^2}\right]^{\frac{1}{2}} = \frac{1}{[1 - v^2/c^2]^{\frac{1}{2}}} \tag{26.19}$$

Equating the coefficient of xt to zero, we get

$$2\left[r^2 v - \frac{c^2 r^2}{v}\left(1-\frac{1}{rr'}\right)\right] = 0 \qquad (26.20)$$

$$\text{or } v^2 - c^2 (1-1/rr')=0 \qquad (26.21)$$

$$(1-\frac{1}{rr'}) = v^2/c^2 \qquad (26,22)$$

$$1/rr'=(1-v^2/c^2)=1/r^2 \qquad (26.23)$$

$$\therefore r = r' \qquad (26.24)$$

we can prove $\quad r = r' = \dfrac{1}{\left(1-\dfrac{v^2}{c^2}\right)^{\frac{1}{2}}}\quad$ Using the value of

that the coefficient of x^2 in Eq 26.16 turns out to zero. Using the values of r and r' we can write the transformation equation as

$$x' = \frac{x - vt}{[1 - v^2/c^2]^{\frac{1}{2}}}$$

$$y' = y$$

$$x' = z$$

$$t' = \frac{(t - vx/c^2)}{[1 - v^2/c^2]^{\frac{1}{2}}} \qquad (26.25)$$

Similarly

$$x = \frac{(x' + vt')}{[1 - v^2/c^2]^{\frac{1}{2}}}$$

$$y' = y$$

$$z = z'$$

$$t' = \frac{(t + vx/c^2)}{[1 - v^2/c^2]^{\frac{1}{2}}} \qquad (26.26)$$

The Eqs.26.25 and 26.26 are called Lorentz-Einstein transformation equations.

If $c \to \infty$ in 26.25 we obtain the Galilean transformation. This is what we see as a first approximation to the Lorentz transformation for finite values of v. It can be seen from Eqs.26.25 and that v can never exceed c. If v exceeds c then $(1-v^2/c^2)$ becomes negative and its root becomes imaginary and the results then would become meaningless. Thus c is the maximum velocity that can ever be observed and all other velocities encountered must be less than c. The significance of Lorentaz transformation can be seen if v is comparable to c under which conditions the special theory of relativity projects new phenomena which cannot be conceived by Newtonian mechanics.

26.4 SPACE CONTRACTION

Consider a body which has length L_0 measured in the direction of x, when at rest, in the intertial reference system S. Let the body be set in motion relative to S at such a speed that it is at rest in the inertial reference frame S'. The length of the body as measured in S' will be L_o we have to findout the length of the body as measured in S relative to which the body is moving with a velocity v. The length of the body can be defined as the distance between two fixed points in S which are occupied by the ends of the body simultaneously i.e. at the same time t. If the coordinates of these points are X_1, and X_2 when the length $L_1 = X_2 - X_1$. If the ends of the body have the coordinates X_2 and X_1 in S_1 then, the length L_0 is given by

$$L_0 = X'_2 - X'_1 \qquad (26.27)$$

Using Lorentz transformation equation

$$x' = \frac{x - Vt}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \qquad (26.28)$$

in Eq. 26.27 we get

$$L_0 = \frac{X_2 - Vt}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} - \frac{X_1 - Vt}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \qquad (26.29)$$

$$L_0 = \frac{X_2 - X_1}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} = \frac{L}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \qquad (26.30)$$

$$L = L_0\left[1 - \frac{V^2}{C^2}\right]^{\frac{1}{2}} \qquad (26.31)$$

Eq. 26.31 indicates that an object measures shorter in terms of a frame relative to which it is moving with a velocity V than it does as measured in a frame relative to which it is at rest. The ratio of shortening is given by $\left[1 - \frac{V^2}{C^2}\right]^{\frac{1}{2}}$ Relative to a single frame any physical body set into motion with a speed V shortens in the direction of its motion in the ratio $\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}$.

Fig.26.1 illustrates the shortening of a ruler as moves approaching the velocity of light.

This indicates that distances shrink with increasing speed and is known as space contraction.

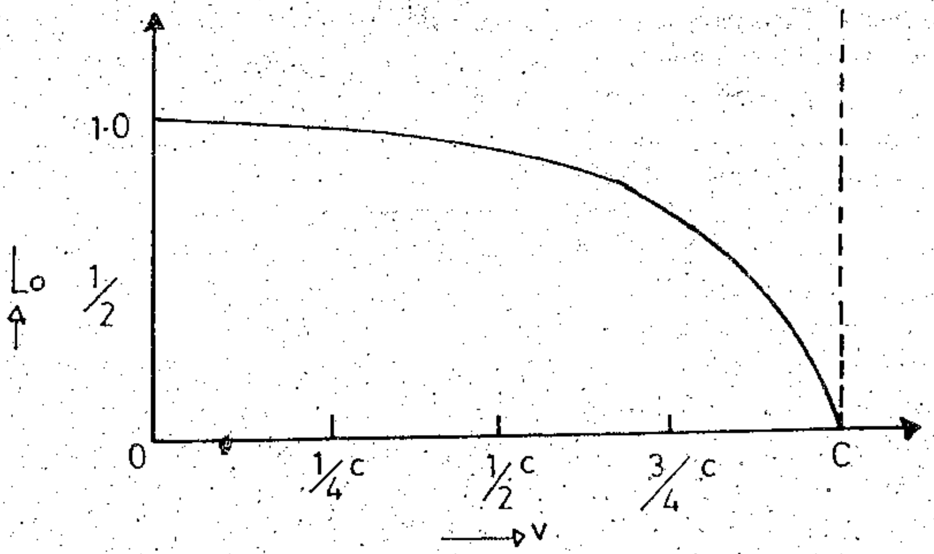


Fig 26.2 Distance shrink with increasing speed. 1 meter ruler pointed in the direction of motion appears to shrink as the speed of the ruler approaches the velocity of light.

This contraction is perhaps not a real one, but as far as effects on surrounding bodies are concerned the contraction is a real one.

26.5 RELATIVITY OF SIMULTANEITY

Let us suppose that two events occur at points X_1 and X_2 in the S system simultaneously so that $t_1 = t_2$. The times at which these events are observed in S' system which is moving with a velocity V relative to S are given by

$$t_1^1 = \frac{t^1 - \frac{vx_1}{c^2}}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.32}$$

and

$$t_2^1 = \frac{t_2 - \frac{vx_2}{c^2}}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.33}$$

The time interval between the events in S' system is given by

$$\Delta t^1 = t^1{}_2 - t^1{}_1 = \frac{t_2 - \frac{vx_2}{c^2}}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} - t^1{}_1 = \frac{t_1 - \frac{vx_1}{c^2}}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

$$\frac{1}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}}\left[(t_2-t_1)-\frac{v}{c^2}(X_2-X_1)\right] \qquad (26.34)$$

Since $t_2=t_1$

$$\Delta t^1 = -\frac{v}{c^2}\cdot\frac{(x_2-x_1)}{\left[1-\frac{v_2}{c_2}\right]^{\frac{1}{2}}} \neq 0 \qquad (26.35)$$

Eq.26.35 indicates that two events which occur at two different points X_1 and X_2 and which are simultaneous for an observer at rest in *S* will no longer appear to be simultaneous to an observer at rest in *S'* but moving relative to *S*. Simultaneity is not an absolute property of a pair of events but is also a function of the state of motion of the observer. There is no such thing as simultaneity between two observers moving with respect to each other. As shown in Fig. 26.3 suppose we in an (X, t) system observe two events *A* and *B* at the same time but separated by a large distance. Then the time of occurence of event *A* equals the time of occurence of event *B*. That is t_A - t_B. The moving observer in the (X_1, t_1) system will say that the two events did not occur at the same time. Because of the tilting of the axes of the moving system as shown in the figure, the observer will say that event ***B*** occured before event *A*. This aspect is also signified by the negative sign in Eq. 26.35

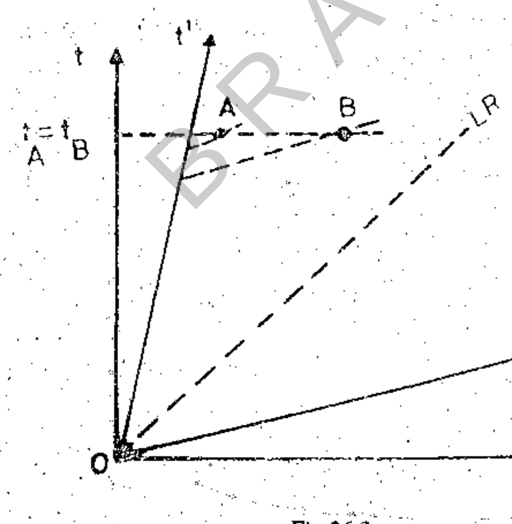


Fig 26.3

Fig.26.3 Relativity of simultancity X_1, *t* system is stationary (x^1, t^1) system is in motion. Its axes and titled towards the 45°. light ray LR.

26.6 TIME DILATION

Consider a clock located at x_1 in the inertial reference system *S*. Let the clock emit signals of some sort at regular intervals Δt given by

$$\Delta t = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 \qquad (26.36)$$

For an observer in the system S^1 which is moving relative to S with a velocity V, the times t_1 and t_2 recorded are as follows.

$$t_1^1 = \frac{t_1 - \frac{vx_1}{c^2}}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.37}$$

$$t_2^1 = \frac{t_2 - \frac{vx_1}{c^2}}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.38}$$

The time interval Δt^1 is given by

$$\Delta t^1 = t_2^1 - t_1^1 = \frac{t_2 - t_1}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.39}$$

or

$$\Delta t^1 > \Delta t$$

Thus the time interval appears to be longer to the moving observer than it does in the system in which he is at rest. Thus when a clock moves with respect to an observer, it appears to him to slow down. The fact that time intervals measured in terms of moving clocks are greater than those measured by clocks at rest with respect to the observer is known as time dialation. This time dilation increases with increase in the velocity of motion is illustrated in Fig. 26.4.

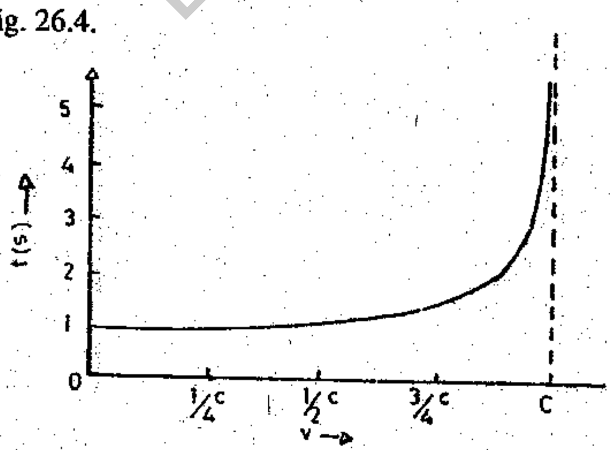


Fig. 26.4

Fig.26.4 Slowing down of time with increasing speed the length of durations appears to increase as the clock measuring this approaches the speed of light.

The famous twin paradox is a consequence of time dilation only one of the twins who goes out in a space ship with a velocity comparable to the velocity of light finds himself younger than his

counterpart after return by a factor given by $\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}$

26.7 VARIATION OF MASS WITH VELOCITY

By considering a collision between two bodies, one can obtain an expression for the variation of mass with velocity. Let us suppose that we have two experimenters one at rest in the inertial reference frame S and the other at rest in another intertial reference frame S'. Let S' be moving with a velocity V in the X-direction relative to S. Let the experimenters have identical and perfectly elastic spheres with them. When the two experimenters cross each other, let each of them project his sphere with a speed **u** in a direction perpendicular to X so that a collision takes place. This is illustrated in Fig.26.5 let u <<v.

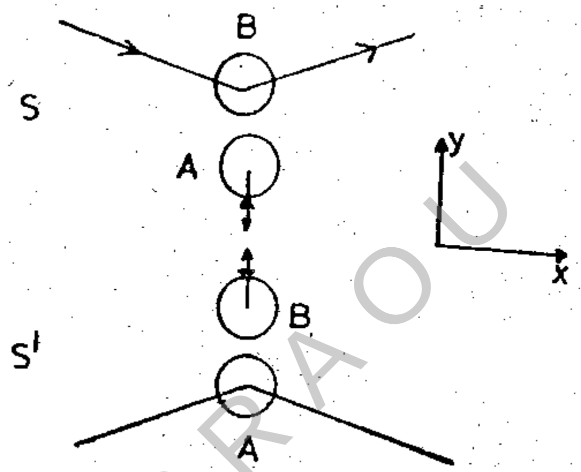


Fig 26.5 Elastic collision of two indentical spheres as viewed from system S and S^1 in relative motion.

Let the ball projected by the experimenter confined to the system S be designated as A. Let the ball projected by the experimenter confined to the system S^1 be designated as B. From the point of view of S the y component of velocity of the ball B is different from u. As per Lorentz transformation equations we have

$$y = y^1 \tag{26.40}$$

and

$$t = \frac{t^1 + \frac{vx^1}{c^2}}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.41}$$

Therefore we have

$$dy = dy^1 \tag{26.42}$$

$$dt = \frac{dt^1}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.43}$$

If w represents the velocity of B as judged by S we have

$$w = \frac{dy}{dt} = \frac{dy^1}{dt^1}\left[1 - \frac{v}{c^2}\right]^{\frac{1}{2}} \qquad (26.44)$$

since $\frac{dy^t}{dt} = u$, we have

$$w = u\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}} \qquad (26.45)$$

Since the collision is between indentical, perfectly elastic spheres, no energy is lost in the collision. The individual velocities of the spheres along their line of centres simply get reversed form the point of view of S the momentum can only be conserved if we put

$$m_o u - mw = mw - m_o u \qquad (26.46)$$

In the above equation m_o represents the mass of sphere A at rest relative to S and m represents the apparent mass of the identical sphere B which is passing S with a velocity v. From Eq.26.46.

$$mw = m_o u \qquad (26.47)$$

Using Eq.26.45 in Eq.26.47

$$mu\left[1 - \frac{v}{c^2}\right]^{\frac{1}{2}} = m_o u \qquad (26.48)$$

$$m = \frac{m_o}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \qquad (26.49)$$

Eq.26.49 indicates that the mass of a moving body to increase with increase in tis velocity. This variation is illustrated in Fig. 26.6

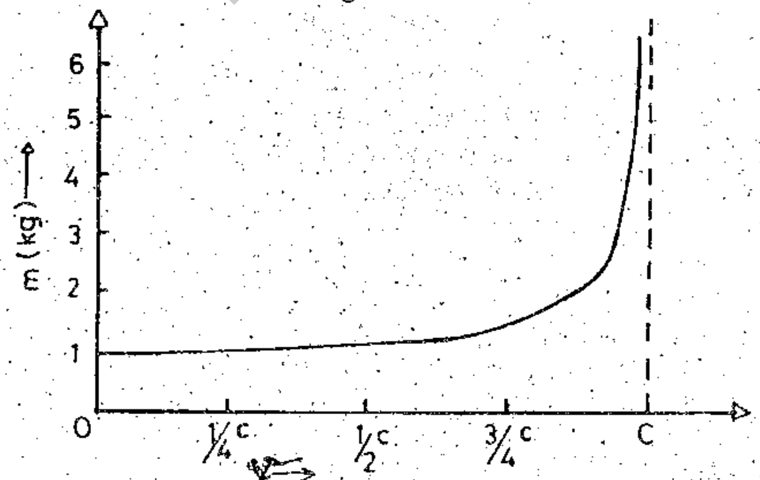


Fig 26.6 Increase in mass with increase in speed the mass of 1 kg object appears to increase as the velocity of the object approaches the velocity of light.

26.8 EINSTEIN'S MASS ENERGY RELATION

The momentum p of a particle moving with a velocity v is given by

$$P = mv \tag{26.50}$$

According to Newton's second law the force acting on a particle whose monentum is P is given by

$$F = \frac{dp}{dt} \tag{26.51}$$

Using Eq.22.25

$$F = \frac{d}{dt}(mv) \tag{26.52}$$

In relativistic mechanics both m and v are dependent on t.

Hence

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \tag{26.53}$$

The kinetic energy k acquired by a particle due to the action of the force F is given by the work done by the force in changing its velocity from 0 to v. Hence

$$dk = F\,ds \tag{26.54}$$

Where ds represents the elemental displacement of the body.

Using Eq.26.53 we get

$$dk = m\frac{dv}{dt}ds + v\frac{dm}{dt}ds \tag{26.55}$$

or

$$dk = m\frac{ds}{dt}dv + v\frac{ds}{dt}dm \tag{26.56}$$

Since $\frac{ds}{dt} = v$ we have

$$dk = mv\,dv + v^2 dm \tag{26.57}$$

The variation of mass with velocity is given by

$$m = \frac{m_o}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \tag{26.58}$$

$$dm = m_o(-1/2)\left[1 - \frac{v^2}{c^2}\right]^{1-2}[(-1/c^2)2vdv] \tag{26.59}$$

$$\therefore dm = \frac{m_o}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \frac{vdv}{c^2\left(1 - \frac{v^2}{c^2}\right)} \tag{26.60}$$

$$dm = m \times \frac{vdv}{c^2\left(1-\frac{v^2}{c^2}\right)} = \frac{mvdv}{c^2 - v^2} \qquad (26.61)$$

$$mvdv = (c^2 - v^2)dm \qquad (26.62)$$

Substituting Eq. 26.62 in Eq. 26.57 we get

$$dk = (C\text{-}V^2)\, dm + V^2\, dm = C^2\, dm \qquad (26.63)$$

If m_o represents the particles rest mass when K = 0 and m represents the mass when K = K then

$$K = \int_{m_o}^{m} C^2 dm = C^2(m - m_o) \qquad (26.64)$$

$$K = mc^2 - m_o c^2 \qquad (26.65)$$

If we define the rest mass energy $E_o = m_o c^2$ then (26.66)

or $K + E_o = mc^2$

$$E = mc^2 \qquad (26.67)$$

where E represents the sum of the kinetic and rest mass energy. It gives the total energy of the particle. Eq. 26.67 is called the Einstein's mass energy relation. The equation implies that if a mass Δm gets converted into energy in any process then the energy released is given by Δmc^2. The interconvertibility of mass into energy and energy into mass is the important outcome of the special theory of relativity. The nuclear energy which is now a days being used for constructive purpose and which can also be employed for destructive purpose is a direct consequence of this famous relation derived by Albert Einstein.

Worked Example 1

Two men who are moving in identical space vehicles carry identical measuring instruments. One of the space ship passes the other at a minimum relative velocity of 2.0 x 10^8 ms^{-1}. One of the men who is stationary in his own system of coordinates meausures the length of the ship to be 10m. Determine the length of this ship as measured by the second man who is also moving relative to the first ship with a speed of 2.0 x 10^8 ms^{-1}.

$$L = L_o\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

As per the problem $L_o = 10$m ; $V = 2.0 \times 10^8\ ms^{-1}$, $C = 3 \times 10^8\ ms^{-1}$

$$L = 10\left[1 - \left(\frac{2x10^8}{3x10^8}\right)^2\right]^{\frac{1}{2}} = 7.8\ \text{m}$$

Worked Example 2

The average life of a low-speed μ-meson as produced in the laboratory is found to be $2.2 x 10^{-6}$ s. The average life of certain high speed μ mesons observed in cosmic rays has been found to be 1.1 x 10^{-5} *s*. Determine the speed of the cosmic ray μ-mesons.

μ-mesons are elementary particles which carry one electronic charge and have rest mass of about 207 electron masses. The low-speed mesons may be considered to be at rest in the laboratory reference frame. The expression for time dilation is given by

$$\Delta t = \frac{\Delta t_o}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

where as per the problem t = 1.1 x 10^{-5} and 2.2 x 10^{-6}s and v represents the velocity of cosmic ray mesons.

$$\left[1-\frac{v^2}{c^2}\right]^1 = \left[\frac{\Delta t_o}{\Delta t}\right]^2$$

$$\frac{v}{c} = \left[1-\left(\frac{\Delta t_o}{\Delta t}\right)^2\right]^{\frac{1}{2}}$$

$$\frac{v}{c} = \left[1-\left(\frac{2.2x10^{-6}}{1.1x10^{-5}}\right)^2\right]^{\frac{1}{2}} = 0.98$$

$$v = 0.98C$$

The speed of the cosmic ray μ mesons is 0.98 times the velocity of light.

Worked Example 3

A particle moving with a velocy V has its kinetic energy 4 times its rest energy. Determine the speed of the particle.

The total energy E of the particle is given by

$$E = E_o + K$$

Where K represents the kinetic energy and E_o represents the rest energy of the particle. As per the problem when the particle moves with a velocity V its kinetic energy is 4 Eo therefore

$$E = E_o + 4E_o = 5E_o$$

According to Einstein mass energy relation if m is the mass of the particle

$$mc^2 = 5m_oc^2$$

The speed of the particle can be determined using the relativistic expression for mass. That is

$$m = \frac{m_0}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}} mc^2 = \frac{m_oc^2}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

or

$$5m_oc^2 = mc^2 = \frac{m_oc^2}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

$$\left(\frac{v}{c}\right)^2 = 1 - 1/25 = \frac{24}{25} = 0.96$$

$v = 0.98c$

26.9 SUMMARY

In the year 1905 Albert Einstein Putforth the special theory of relativity by proposing two postulates namely (i) the laws of physics have the same mathematical form in all inertial reference systems and (ii) the speed of light in vaccum is constant and the same for all observers, independent of the motion of the source and the observer.

Einstein derived the transformation equations pertaining to two inertial systems in relative motion by applying the postulates of special theory of relativity. These equations are called Lorentz-Einstein or simply Lorentz transformation equations.

It S and S′ are two inertial reference systems in which S′ is moving with uniform velocity v relative to S and if (x′, y′, z′ ,t′) represent any event in the systems S′ and S respectively then the Lorentz-Einstein transformation equations are :

$$x' = \frac{x - vt}{\left[1 - v^2/c^2\right]^{\frac{1}{2}}} \quad : \quad x = \frac{x' + vt}{\left[1 - v^2/c^2\right]^{\frac{1}{2}}}$$

$$y' = y \quad : \quad y = y'$$

$$z' = z \quad : \quad z = z'$$

$$t^1 = \frac{t - vx/c^2}{\left[1 - v^2/c^2\right]^{\frac{1}{2}}} \quad : \quad t = \frac{t' + vx'/c^2}{\left[1 - v^2/c^2\right]^{\frac{1}{2}}}$$

When a body is moving with a velocity comparable to the velocity of light relativistic effects such as (i) space contraction, (ii) time dilation and (iii) increase in mass are observed.

An object of length L_0 moving with a velocity V appears to be contracted to an observer who is at rest. The length of the object as observed by an observer in a stationary reference frame is given by

$$L = Lo\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}$$

Where c represents the velocity of light.

A moving clock appears to be slowed down as observed by a stationary observer. The increase in the duration of events in moving frame as observed by a stationary observer is called time dialation. It is given by

$$Ct' = \frac{Ct}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

There is nothing like absolute simultaneity. Two events separated by a large distance in a stationary reference frame and which occur at the same instant of time appear to have occured at different times for an observer at rest in a moving reference frame.

The mass m of a body increases with increase in velocity and is given by

$$m = \frac{m_o}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}}$$

Where m_o represents the rest mass.

Mass and energy are interconvertible. The mass-energy relation derived by Einstein is an important consequence of the special theory of relativity. It is given by

$$E - mc^2$$

Where c represents the velocity of light.

The ultimate speed any object can realize is the velocity of light.

26.10 MODEL EXAMINATION QUESTIONS

I. Answer the Following in Detail

1. Derive Lorentz transformation equation based on the positulates of special theory of relativity.

2. Derive Einstein's mass energy relation. Discuss its significance.

II. Answer the Following in Briefly.

1. Show that the length of a moving ruler appears to get shrunk for a stationary observer.

2. What is time dilation?

3. Show that the mass of moving body increases with velocity based on relativistic concepts.

4. Write a note on the significance of Lorentz transformation equations. Under what conditions the equation resumble Galilean transformation equations.

III. Solve the Following Problems.

1. A space ship of 2 m length moves with a velocity of 0.8 c with respect to a stationary observer. Determine the length of the ship as seen by the stationary observer.
(Ans : 1.2 m)

2. A clock is a space ship emits signals at intervals of 1s as observed by an astronaut in

the space ship. The space ship is moving with a velocity of 0.8 c. Determine the interval between successive signals as seen by an observer at the control centre on the ground.

(Ans : 1.67s)

3. The life time of a certain particle is found to be 4×10^{-6} s as measured in the system at rest with respect to the particle. The particle moves with a speed of 0.6 c relative to the laboratory. Determine the life time of the particle as measured by the observer at rest with respect to the laboratory. (Ans : 5×10^{-6}s)

4. The average life of low speed μ-meson is produced in the laboratory is 2.2×10^{-6} s. The μ-mesons observed in cosmic are found to move with a speed of 0.98 c. Determine the distance travelled by the cosmic ray μ-meson through the atmosphere.

(Ans : 3235 m)

5. Determine the velocity of a particle which has a mass equal to three times the rest mass.

(Ans : 0.943 c)

6. Determine the mass of an electron moving with a velocity of 0.98 c.

(Ans: 45.5×10^{-31} kg.)

7. A train at rest has a length of 1000 m. What is the observed length of this train with respect a system at rest when it is moving so fast that is mass is twice its rest mass.

(Ans : 500 m)

8. Determine the speed of a particle if it were to have kinetic energy equal to its rest energy.

(Ans : 0.98 c)

26.11 GLOSSARY

Lorentz transformation	The relations between space and time coordinates derived based on the postulates of special theory of relativity in systems which move with uniform velocities relative to each other.
Inertial reference system	A reference system in which the law of inertia holds good.
Meson	An elementary particle whose rest mass lies in between the rest mass of an electron and rest mass of a proton.
Rest mass	The mass of a particle or a body as determined in a system of reference in which it is at rest.
Rest mass energy (E_o).	It is the energy associated with the rest mass m o of a body. According to Eintein relation $F_o = m_o c^2$.
Life time of a particle	The time interval between production and decay of a particle.

26.12 RECOMMENDED BOOKS

1.	Wangsness, R.K.	Introductory Topics in Theoretical Physics	John Wiley and Sons Inc. New York.
2.	Carr, H.Y. and Weidner, R.T.	Physics from the Ground up	Mc Graw Hill Book Co., New York.
3.	Kaufmann, W.J.	Relativity and Cosmology	Harper & Row Publishers, London.
4.	Richtmyer, F.K. Kennard, E.H. and Lauritsen, T.	Introduction to Modern Physics.	Mc Graw Hill Book Co., New York.
5.	French, A.P.	Principles of Modern Physics	John Wiley & Sons Inc., New York.
6.	Renick, R.	Introduction to Special Relativity	Wiley Eastern Ltd., New Delhi.
7.	Pauli, W.	Theory of Relativity	BI Publications, Bombay.
8.	Shadowitz, A.	Special relativity	W.B.Saunders Co., London.
9.	Liverhant, S.E.	Outline of Atomic Physics	Regents Publishing Co. Inc., New York.
10.	French, A.P.Special	The English Language Relativity	Book Society and Nelson.

Dr. B.R.AMBEDKAR OPEN UNIVERSITY

(Undergraduate Programme)

SECOND YEAR SYLLABUS

PHYSICS - COURSE - I

MECHANICS WAVES AND OPTICS

BLOCK - I	**:**	**CONSERVATION OF LINEAR MOMENTUM**
UNIT - 1	:	Centre of Mass, Motion of Centre of Mass, reduced mass
UNIT - 2	:	Linear Momentum of a system of particles, Conservation of Linear momentum.
BLOCK - II	**:**	**ROTATIONAL DYNAMICS**
UNIT - 3	:	Kinematics.
UNIT - 4	:	Torque and Rotational Motion.
UNIT - 5	:	Conservation of Angular Momentum.
BLOCK - III	**:**	**GRAVITATION**
UNIT - 6	:	Motion of Planets and Satellites - Keples Laws. (Polar coordinates)
UNIT - 7	:	Gravitational Field and Gravitational Potential.
BLOCK - IV	**:**	**COLLISIONS**
UNIT - 8	:	Collision Cross section and Application to Atomic Collisions.
BLOCK - V	**:**	**OSCILLATIONS**
UNIT - 9	:	Simple Harmonic motion.
UNIT - 10	:	Damped and Forced Oscillations and Resonance.
BLOCK - VI	**:**	**WAVES IN ELASTIC MEDIA**
UNIT - 11	:	Progressive Waves - Principles of superposition and its Interference
UNIT - 12	:	Doppler effect
BLOCK - VII	**:**	**INTERFERENCE**
UNIT - 13	:	Huygen's Principle, Young's experiment and qualitative treatment - Interference pattern
UNIT - 14	:	Newtons Rings
BLOCK - VIII	**:**	**DIFFRACTION**
UNIT - 15	:	Fresnal and Fraunhofer diffraction.
UNIT - 16	:	Fresnel diffraction at a straight edge
UNIT - 17	:	Plane diffraction grating measurement of wavelength of light
UNIT - 18	:	Resolving power and dispersion of a Grating.
UNIT - 19	:	X-ray diffraction
UNIT - 20	:	Holography
BLOCK - IX	**:**	**POLARISATION**
UNIT - 21	:	Plane Polarisation - Polaroid, Polarisation by Reflection, Double Refraction
UNIT - 22	:	Production and analysis of different types of Polarized light.
UNIT - 23	:	Rotary Polarization.
BLOCK - X	**:**	**RELATIVITY**
UNIT - 24	:	Galilean Transformation
UNIT - 25	:	Michelson - Morely experiment. Postulates of special theory of Relativity.
UNIT - 26	:	Lorentz Transformation.

Dr. B.R.AMBEDKAR OPEN UNIVERSITY

FACULTY OF SCIENCES

B.S.c.II Year (3 Y.D.C) Annual Examination,

Model Question Paper

PHYSICS - COURSE -I

MECHANICS WAVES AND OPTICS

TIME : THREE HOURS] [MAX. MARKS : 70

[MIN. MARKS : 25

SECTION - A

(MARKS : 3 X 15 = 45)

Instructions to the Candidate

*1) Answer **any three** of the folowing questions in about 30 lines each.*

*2) Each question carries **fifteen** marks.*

1. Derive an expression for the moment of Inertia of a solid cylinder about an axis perpendicular to the axis of the cylinder and passing through it centre.

2. Derive Einstein's Mass Energy relation. Discuss its Significance.

3. Explain the phenomenon of Doppler effect and clearly discuss the cases.

4. How do the following Quantities vary with time in the case of simple harmonic motion.

   (a) displacement (b) velocity (c) Acceleration

   (d) potential energy (e) kinetic energy

   Depict the variation graphically.

5. Explain the conditions under which the phenomenon of total internal reflection can be observed. Describe a few applications of the phenomenon of total internal reflection.

6. Derive the equation which represents the variations of intensity with angle which describe the diffraction pattern of a single slit.

SECTION - B

(MARKS : 5 X 5 = 25)

Instructions to the Candidate

*1) Answer **any five** of the folowing questions in about 30 lines each.*

*2) Each question carries **five** marks.*

7. Explain Parallel axis theorem.

8. The rest mass of a body is 0.010kg. What is its mass when it moves at a speed of 3.0×10^7 m/sec relative to the observes.

9. State Keplers laws of planetary motion.

10. On what factors depend the ability of a Prism to resolve in the neighbouring Wave fronts.

11. Distinguish between progressive waves & stationary waves.

12. Explain the Principles of super position of Waves.

13. Derive Bragg's Law.

14. Explain the phenomenon of double refraction

15. Discuss the construction and working of a polaroid.

16. Mention the applications of Michelson interferometer.